A PRICE-VOLUME MODEL FOR A SINGLE-PERIOD STOCK MARKET

by

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A thesis submitted in partial fulfillment of the requirements
for the Honors in the Major Program in Mathematics
in the College of Sciences
and in the Burnett Honors College
at the University of Central Florida
Orlando, Florida

Fall Term, 2014

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ABSTRACT

The intention of this thesis is to provide a primitive mathematical model for a financial market in which tradings affect the asset prices. Currently, the idea of a price-volume relationship is typically used in the form of empirical models for specific cases. Among the theoretical models that have been used in stock markets, few included the volume parameter. The thesis provides a general theoretical model with the volume parameter for the intention of a broader use.

The core of the model is the correlation between trading volume and stock price, indicating that volume should be a function of the stock price and time. This function between price and time was made visible by the use of the trading volume process, also known as the Limit Order book.

The development of this model may be of some use to investors, who could build their wealth process based on the dynamics of the process found through a Limit Order Book. This wealth process can help them build an optimal trading strategy design.

DEDICATION

This paper is dedicated to my enduring husband and loving parents.

ACKNOWLDGEMENTS

I would like to express my appreciation for the contributions of Dr. Jiongmin Yong for mentoring me and guiding me with endless insight and wisdom. I would also like to thank Dr. Liquang Ni and Dr. Zuhair Nashed for their patience and sharing their invaluable advice throughout the course of this thesis' development.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF EQUATIONS

1 INTRODUCTION

The purpose of this research is to provide a primitive mathematical model for a financial market in which tradings affect the asset prices. From an investor's point of view, a common way of maximizing the return of a given stock is to make a prediction about its price so that an optimal trading time and price can be determined. In checking the price of a given stock, investors usually look at metrics that they think can help predict the value of that stock, such as the earnings per share, the price-to-earnings ratio, the dividend yield, or the shares outstanding. Albeit considered for the purpose of understanding the price volatility and the information regarding a given stock, most investors do not use the volume parameter as their primary indicator of price [1].

Although a price-volume relationship has been well-established and supported by several important studies, which will be further explored in the Literature Review chapter of this paper, most of the models presented in these studies are empirical. Admittedly, empirical models are efficient and satisfactorily accurate for specific cases, but they cannot be used in the general case like theoretical models. Among theoretical models that have been used in stock markets, few included the volume parameter. For example, the Black-Scholes Model and the ARCH Model do not include the volume parameter [2], and the study by Lamoureux and Lastrapes [3] used the volume parameter for the sole purpose of quantifying the arrival of information.

Since there exists a correlative relationship between the volume and the stock price, we can define the trading volume as a function of the stock price and the time. The theoretical model can then be simplified down to one random process. This model will be useful for investors because it introduces a new perspective to analyzing the relationship between price and volume by starting with a theoretical model instead of an empirical model. This functionality will give investors the tool to fit the model to more general cases instead of relying on specific models for each specific case.

The remainder of this paper is arranged as follows. Chapter 2 provides a brief review on the literature used to understand the price-volume relationship. Chapter 3 presents the hypothesized market in which we will build the theoretical model. We then present the theoretical model that describes the trading process in a single-period market in Chapter 4. We will finally conclude our findings and present possible uses of our model in Chapter 5.

2 LITERATURE REVIEW

As early as 1959, Osborne [4] attempted to find the price-volume relationship in his seminal work, in which he modeled the stock price change as a diffusion process with variance depending on the number of transactions. However, Osborne assumed that transactions formed a uniform distribution over time, and hence he expressed the price process in time intervals instead of directly addressing the volume-price issue.

In 1966, Ying [5] concluded a more detailed relationship between security prices and trading volume with empirical methods on a six-year, daily series of price and volume. The data Ying chose for his investigation consist of Standard and Poor's 500 composite stocks daily closing price indexes and daily volume of stock sales on the New York Stock Exchange. His conclusions mainly were:

- A small daily volume of stock sales is usually accompanied by a fall in the daily closing price index
- A large daily volume of stock sales is usually accompanied by a rise in the daily closing price index
- A large increase in the daily volume of stock sales is usually accompanied by a large change in the daily closing price index
- A large daily volume of stock sales is usually followed by a rise in the daily closing price index
- If the daily volume of stock sales has decreased (increased) five straight trading days, the daily closing price index will tend to fall (rise) over the next four trading days.

Critics to Ying's work have argued that his price series and volume series were not necessarily comparable. In addition, his adjustments to the data were questionable since he adjusted the daily price series by quarterly dividends while the daily volume by monthly outstanding shares. However, he was still the first to document both price-volume correlations in the same data set.

In 1987, Karpoff [6] conducted a survey to study how prices and volume move together. He synthesized previous research and determined empirically that large volume is accompanied with large absolute change, although no conclusion can be drawn for the direction of the price change. Below is a summary of empirical studies about correlation of the absolute value of the price change ($|\delta p|$) with trading volume (V):

Table 2.1, Summary of empirical studies about correlation of the absolute value of the price change (|δp|) with trading volume (V)

Author(s)	Years of Study	Sample Data	Sample Period	Differencing Period	Support Positive $(\lvert\delta p\rvert, V)$ Correlation?
Godfrey, Granger, and Morgenstem	1964	Stock market aggregates, 3 common stocks	1859-63, 1951-53, 1963	weekly, daily transactions	No
Ying	1966	Stock market aggregates	1957-62	daily	Yes
Crouch	1970	5 common stocks	1963-67	daily	Yes
Crouch	1970	Stock market aggregates, 3 common stocks	1966-68	hourly and daily	Yes
Clark	1973	Cotton future contracts	1945-58	daily	Yes
Epps and Epps	1976	20 common stocks	1971	transactions	Yes
Morgan	1976	17 common stocks, and 44 common stocks	1962-65, 1926-68	4-days, monthly	Yes
Westerfield	1977	315 common stocks	1968-69	daily	Yes
Cornell	1981	Futures contracts for 17 commodities	1968-79	daily	Yes
Harris	1983	16 common stocks	1968-69	daily	Yes
Tauchen and Pitts	1983	T-bill futures contracts	1976-79	daily	Yes
Comiskey, Walking, and Weeks	1984	211 common stocks	1976-79	yearly	Yes
Harries	1984	50 common stocks	1981-83	transactions, daily	Yes
Rutledge	1984	Futures contracts for 13 commodities	1973-76	daily	Yes
Wood, Micinish, and Ord	1986	946 common stocks, 1138 common stocks	1971-72	minutes	Yes
Harris	1986	479 common stocks	1976-77	daily	Yes
Jain and Joh	1986	Stock market aggregates	1979-83	hourly	Yes
Richardson, Sefcik, and Thompson	1987	106 common stocks	1973-82	weekly	Yes

Karpoff also discussed how volume changed during upticks and downticks. He stated that the ratio of volume to a positive price change (when bulls' demands increase) was greater than the absolute value of the ratio of volume to a negative price change (when the bears' demands decrease).

In 1993, Campbell et al. [7] raised questions about the behavior of price changes after periods of abnormal changes in volume. They considered a model in which risk-averse market-makers accommodate buying or selling pressure from liquidity or non-informational traders. Campbell et al. found that price changes due to high volume tend to be reversed over time.

The assumption of Campbell's model is that there are two kinds of price changes: informational and non-informational. If the price change is informational, which means that the information has been spread to the public, then little trading will occur since there is a general agreement concerning the new value of the security. On the other hand, if the price change is caused by non-informational investors, the trading volume will significantly increase due to an unbalanced perceived new value of the security. Based on this assumption, in the case of non-informational trading, the price changes due to high volume will be reversed once the non-informational traders have completed their transactions.

In 2006, Lo and Wang [8] implemented empirically an intertemporal equilibrium model using weekly returns and trading volume of NYSE and AMEX stocks from 1962 to 2004. With the model, they identified and constructed a hedging portfolio that has considerable forecast power in predicting future returns of the market portfolio.

In Lo and Wang's model, a trader holds two distinct portfolios. He/she holds a market portfolio to adjust their exposure to market risk as well as a hedging portfolio to hedge the risk of changes in market conditions. In equilibrium, the return exhibits a two factor linear model, where the two factors are the returns on the two portfolios. Similarly, the trading volume also holds a two-factor structure, where the two factors are the trading volumes in the two portfolios. With this two-factor setting, Lo and Wang derived the intertemporal equilibrium model. By empirically implementing the model they constructed a hedging portfolio that has a strong forecasting power according to empirical analysis.

In conclusion, a price-volume relationship is well-established and supported by a large quantity of empirical studies. Objectively, empirical models are efficient and accurate when used for particular cases, but they cannot be used in the general case, making each empirical model only useful within their limited parameters. Since there is a correlation between the volume and the stock price, naturally trading volume should be a function of the stock price and the time. With such an idea, we introduce the available trading volume process, which is the so-called Limit Order Book (LOB) as a function of the price and the time. Investors can then build their wealth process based on the dynamics of the LOB process, followed by the optimal trading strategy design.

3 PROPOSED MODEL IN A SINGLE-PERIOD MARKET

In this section, we develop a model of a single-period market with one investor. Such a model is mathematically simple and can be served as a building block for a model in a multi-period market.

3.1 Hypothesized Market -- A Limit Order Book

We consider a stock market defined on a set of discrete time $t = 0,1$ with k risky assets ($k \in \mathbb{Z}, k > 1$). At every time t the investor is allowed to makes a trade. Let t^- denote the time right before t and t^+ the time right after t. We denote $X_i(t,p)$ to be the volume of the i^{th} stock available to be traded at time t and price p, where

Equation 3.1

$$\begin{cases} X_i(t,p) > 0, \text{denotes the volume of the } i^{\text{th}} \text{ stock available for purchase} \\ X_i(t,p) < 0, \text{denotes the volume of the } i^{\text{th}} \text{ stock available for sale} \end{cases}$$

and i $\in \{1,2,3, \dots k\}$

The family $\{X_i(t,p)\}$ is called a limit order book (LOB, for short), in which all prices of the risky asset (or stock) are listed from the highest to the lowest. For convenience, we let $p \in \mathbb{N}$ (the set of all natural numbers), and the difference in price between adjacent orders is 1. Then we can define the lowest ask price p_i^a and the highest bid price p_i^b as follows:

Equation 3.2

$$\begin{cases} p_i^a(t) = \min\{p \in \mathbb{N} | X_i(t,p) > 0\} \\ p_i^b(t) = \max\{p \in \mathbb{N} | X_i(t,p) < 0\} \end{cases}$$

It is clear that $p_i^b(t) < p_i^a(t)$. Then the higher ask prices of the i^{th} stock are $p_i^a(t) + 1$, $p_i^a(t) + 2, \dots$, and the lower bid prices are $p_i^b(t) - 1,\ p_i^b(t) - 2, \dots$

In order to perform trading analysis, we define $\mathbb{X}_i^a(t,p)$ to be the total volume of the i^{th} stock that is available for purchase with the price no more than the price p, and $\left|\mathbb{X}_i^b(t,p)\right|$ to be the total volume of the i^{th} stock that is available for sale with the price no less than the price p, i.e.,

Equation 3.3

$$\begin{cases} \mathbb{X}_i^a(t,p) = \displaystyle\sum_{p_i^a(t) \le q \le p} X_i(t,q) \\ \mathbb{X}_i^b(t,p) = \displaystyle\sum_{\mathrm{p} \le \mathrm{q} \le p_i^b(t)} X_i(t,q) \end{cases}$$

When an investor enters the market at time $t = 0^-$, we define his initial portfolio as

Equation 3.4

$$\pi(0^-) \equiv (\pi_0(0^-), \pi_1(0^-), \dots, \pi_k(0^-))$$

where $\pi_0(0^-)$ is the dollar amount in a cash account, and $\pi_i(0^-)$ is the share number held for the i^{th} stock. After he trades at time $t = 0$, the portfolio becomes

Equation 3.5

$$\pi(0^+) \equiv (\pi_0(0^+), \pi_1(0^+), \dots, \pi_k(0^+))$$

We let $\pi_i(0^+) = \pi_i(0^-) + u_i(0)$ for $1 \le \mathrm{i} \le k$ where

Equation 3.6

$$\begin{cases} u_i(0) > 0, \text{the investor purchased } u_i(0) \text{ shares of the } i^{th} \text{ stock} \\ u_i(0) < 0, \text{the investor purchased } |u_i(0)| \text{ shares of the } i^{th} \text{ stock} \end{cases}$$

3.2 Cost for Each Transaction $u_i(0)$

When $u_i(0) > 0$, i.e., the investor has purchased $u_i(0)$ shares of the i^{th} stock, since at time $t = 0^-$ and price p, the investor can only buy at most $X_i(t,p)$ shares of the i^{th} stock, we need to define the "highest" ask price of the i^{th} stock in the transaction. We denote this price by $\bar{p}_i^a(u_i(0))$, defined by the following:

Equation 3.7

$$\bar{p}_i^a(u_i(0)) = \min\{\, p \geq p_i^a(0^-) | u_i(0) \leq \mathbb{X}_i^a(0^-, p)\}$$

The LOB of positive volumes at time $t = 0^-$ can be summarized as below:

Table 3.1, The LOB of positive volumes at time t=0-

Price p	Volume at $t = 0^-$ $X_i(0^-, p)$
⋮	⋮
$\bar{p}_i^a(u_i(0)) + 1$	$X_i(0^-, \bar{p}_i^a(u_i(0)) + 1)$
$\bar{p}_i^a(u_i(0))$	$X_i(0^-, \bar{p}_i^a(u_i(0)))$
$\bar{p}_i^a(u_i(0)) - 1$	$X_i(0^-, \bar{p}_i^a(u_i(0)) - 1)$
$\bar{p}_i^a(u_i(0)) - 2$	$X_i(0^-, \bar{p}_i^a(u_i(0)) - 2)$
⋮	⋮
⋮	⋮
$p_i^a(0^-) + 2$	$X_i(0^-, p_i^a(0^-) + 2)$
$p_i^a(0^-) + 1$	$X_i(0^-, p_i^a(0^-) + 1)$
$p_i^a(0^-)$	$X_i(0^-, p_i^a(0^-))$

Therefore, we can calculate the cost of the transaction $u_i(0)$ as follows:

Equation 3.8

$$\begin{aligned} C(u_i(0)) &= c_i + g_i u_i(0) + \sum_{p < \bar{p}_i^a(u_i(0))} p X_i(0^-, p) + \bar{p}_i^a(u_i(0)) \left[u_i(0) - \sum_{p < \bar{p}_i^a(u_i(0))} X_i(0^-, p) \right] \\ &= c_i + g_i u_i(0) + \bar{p}_i^a(u_i(0)) u_i(0) - \sum_{p < \bar{p}_i^a(u_i(0))} (\bar{p}_i^a(u_i(0)) - p)\, X_i(0^-, p), \end{aligned}$$

where c_i is a fixed cost which will appear as long as $u_i(0) \neq 0$ and g_i is a proportional cost rate.

When $u_i(0) < 0$, i.e., the investor has sold $|u_i(0)|$ shares of the i^{th} stock, since at time $t = 0^-$ and price p, the investor can only sell at most $|X_i(t, p)|$ shares of the i^{th} stock, we need to define the "lowest" bid price in the transaction. We denote this price as $\bar{p}_i^b(u_i(0))$, where

Equation 3.9

$$\bar{p}_i^b(u_i(0)) = \max\{p \leq p_i^b(0^-) | |u_i(0)| \leq |\mathbb{X}_i^b(0^-, p)|\}$$

The LOB of negative volumes at time $t = 0^-$ can be summarized as below:

Table 3.2, The LOB of negative volumes at time t=0-

Price p	Volume at $t = 0^-$ $X_i(0^-, p)$
$\boldsymbol{p_i^b(0^-)}$	$X_i\left(0^-, p_i^b(0^-)\right)$
$\boldsymbol{p_i^b(0^-) - 1}$	$X_i\left(0^-, p_i^b(0^-) - 1\right)$
$\boldsymbol{p_i^b(0^-) - 2}$	$X_i\left(0^-, p_i^b(0^-) - 2\right)$
$\vdots$	$\vdots$
$\vdots$	$\vdots$
$\boldsymbol{\bar{p}_i^b(u_i(0)) + 2}$	$X_i\left(0^-, \bar{p}_i^b(u_i(0)) + 2\right)$
$\boldsymbol{\bar{p}_i^b(u_i(0)) + 1}$	$X_i\left(0^-, \bar{p}_i^b(u_i(0)) + 1\right)$
$\boldsymbol{\bar{p}_i^b(u_i(0))}$	$X_i\left(0^-, \bar{p}_i^b(u_i(0))\right)$
$\boldsymbol{\bar{p}_i^b(u_i(0)) - 1}$	$X_i\left(0^-, \bar{p}_i^b(u_i(0)) - 1\right)$
$\vdots$	$\vdots$

Therefore, we can calculate the cost of the transaction $u_i(0)$ as following:

Equation 3.10

$$C\left(u_i(0)\right) = c_i + g_i u_i(0) - \sum_{p > \bar{p}_i^b(u_i(0))} p|X_i(0^-, p)| - \bar{p}_i^b\left(u_i(0)\right)\left|u_i(0) - \sum_{p < \bar{p}_i^b(u_i(0))} X_i(0^-, p)\right|$$
$$= c_i + g_i u_i(0) + \bar{p}_i^b(u_i(0))u_i(0) - \sum_{p > \bar{p}_i^b(u_i(0))} (\bar{p}_i^b\left(u_i(0)\right) - p)\, X_i(0^-, p),$$

where c_i and g_i are defined same as the case $u_i(0) > 0$.

As long as $u_i(0) \neq 0$ for some i, the cash account $\pi(0^+)$ changes after the transaction at time $t = 0$, and from the above, we have:

Equation 3.11

$$\pi_0(0^+) = \pi_0(0^-) - \sum_{u_i(0) \neq 0} C\left(u_i(0)\right)$$

3.3 Limit Order Book at Time t=0

When $u_i(0) > 0$, i.e., the investor has bought $u_i(0)$ shares of the i^{th} stock, he has purchased all the stocks available in the market that are corresponding to the price p, where

$p_i^a(0^-) \le p \le \bar{p}_i^a(u_i(0))$. Therefore, the LOB at time $t = 0^-$ and $t = 0$ can be summarized as below:

Table 3.3, The LOB at time t=0- and t=0 when ui(0)>0

Price p	Volume at $t = 0^-$ $X_i(0^-, p)$	Volume at $t = 0$ $X_i(0, p)$
⋮	⋮	⋮
$\bar{p}_i^a(u_i(0)) + 1$	$X_i(0^-, \bar{p}_i^a(u_i(0)) + 1)$	$X_i(0^-, \bar{p}_i^a(u_i(0)) + 1)$
$\bar{p}_i^a(u_i(0))$	$X_i(0^-, \bar{p}_i^a(u_i(0)))$	$\mathbb{X}_i^a(0^-, \bar{p}_i^a(u_i(0))) - u_i(0)$
$\bar{p}_i^a(u_i(0)) - 1$	$X_i(0^-, \bar{p}_i^a(u_i(0)) - 1)$	0
$\bar{p}_i^a(u_i(0)) - 2$	$X_i(0^-, \bar{p}_i^a(u_i(0)) - 2)$	0
⋮	⋮	0
⋮	⋮	0
$p_i^a(0^-) + 2$	$X_i(0^-, p_i^a(0^-) + 2)$	0
$p_i^a(0^-) + 1$	$X_i(0^-, p_i^a(0^-) + 1)$	0
$p_i^a(0^-)$	$X_i(0^-, p_i^a(0^-))$	0
PRICE GAP		
$p_i^b(0^-)$	$X_i(0^-, p_i^b(0^-))$	$X_i(0^-, p_i^b(0^-))$
$p_i^b(0^-) - 1$	$X_i(0^-, p_i^b(0^-) - 1)$	$X_i(0^-, p_i^b(0^-) - 1)$
$p_i^b(0^-) - 2$	$X_i(0^-, p_i^b(0^-) - 2)$	$X_i(0^-, p_i^b(0^-) - 2)$
⋮	⋮	⋮
⋮	⋮	⋮
$\bar{p}_i^b(u_i(0)) + 2$	$X_i(0^-, \bar{p}_i^b(u_i(0)) + 2)$	$X_i(0^-, \bar{p}_i^b(u_i(0)) + 2)$
$\bar{p}_i^b(u_i(0)) + 1$	$X_i(0^-, \bar{p}_i^b(u_i(0)) + 1)$	$X_i(0^-, \bar{p}_i^b(u_i(0)) + 1)$
$\bar{p}_i^b(u_i(0))$	$X_i(0^-, \bar{p}_i^b(u_i(0)))$	$X_i(0^-, \bar{p}_i^b(u_i(0)))$
$\bar{p}_i^b(u_i(0)) - 1$	$X_i(0^-, \bar{p}_i^b(u_i(0)) - 1)$	$X_i(0^-, \bar{p}_i^b(u_i(0)) - 1)$
⋮	⋮	⋮

From the preceding table, we can conclude that

Euation 3.12

$$X_i(0,p) = \begin{cases} 0, & p_i^a(0^-) \le p < \bar{p}_i^a(u_i(0)) \\ \mathbb{X}_i\left(0^-, \bar{p}_i^a(u_i(0))\right) - u_i(0), & p = \bar{p}_i^a(u_i(0)) \\ X_i(0^-, p), & p > \bar{p}_i^a(u_i(0)), \text{or } p \le \bar{p}_i^b(u_i(0)) \end{cases}$$

When $u_i(0) < 0$, i.e., the investor has sold $|u_i(0)|$ shares of the i^{th} stock, he has sold all the stocks available in the market that are corresponding to the price p, where $\bar{p}_i^b(u_i(0)) \le p \le p_i^b(0^-)$. Therefore, the LOB at time $t = 0^-$ and $t = 0$ can be summarized as below:

Table 3.4, The LOB at time t=0⁻ and t=0 when $u_i(0)<0$

Price p	Volume at $t=0^-$ $X_i(0^-,p)$	Volume at $t=0$ $X_i(0,p)$
⋮	⋮	⋮
$\bar{p}_i^a(u_i(0))+1$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))+1)$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))+1)$
$\bar{p}_i^a(u_i(0))$	$X_i(0^-,\ \bar{p}_i^a(u_i(0)))$	$X_i(0^-,\ \bar{p}_i^a(u_i(0)))$
$\bar{p}_i^a(u_i(0))-1$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))-1)$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))-1)$
$\bar{p}_i^a(u_i(0))-2$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))-2)$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))-2)$
⋮	⋮	⋮
⋮	⋮	⋮
$p_i^a(0^-)+2$	$X_i(0^-,\ p_i^a(0^-)+2)$	$X_i(0^-,\ p_i^a(0^-)+2)$
$p_i^a(0^-)+1$	$X_i(0^-,\ p_i^a(0^-)+1)$	$X_i(0^-,\ p_i^a(0^-)+1)$
$p_i^a(0^-)$	$X_i(0^-,\ p_i^a(0^-))$	$X_i(0^-,\ p_i^a(0^-))$
PRICE GAP		
$p_i^b(0^-)$	$X_i(0^-,\ p_i^b(0^-))$	0
$p_i^b(0^-)-1$	$X_i(0^-,\ p_i^b(0^-)-1)$	0
$p_i^b(0^-)-2$	$X_i(0^-,\ p_i^b(0^-)-2)$	0
⋮	⋮	0
⋮	⋮	0
$\bar{p}_i^b(u_i(0))+2$	$X_i(0^-,\ \bar{p}_i^b(u_i(0))+2)$	0
$\bar{p}_i^b(u_i(0))+1$	$X_i(0^-,\ \bar{p}_i^b(u_i(0))+1)$	0
$\bar{p}_i^b(u_i(0))$	$X_i(0^-,\ \bar{p}_i^b(u_i(0)))$	$\mathbb{X}_i^b(0^-,\ \bar{p}_i^b(u_i(0)))-u_i(0)$
$\bar{p}_i^b(u_i(0))-1$	$X_i(0^-,\ \bar{p}_i^b(u_i(0))-1)$	$X_i(0^-,\ \bar{p}_i^b(u_i(0))-1)$
⋮	⋮	⋮

From the preceding table, we can conclude that

Equation 3.13

$$X_i(0,p)=\begin{cases}0, & \bar{p}_i^b(u_i(0))<p\le p_i^b(0^-)\\ \mathbb{X}_i\left(0^-,\bar{p}_i^b(u_i(0))\right)-u_i(0), & p=\bar{p}_i^b(u_i(0))\\ X_i(0^-,p), & p<\bar{p}_i^b(u_i(0)),\text{or } p\ge\bar{p}_i^a(0^-)\end{cases}$$

By the definition of $p_i^a(t)$ and $p_i^b(t)$, when $u_i(0)>0$,

Equation 3.14

$$\begin{cases}p_i^a(0)\ge\bar{p}_i^a(u_i(0))\\ p_i^b(0)=p_i^b(0^-)\end{cases}$$

When $u_i(0)<0$,

Equation 3.15

$$\begin{cases} p_i^a(0) = p_i^a(0^-) \\ p_i^b(0) \le \bar{p}_i^b(u_i(0)) \end{cases}$$

Therefore, we can conclude that for $u_i(0) \neq 0$,

Equation 3.16

$$p_i^a(0) - p_i^b(0) \ge \begin{cases} \bar{p}_i^a(u_i(0)) - p_i^b(0^-) > p_i^a(0^-) - p_i^b(0^-), & \text{if } u_i(0) > 0 \\ p_i^a(0^-) - \bar{p}_i^b(u_i(0)) > p_i^a(0^-) - p_i^b(0^-), & \text{if } u_i(0) < 0 \end{cases}$$

3.4 Limit Order Book at time t=0+

After the investor finished trading, how will the stock market change? Let us first discuss the case in which the investor purchased $u_i(0)$ shares of stocks at time $= 0$.

Immediately after the investor finished buying $u_i(0)$ shares of the i^{th} stock, the sell limit orders from the price $p_i^a(0^-)$ to $\bar{p}_i^a(u_i(0))$ were removed. In other words, another investor who enters the market at $t = 0$ is not able to purchase the i^{th} stock with a price lower than $\bar{p}_i^a(u_i(0))$. Therefore, more buy limit orders (the i^{th} stock that is available for sale) will be filled in the market with the price above $p_i^a(0^-)$, and more sell limit orders (the i^{th} stock that is available for purchase) will be filled in with the prices under $\bar{p}_i^a(u_i(0))$, to meet the needs. Eventually, the lowest ask price and highest bid price of new orders will get close in the middle of the prices $p_i^b(0)$ and $p_i^a(0)$.

We define two middle prices $p_i^{m_+}(t)$ and $p_i^{m_-}(t)$ between the highest ask price $p_i^a(t)$ and the lowest bid price $p_i^b(t)$ by

Equation 3.17

$$\begin{cases} p_i^{m_+}(t) = \left\lfloor \dfrac{p_i^a(t) + p_i^b(t) + 1}{2} \right\rfloor \\ p_i^{m_-}(t) = \left\lfloor \dfrac{p_i^a(t) + p_i^b(t)}{2} \right\rfloor \end{cases}$$

where $\lfloor x \rfloor = \max\{m \in \mathbb{Z} | m \leq x\}, \forall x \in \mathbb{R}$. Therefore, in the case that $p_i^a(t) + p_i^b(t)$ is even, $p_i^{m_+}(t) = p_i^{m_-}(t)$. Otherwise, $p_i^{m_+}(t) = p_i^{m_-}(t) + 1$.

Using the definition of middle price, relationship between $p_i^{m-}(0)$ and $p_i^{m-}(0^-)$, as well as relationship $p_i^{m_+}(0)$ and $p_i^{m_+}(0^-)$, in the case $u_i(0) > 0$ are:

Equation 3.18

$$\begin{cases} p_i^{m_+}(0) = \left\lfloor \dfrac{p_i^a(0) + p_i^b(0) + 1}{2} \right\rfloor \geq \dfrac{\bar{p}_i^a\big(u_i(0)\big) + p_i^b(0^-)}{2} > \dfrac{p_i^a(0^-) + p_i^b(0^-)}{2} = p_i^{m_+}(0^-) \\ p_i^{m_-}(0) = \left\lfloor \dfrac{p_i^a(t) + p_i^b(t)}{2} \right\rfloor \geq \dfrac{\bar{p}_i^a\big(u_i(0)\big) + p_i^b(0^-)}{2} > \dfrac{p_i^a(0^-) + p_i^b(0^-)}{2} = p_i^{m_-}(0^-) \end{cases}$$

We assume a constant gap between the lowest ask price and highest bid price at time $t = 0^-$ and $t = 0^+$, i.e.,

Equation 3.19

$$p_i^a(0^+) - p_i^b(0^+) = p_i^a(0^-) - p_i^b(0^-) = 2\delta_i$$

for some $\delta_i \in \mathbb{N}$.

Note that the transaction is made at $t = 0$, which leads changes of the limit order book between $t = 0^-$ and $t = 0$, but there is no transaction between $t = 0$ and $t = 0^+$. Therefore, we assume that $p_i^{m_+}(0) = p_i^{m_+}(0^+)$, $p_i^{m_-}(0) = p_i^{m_-}(0^+)$ and such an assumption leads to the following:

Equation 3.20

$$\begin{cases} p_i^a(0^+) = p_i^{m_+}(0) + \delta_i \\ p_i^b(0^+) = p_i^{m_-}(0) - \delta_i \end{cases}$$

Hence in the case $u_i(0) > 0$,

Equation 3.21

$$\begin{cases} p_i^a(0^+) = p_i^{m_+}(0) + \delta_i = \left\lceil \dfrac{p_i^a(0) + p_i^b(0) + 1}{2} \right\rceil + \delta_i > p_i^{m_+}(0^-) + \delta_i = p_i^a(0^-) \\ p_i^b(0^+) = p_i^{m_-}(0) - \delta_i = \left\lfloor \dfrac{p_i^a(0) + p_i^b(0)}{2} \right\rfloor - \delta_i > p_i^{m_-}(0^-) + \delta_i = p_i^b(0^-) \end{cases}$$

This means that in the case $u_i(0) > 0$ the overall price is increasing. As we mentioned before, the lowest ask price $p_i^a(0^+)$ and highest bid price $p_i^b(0^+)$ will approach in the middle of the prices $p_i^a(0)$ and $p_i^b(0)$. We also need to take into account the fact that $X_i(0,p) = 0$, for $p \in \{p_i^a(0^-),\ p_i^a(0^-) + 1, \dots,\ \bar{p}_i^a(u_i(0))\}$. To this end, we partition the prices into four sets, namely, $\{\dots,\ p_i^b(0^-) - 1,\ p_i^b(0^-)\}$, $\{p_i^a(0^-),\ p_i^a(0^-) + 1, \dots,\ p_i^b(0^+)\}$, $\{p_i^a(0^+), p_i^a(0^+) + 1, \dots, \bar{p}_i^a(u_i(0))\}$, and $\{\bar{p}_i^a(u_i(0)) + 1, \bar{p}_i^a(u_i(0)) + 2, \dots\}$. We then discuss the volume $X_i(0^+, p)$ for each case.

We first look at the sets $\{\dots,\ p_i^b(0^-) - 1,\ p_i^b(0^-)\}$ and $\{p_i^a(0^-),\ p_i^a(0^-) + 1, \dots,\ p_i^b(0^+)\}$. When $p \in \{p_i^a(0^-),\ p_i^a(0^-) + 1, \dots,\ p_i^b(0^+)\}$, $X_i(0,p) = 0$. Hence, we need to fill in buy limit orders at $t = 0^+$ in this price set. Consider $X_i(0,p)\ \forall\ p \in \{\dots,\ p_i^b(0^-) - 1,\ p_i^b(0^-)\}$, which are the buy limit orders at $t = 0$. They were not traded at $t = 0$, and thus kept the distribution of all buy limit orders when the market was not affected by any transaction. Therefore, we shift up all buy limit orders at $t = 0$ by $p_i^b(0^+) - p_i^b(0^-)$ units to fill the LOB at $t = 0^+$ in the price set $\{\dots,\ p_i^b(0^-) - 1,\ p_i^b(0^-)\} \cup \{p_i^a(0^-),\ p_i^a(0^-) + 1, \dots,\ p_i^b(0^+)\} = \{\dots,\ p_i^b(0^+) - 1,\ p_i^b(0^+)\}$. In other words, we let

Equation 3.22

$$\begin{aligned} X_i(0,p) &= X_i\left(0^+, p + p_i^b(0^+) - p_i^b(0^-)\right) \\ &= X_i\left(0^+, p + p_i^{m-}(0) - \delta_i - p_i^b(0^-)\right) \end{aligned}$$

$$= X_i\left(0^+, p + \left\lfloor\frac{p_i^a(0) + p_i^b(0)}{2}\right\rfloor - p_i^b(0^-) - \delta_i\right)$$

$$= X_i\left(0^+, p + \left\lfloor\frac{p_i^a(0) + p_i^b(0)}{2}\right\rfloor - \delta_i\right), \forall\, p \in \{\dots,\ p_i^b(0^+) - 1,\ p_i^b(0^+)\}$$

Thus we finished modeling the volume $X_i(0^+, p)$ for the price set $\{\dots,\ p_i^b(0^+) - 1,\ p_i^b(0^+)\}$.

As for the volumes corresponding to the price set $\{p_i^a(0^+),\ p_i^a(0^+) + 1, \dots\}$, we do not shift down the sell limit orders at $t = 0$ by $(\bar{p}_i^a(u_i(0)) - p_i^a(0^+) + 1)$ units to fill the LOB at $t = 0^+$, as we did to model the volumes corresponding to the price set $\{\dots,\ p_i^b(0^+) - 1,\ p_i^b(0^+)\}$. The reasoning is that the distribution of the sell limit orders at $t = 0$ were affected by the transaction since the orders in the price set $\{p_i^a(0^-),\ p_i^a(0^-) + 1, \dots,\ \bar{p}_i^a(u_i(0))\}$ were removed. Therefore, we need to discuss the volume $X_i(0^+, p)$ for the price sets $\{p_i^a(0^+),\ p_i^a(0^+) + 1, \dots,\ \bar{p}_i^a(u_i(0))\}$ and $\{\bar{p}_i^a(u_i(0)) + 1,\ \bar{p}_i^a(u_i(0)) + 2, \dots, \}$ separately. We let

Equation 3.23

$$X_i(0, p) = X_i(0^+, p), \forall\, p \in \{\bar{p}_i^a(u_i(0)) + 1,\ \bar{p}_i^a(u_i(0)) + 2, \dots, \},$$

since they were not traded.

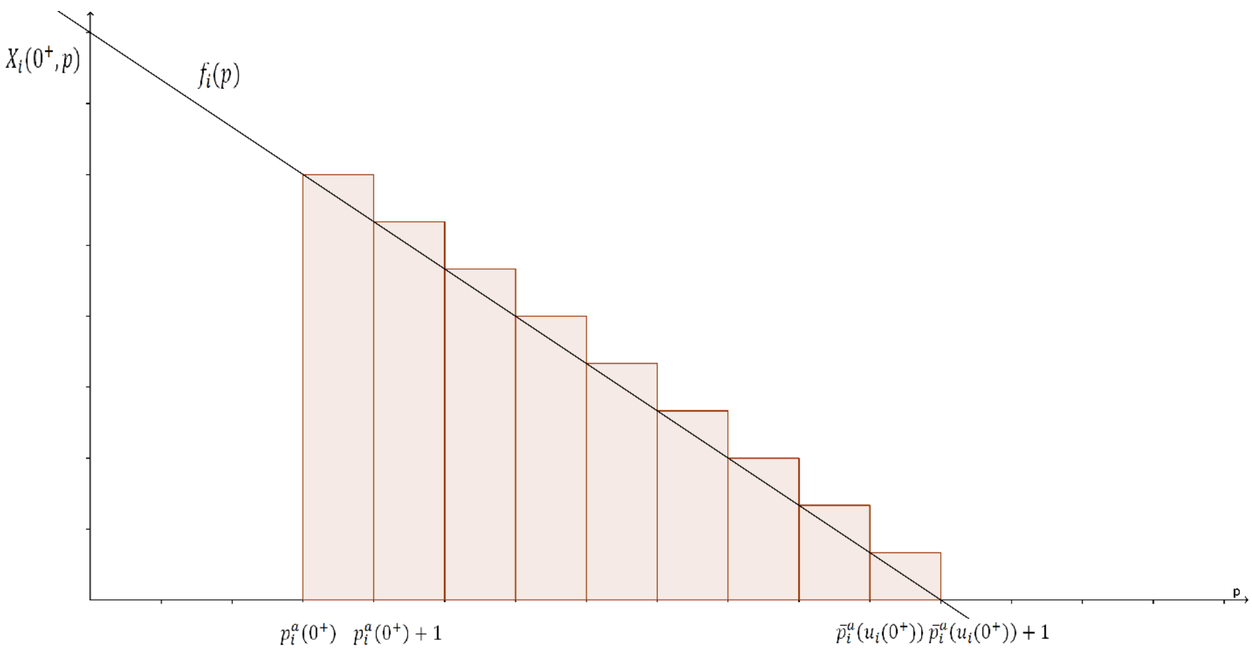
As for the volumes corresponding to the price set $\{p_i^a(0^+),\ p_i^a(0^+) + 1, \dots,\ \bar{p}_i^a(u_i(0))\}$, we assume that a total amount of $\frac{1}{2}u_i(0)$ shares of the i^{th} stock are filled in. In order for the external addition to form a similar distribution of sell limit orders that are not affected by the transaction, we need to make sure that a greater percentage of the external orders will be added to a price that is closer $p_i^a(0^+)$. Therefore, $X_i(0^+, p)$ in this price set seems to behave like a negative linear function

Equation 3.24

$$f_i(p) = \lambda_i(p - \bar{p}_i^a(u_i(0)) - 1)$$

, with some $\lambda_i < 0$, for $p \in [\, p_i^a(0^+), \bar{p}_i^a(u_i(0)) + 1]$. Since the prices are discrete and different by 1, according to our definition of prices in Section 3.1, we can model $X_i(0^+, p)$ with the area of a rectangle whose width is 1 and length is $f_i(p)$, as shown in Figure 3.1, and the sum of all the rectangular areas is, by definition [9], the left Riemann sum of $f_i(p)$ over the interval $[\, p_i^a(0^+), \bar{p}_i^a(u_i(0)) + 1]$.

Figure 3.1, Left Riemann Sum of $f_i(p)$



Now we solve λ_i. We can see that

Equation 3.25

$$\sum_{p = p_i^a(0^+)}^{\bar{p}_i^a(u_i(0))} (X_i(0^+, p)) = \frac{1}{2} u_i(0)$$

can be approximated by the area under $f_i(p)$ over the interval $[\, p_i^a(0^+), \bar{p}_i^a(u_i(0)) + 1]$, where

Equation 3.26

$$f_i(\, p_i^a(0^+)) = \frac{2 \times \frac{1}{2} u_i(0)}{\bar{p}_i^a(u_i(0)) + 1 - \, p_i^a(0^+)} = \frac{u_i(0)}{\bar{p}_i^a(u_i(0)) - \, p_i^a(0^+) + 1}$$

Hence, the two points that determine λ_i are $\left(p_i^a(0^+), \frac{u_i(0)}{\bar{p}_i^a(u_i(0)) - \, p_i^a(0^+) + 1} \right)$ and $(\bar{p}_i^a(u_i(0)) + 1, 0)$. With a few algebraic computations, we have

Equation 3.27

$$\lambda_i = \frac{u_i(0)}{\left(\bar{p}_i^a(u_i(0)) - \, p_i^a(0^+) + 1\right)^2}$$

Now we have discussed $X_i(0^+, p)$ for all cases, and the conclusion is as follows:

Equation 3.28

$$X_i(0^+, p) = \begin{cases} X_i\left(0^-, p - \left\lfloor \frac{p_i^a(0) - \, p_i^b(0)}{2} \right\rfloor + \delta i\right), & p \leq \, p_i^b(0^+) \\ X_i(0, p), & p > \bar{p}_i^a(u_i(0)) \\ \lambda_i(p - \, p_i^a(u_i(0)) - 1), & p_i^a(0^+) \leq p \leq \, \bar{p}_i^a(u_i(0)) \end{cases}$$

where

Equation 3.29

$$\lambda_i = \frac{-u_i(0)}{\left(\bar{p}_i^a(u_i(0)) - \, p_i^a(0^+) + 1\right)^2}$$

The LOB at time $t = 0^-$, $t = 0$, and $t = 0^+$ can be summarized in the following:

Table 3.5, The LOB at time t=0⁻, t=0, and t=0⁺ when $u_i(0)>0$

Price p	Volume at $t=0^-$ $X_i(0^-,p)$	Volume at $t=0$ $X_i(0,p)$	Volume at $t=0^+$ $X_i(0^+,p)$
⋮	⋮	⋮	⋮
$\boldsymbol{\bar{p}_i^a(u_i(0))+1}$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))+1)$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))+1)$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))+1)$
$\boldsymbol{\bar{p}_i^a(u_i(0))}$	$X_i(0^-,\ \bar{p}_i^a(u_i(0)))$	$\mathbb{X}_i^a(0^-,\ \bar{p}_i^a(u_i(0)))-u_i(0)$	$-\lambda_i$
$\boldsymbol{\bar{p}_i^a(u_i(0))-1}$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))-1)$	0	$-2\lambda_i$
$\boldsymbol{\bar{p}_i^a(u_i(0))-2}$	$X_i(0^-,\ \bar{p}_i^a(u_i(0))-2)$	0	$-3\lambda_i$
⋮	⋮	⋮	⋮
$\boldsymbol{p_i^a(0^+)}$	$X_i(0^-,\ p_i^a(0^+))$	0	$-\lambda_i p_i^a(0^+)-\ \bar{p}_i^a(u_i(0)-1)$
PRICE GAP			
$\boldsymbol{p_i^b(0^+)}$	$X_i(0^-,p_i^b(0^+))$	0	$X_i(0^-,\ p_i^b(0^-))$
$\boldsymbol{p_i^b(0^+)-1}$	$X_i(0^-,p_i^b(0^+)-1)$	0	$X_i(0^-,\ p_i^b(0^-)-1)$
$\boldsymbol{p_i^b(0^+)-2}$	$X_i(0^-,p_i^b(0^+)-2)$	0	$X_i(0^-,\ p_i^b(0^-)-2)$
⋮	⋮	⋮	⋮
$\boldsymbol{p_i^a(0^-)+2}$	$X_i(0^-,\ p_i^a(0^-)+2)$	0	⋮
$\boldsymbol{p_i^a(0^-)+1}$	$X_i(0^-,\ p_i^a(0^-)+1)$	0	⋮
$\boldsymbol{p_i^a(0^-)}$	$X_i(0^-,\ p_i^a(0^-))$	0	⋮
⋮	⋮	⋮	⋮
PRICE GAP			
$\boldsymbol{p_i^b(0^-)}$	$X_i(0^-,\ p_i^b(0^-))$	$X_i(0^-,\ p_i^b(0^-))$	⋮
$\boldsymbol{p_i^b(0^-)-1}$	$X_i(0^-,\ p_i^b(0^-)-1)$	$X_i(0^-,\ p_i^b(0^-)-1)$	⋮
$\boldsymbol{p_i^b(0^-)-2}$	$X_i(0^-,\ p_i^b(0^-)-2)$	$X_i(0^-,\ p_i^b(0^-)-2)$	⋮
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮

Similarly, in the case $u_i(0)<0$, the buy limit orders from the price $p_i^b(0^-)$ to $\bar{p}_i^b(u_i(0))$ were removed. Therefore, more sell limit will be filled in the LOB with the price below $p_i^a(0^-)$ and more limit buy orders will be filled with the prices above $\bar{p}_i^b(u_i(0))$ to meet needs. Eventually, the lowest ask price and the lowest bid price of new orders will get close in the middle of prices $p_i^b(0)$ and $p_i^a(0)$. The middle price is defined in the same way and the relationship between $p_i^{m-}(0)$ and $p_i^{m-}(0^-)$, as well as relationship $p_i^{m+}(0)$ and $p_i^{m+}(0^-)$, are

Equation 3.30

$$\begin{cases} p_i^{m_+}(0) = \left\lfloor \dfrac{p_i^a(0) + \ p_i^b(0) + 1}{2} \right\rfloor \le \dfrac{p_i^a(0^-) + \ \bar{p}_i^b\big(u_i(0)\big)}{2} < \dfrac{p_i^a(0^-) + \ p_i^b(0^-)}{2} = \ p_i^{m_+}(0^-) \\ p_i^{m_-}(0) = \left\lfloor \dfrac{p_i^a(0) + \ p_i^b(0)}{2} \right\rfloor \le \dfrac{p_i^a(0^-) + \ \bar{p}_i^b\big(u_i(0)\big)}{2} < \dfrac{p_i^a(0^-) + \ p_i^b(0^-)}{2} = \ p_i^{m_-}(0^-) \end{cases}$$

Also

Equation 3.31

$$\begin{cases} p_i^a(0^+) = p_i^{m_+}(0) + \delta i \\ p_i^b(0^+) = \ p_i^{m_-}(0) - \delta i \end{cases}$$

When $u_i(0) < 0$,

Equation 3.32

$$\begin{cases} p_i^a(0^+) = \ p_i^{m_+}(0) + \delta i = \left\lfloor \dfrac{p_i^a(0) + \ p_i^b(0) + 1}{2} \right\rfloor + \delta i < \ p_i^{m_+}(0^-) + \delta i = \ p_i^a(0^-) \\ p_i^b(0^+) = \ p_i^{m_-}(0) - \delta i = \left\lfloor \dfrac{p_i^a(0) + \ p_i^b(0)}{2} \right\rfloor - \delta i < \ p_i^{m_-}(0^-) - \delta i = \ p_i^b(0^-) \end{cases}$$

This means that in the case $u_i(0) < 0$ the overall price is decreasing. As we mentioned before, the lowest ask price $p_i^a(0^+)$ and highest bid price $p_i^b(0^+)$ will approach in the middle of the prices $p_i^a(0)$ and $p_i^b(0)$. We also need to take into account the fact that $X_i(0, p) = 0$, for $p \in \{\, \bar{p}_i^b\big(u_i(0)\big),\ \bar{p}_i^b(u_i(0) + 1), \dots,\ p_i^b(0^-)\,\}$. Similar to before, we partition the prices into four sets, namely, $\{\dots,\ \bar{p}_i^b\big(u_i(0)\big) - 2,\ \bar{p}_i^b\big(u_i(0)\big) - 1\}$, $\{\, \bar{p}_i^b\big(u_i(0)\big),\ \bar{p}_i^b\big(u_i(0)\big) + 1, \dots,\ p_i^b(0^+)\,\}$, $\{\, p_i^a(0^+),\ p_i^a(0^+) + 1, \dots,\ p_i^b(0^-)\}$, $\{\, p_i^a(0^-),\ p_i^a(0^-) + 1, \dots\}$, and discuss the volume $X_i(0^+, p)$ for each case.

When $p \in \{\, p_i^a(0^+),\ p_i^a(0^+) + 1, \dots,\ p_i^b(0^-)\,\}$, $X_i(0, p) = 0$. Hence, we need to fill in sell limit orders at $t = 0^+$ in this price set. Consider $X_i(0, p)$, $\forall\, p \in \{\, p_i^a(0^-),\ p_i^a(0^-) + 1, \dots\})$, which are the sell limit orders at $t = 0$. They were not traded at $t = 0$, and thus kept the distribution of all sell limit orders when the market was not affected by any transaction.

Therefore, we shift down all buy limit orders at $t = 0$ by $p_i^a(0^-) - p_i^a(0^+)$ units to fill the LOB at $t = 0$ in the price set

Equation 3.33

$$\{ p_i^a(0^+),\ p_i^a(0^+) + 1, \dots,\ p_i^b(0^-) \} \cup \{ p_i^a(0^-),\ p_i^a(0^-) + 1, \dots \} = \{ p_i^a(0^+),\ p_i^a(0^+) + 1, \dots \}$$

In other words, we let

Equation 3.34

$$\begin{aligned} X_i(0,p) &= X_i\left(0^+, p - p_i^a(0^-) + p_i^a(0^+)\right) \\ &= X_i\left(0^+, p - p_i^a(0^-) + p_i^{m_+}(0) + \delta i\right) \\ &= X_i\left(0^+, p - p_i^a(0^-) + \left\lfloor \frac{p_i^a(0) + p_i^b(0) + 1}{2} \right\rfloor + \delta i\right) \\ &= X_i\left(0^+, p - \left\lfloor \frac{p_i^a(0) - p_i^b(0)}{2} \right\rfloor + \delta i\right), \forall\, p \in \{ p_i^a(0^+),\ p_i^a(0^+) + 1, \dots \} \end{aligned}$$

Thus we finished modeling the volume $X_i(0^+, p)$ for the price set $\{ p_i^a(0^+),\ p_i^a(0^+) + 1, \dots \}$.

As for the volumes corresponding to the price set $\{ \dots,\ p_i^b(0^+) - 1,\ p_i^b(0^+) \}$, we do not shift up the buy limit orders at $t = 0$ by $\left(p_i^b(0^+) - \bar{p}_i^b\left(u_i(0)\right) + 1\right)$ units to fill the LOB at $t = 0^+$, as we did to model the volumes corresponding to the price set $\{ p_i^a(0^-),\ p_i^a(0^-) + 1, \dots \}$. The reason is that the distribution of the buy limit orders at $t = 0$ were affected by the transaction since the orders in the price set $\{ \bar{p}_i^b\left(u_i(0)\right),\ \bar{p}_i^b(u_i(0) + 1), \dots,\ p_i^b(0^+) \}$ were removed. Therefore, we need to discuss the volume $X_i(0^+, p)$ for the price sets $\{ \dots,\ \bar{p}_i^b\left(u_i(0)\right) - 2,\ \bar{p}_i^b\left(u_i(0)\right) - 1 \}$ and $\{ \bar{p}_i^b\left(u_i(0)\right),\ \bar{p}_i^b(u_i(0) + 1), \dots,\ p_i^b(0^+) \}$ separately. We let

Equation 3.35

$$X_i(0,p) = X_i(0^+, p), \forall\, p \in \{ \dots,\ \bar{p}_i^b\left(u_i(0)\right) - 2,\ \bar{p}_i^b\left(u_i(0)\right) - 1 \}$$

since they were not traded.

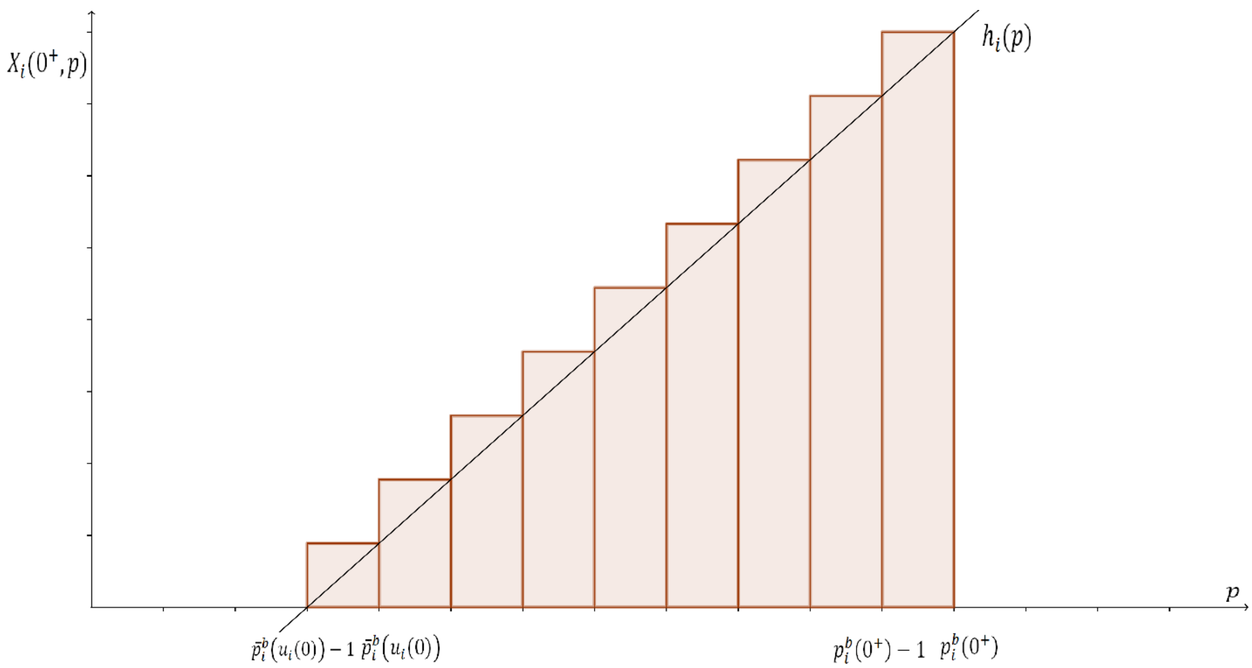
As for the volumes corresponding to the price set $\{ \bar{p}_i^b\big(u_i(0)\big),\ \bar{p}_i^b(u_i(0)+1), \ldots,\ p_i^b(0^+) \}$, we assume that a total amount of $\frac{1}{2}u_i(0)$ shares of the i^{th} stock are filled in. In order for the external addition to form a similar distribution of limit sell orders that are not affected by the transaction, we need to make sure that a greater percentage of the external orders will be added to a price that is closer $p_i^b(0^+)$. Therefore, $X_i(0^+, p)$ in this price set seems to behave like a function

Equation 3.36

$$h_i(p) = \epsilon_i\left(p - \bar{p}_i^b(u_i(0)+1)\right)$$

with some $\epsilon_i > 0$, for $p \in \left[\bar{p}_i^b\big(u_i(0)\big) - 1,\ p_i^b(0^+)\right]$. Since the prices are discrete and different by 1, according to our definition of prices in section 3.1, we can model $X_i(0^+, p)$ with the area of a rectangle whose width is 1 and length is $h_i(p)$, as shown in Figure 3.2, and the sum of all the rectangular areas is, by definition [9], the right Riemann sum of $h_i(p)$ over the interval $\left[\bar{p}_i^b\big(u_i(0)\big) - 1,\ p_i^b(0^+)\right]$.

Figure 3.2, Right Riemann Sum of h_i(p)



Now we determine ϵ_i. Since

Equation 3.37

$$\sum_{p=\bar{p}_i^b(u_i(0))}^{p_i^b(0^+)} X_i(0^+, p) = \frac{1}{2} u_i(0)$$

which can be approximated by the area under $h_i(p)$ over the interval $[\bar{p}_i^b(u_i(0)) - 1,\ p_i^b(0^+)]$,

Equation 3.38

$$h_i\left(p_i^b(0^+)\right) = \frac{2 \times \frac{1}{2} u_i(0)}{p_i^b(0^+) - \bar{p}_i^b(u_i(0)) - 1} = \frac{u_i(0)}{p_i^b(0^+) - \bar{p}_i^b(u_i(0)) - 1}$$

Hence, the two points that determine ϵ_i are $\left(p_i^b(0^+), \frac{u_i(0)}{p_i^b(0^+) - \bar{p}_i^b(u_i(0)) - 1}\right)$ and $\left(\bar{p}_i^b(u_i(0)) - 1, 0\right)$. With a few algebraic manipulation, we have

Equation 3.39

$$\epsilon_i = \frac{-u_i(0)}{\left(\bar{p}_i^b(u_i(0)) - p_i^b(0^+) - 1\right)^2}$$

Now we have discussed $X_i(0^+, p)$ for all cases, and the conclusion is as follows:

Equation 3.40

$$X_i(0^+, p) = \begin{cases} X_i\left(0^-, p + \left\lceil \frac{p_i^a(0) - p_i^b(0)}{2} \right\rceil - \delta i\right), & p \geq p_i^a(0^+) \\ X_i(0, p), & p < \bar{p}_i^b(u_i(0)) \\ \epsilon_i\left(p - \bar{p}_i^b(u_i(0)) + 1\right), & \bar{p}_i^b(u_i(0)) \leq p \leq p_i^b(0^+) \end{cases}$$

where ϵ_i is defined by Equation 3.37.

The LOB at time $t = 0^-$, $t = 0$, and $t = 0^+$ can be summarized as below:

Table 3.6, The LOB at time t=0-, t=0, and t=0+ when ui(0)<0

Price p	Volume at $t = 0^-$ $X_i(0^-, p)$	Volume at $t = 0$ $X_i(0, p)$	Volume at $t = 0^+$ $X_i(0^+, p)$
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮
$\boldsymbol{p_i^a(0^-) + 2}$	$X_i(0^-, p_i^a(0^-) + 2)$	$X_i(0^-, p_i^a(0^-) + 2)$	⋮
$\boldsymbol{p_i^a(0^-) + 1}$	$X_i(0^-, p_i^a(0^-) + 1)$	$X_i(0^-, p_i^a(0^-) + 1)$	⋮
$\boldsymbol{p_i^a(0^-)}$	$X_i(0^-, p_i^a(0^-))$	$X_i(0^-, p_i^a(0^-))$	⋮
PRICE GAP			
$\boldsymbol{p_i^b(0^-)}$	$X_i\left(0^-, p_i^b(0^-)\right)$	0	⋮
$\boldsymbol{p_i^b(0^-) - 1}$	$X_i(0^-, p_i^b(0^-) - 1)$	0	⋮
$\boldsymbol{p_i^b(0^-) - 2}$	$X_i(0^-, p_i^b(0^-) - 2)$	0	⋮
⋮	⋮	⋮	⋮
$\boldsymbol{p_i^a(0^+) + 2}$	$X_i(0^-, p_i^a(0^+) + 2)$	0	$X_i(0^-, p_i^a(0^-) + 2)$
$\boldsymbol{p_i^a(0^+) + 1}$	$X_i(0^-, p_i^a(0^+) + 1)$	0	$X_i(0^-, p_i^a(0^-) + 1)$
$\boldsymbol{p_i^a(0^+)}$	$X_i(0^-, p_i^a(0^+))$	0	$X_i(0^-, p_i^a(0^-))$
PRICE GAP			
$\boldsymbol{p_i^b(0^+)}$	$X_i\left(0^-, p_i^b(0^+)\right)$	0	$\epsilon_i\left(p_i^b(0^+) - \bar{p}_i^b(u_i(0)) + 1\right)$
⋮	⋮	⋮	⋮
$\boldsymbol{\bar{p}_i^b(u_i(0)) + 2}$	$X_i(0^-, \bar{p}_i^b(u_i(0)) + 2)$	0	$3\epsilon_i$
$\boldsymbol{\bar{p}_i^b(u_i(0)) + 1}$	$X_i(0^-, \bar{p}_i^b(u_i(0)) + 1)$	0	$2\epsilon_i$
$\boldsymbol{\bar{p}_i^b(u_i(0))}$	$X_i\left(0^-, \bar{p}_i^b(u_i(0))\right)$	$\mathbb{X}_i^b\left(0^-, \bar{p}_i^b(u_i(0))\right) - u_i(0)$	ϵ_i
$\boldsymbol{\bar{p}_i^b(u_i(0)) - 1}$	$X_i(0^-, \bar{p}_i^b(u_i(0)) - 1)$	$X_i(0^-, \bar{p}_i^b(u_i(0)) - 1)$	$X_i(0^-, \bar{p}_i^b(u_i(0)) - 1)$
⋮	⋮	⋮	⋮

3.5 Limit Order Book at time t=1⁻

Looking at the a possible change of the LOB from $t = 0^+$ to $t = 0^-$, we propose the following model:

Equation 3.41

$$X_i(1^-, p) = [1 + \xi_i(p)]X_i(0^+, p)$$

in which $\xi_i(p)$ is the relative rate of change. Intuitively, the map $p \mapsto \xi_i(p)$ should have the following properties:

1. For p far way from $\left[\, p_i^b(0^+),\ p_i^a(0^+)\right]$ the values of $\xi_i(p)$ should be very small, since the farther p is away from $\left[\, p_i^b(0^+),\ p_i^a(0^+)\right]$, the less correspondent volumes are likely to be traded at $t = 1^-$.
2. If $u_i(0) > 0$, there should be a relatively bigger change near $p_i^a(0)$, and if $u_i(0) < 0$, there should be a relatively bigger change near $p_i^b(0)$.

We define $\xi_i(p) = \eta_i(p) + \sigma_i(p)\zeta$, where $\zeta \sim N(0,1), \eta_i(p)$ is the expected value and $\sigma_i(p)$ is the volatility. Now let us introduce a function $\eta_i(p)$, with which we can describe the above two properties.

For the first property, we let

Equation 3.42

$$\eta_i(p) = \begin{cases} o, & p \in \left[\, p_i^b(0^+) - \beta i,\ p_i^a(0^+) + \beta_i\right] \\ u_i, & p \in \left[\, p_i^b(0^+) - \beta i,\ p_i^a(0^+) + \beta_i\right] \end{cases}$$

for some $\beta_i > 0$, and $u_i \in (0,1)$.

Equation 3.43

$$\eta_i(p) = \begin{cases} 0, & p \notin \left[\, p_i^b(0^+) - \beta_i,\ p_i^a(0^+) + \beta_i\right] \\ u_i - v_i[p - p_i^a(0)]^2,\ u_i(0) > 0, & p \in \left[\, p_i^b(0^+) - \beta_i,\ p_i^a(0^+) + \beta_i\right] \\ u_i - v_i\left[p - p_i^b(0)\right]^2\ u_i(0) < 0, & p \in \left[\, p_i^b(0^+) - \beta_i,\ p_i^a(0^+) + \beta_i\right] \end{cases}$$

where $v_i \in \mathbb{R}^+$.

4 TRADING ANALYSIS IN THE HYPOTHESIZED MARKET

In this chapter, we calculate the return by subtracting the investor's initial wealth at $t = 0^-$ from his final wealth at $t = 1^-$. The initial wealth can be calculated using the information in the *given* LOB at $t = 0^-$, while the final wealth can only be estimated with the information in the *predicted* LOB at $t = 1^-$. Recall from Chapter 3 that the investor's portfolio at time t is defined to be $\pi(t) = (\pi_0(t),\ \pi_1(t), \dots, \pi_k(t))$, where $\pi_0(t)$ is the dollar amount in a cash account, and for $i = 1,2,\dots,k,\ \pi_i(t)$ is the share number held for the i^{th} stock.

Then at each time t, we define the investor's wealth

Equation 4.1

$$W(t) = \pi_0(t) + \sum_{i=1}^{k} V_i(t)$$

where $V_i(t)$ is the value of the i^{th} stock that the investor holds at time t. We set a restriction for the LOB that an investor cannot short stocks at any time t, i.e. $\pi_i(t) \geq 0, \forall t \in \{0,1\}$. When $\pi_i(t) \geq 0$, we define the value as the total amount of money that the investor will acquire if he sells $\pi_i(t)$ shares of the i^{th} stock at time t in the hypothesized market. The exact expressions of $V_i(0^-)$ and $V_i(1^-)$ will be given in Section 4.1 and 4.2.

4.1 Initial Wealth and Final Wealth

Based on our definition of $V_i(t)$, we need to calculate the cash value of the investor's i^{th} stock at time $t = 0^-$. Since at time $t = 0^-$and price p, the investor can only sell at most $|X_i(0^-, p)|$ shares of the i^{th} stock, we need to define the "lowest" bid price in the transaction. We denote this price as $\bar{p}_i^b(\pi_i(0^-))$, where

Equation 4.2

$$\bar{p}_i^b\big(\pi_i(0^-)\big) = \max\{p \le p_i^b(0^-) | \pi_i(0^-) \le -\mathbb{X}_i^b(0^-, p)\}.$$

Recall that $\bar{p}_i^b(u_i(0))$, defined in section 3.2, is the "lowest" bid price in the transaction if the investor decides to sell $u_i(0)$ shares of stocks. Since we set the restriction for shorting stocks, we always have

Equation 4.3

$$\bar{p}_i^b\big(\pi_i(0^-)\big) \ge \bar{p}_i^b(u_i(0))$$

For illustration, the LOB of negative volumes at time $t = 0^-$ is summarized as below:

Table 4.1, The LOB of Negative Volumes at time t-0⁻

Price p	Volume at $t = 0^-$ $X_i(0^-, p)$
$p_i^b(0^-)$	$X_i(0^-, p_i^b(0^-))$
$p_i^b(0^-) - 1$	$X_i(0^-, p_i^b(0^-) - 1)$
$p_i^b(0^-) - 2$	$X_i(0^-, p_i^b(0^-) - 2)$
$\vdots$	$\vdots$
$\vdots$	$\vdots$
$\bar{p}_i^b\big(\pi_i(0^-)\big)+2$	$X_i(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2)$
$\bar{p}_i^b\big(\pi_i(0^-)\big)+1$	$X_i(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1)$
$\bar{p}_i^b\big(\pi_i(0^-)\big)$	$X_i(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big))$
$\bar{p}_i^b\big(\pi_i(0^-)\big) - 1$	$X_i(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) - 1)$
$\vdots$	$\vdots$

Then, when $t = 0^-$, the initial wealth is

Equation 4.4

$$W(0^-) = \pi_0(0^-) + \sum_{i=1}^{k} V_i(0^-)$$

where $\pi_0(0^-)$ is the total amount in his cash account at $t = 0^-$, and

Equation 4.5

$$V_i(0^-) = \sum_{p > \bar{p}_i^b(\pi_i(0^-))} p|X_i(0^-, p)| + \bar{p}_i^b\big(\pi_i(0^-)\big)\left[|\pi_i(0^-)| - \sum_{p > \bar{p}_i^b(\pi_i(0^-))} |X_i(0^-, p)|\right]$$

$$= \bar{p}_i^b(\pi_i(0^-))\pi_i(0^-) + \sum_{p > \bar{p}_i^b(\pi_i(0^-))} (\bar{p}_i^b(\pi_i(0^-)) - p)X_i(0^-, p)$$

Similarly, we defined the "lowest" bid price for $\pi_i(1^-)$ shares of the i^{th} stock before we compute the investor's final wealth $W(1^-)$. Since no trade was made between $t = 0^+$ and $t = 1^-$, $\pi(1^-) = \pi_i(0^+) = \pi_i(0^-) + u_i(0)$. We denote the price as $\bar{p}_i^b(\pi_i(1^-))$, where

Equation 4.6

$$\bar{p}_i^b(\pi_i(1^-)) = max\{p \leq p_i^b(1^-) | \pi_i(0^-) + u_i(0) \leq -\mathbb{X}_i^b(0^-, p)\}.$$

Then when $t = 1^-$, the market value of the final wealth is

Equation 4.7

$$W(1^-) = \pi_0(1^-) + \sum_{i=1}^{k} V_i(1^-)$$

where $\pi_0(0^-)$ is the total amount in his cash account at $t = 0^-$, and

Equation 4.8

$$V_i(1^-) = \bar{p}_i^b(\pi_i(1^-))\pi_i(1^-) + \sum_{p > \bar{p}_i^b(\pi_i(1^-))} (\bar{p}_i^b(\pi_i(1^-)) - p)X_i(1^-, p).$$

4.2 Calculation of the Wealth Process

In this section we will illustrate how an investor can build his wealth process based on the dynamics of the LOB process in the case that $u_i(0) < 0$. The calculation of wealth process when $u_i(0) > 0$ can be calculated with a similar method.

In this case, the investor owned $\pi_i(0^-)$ shares of the i^{th} stock at $t = 0^-$ and he sold $-u_i(0)$ shares of this stock. Also, the amount of stocks he sold was less than the amount of stocks he owned, i.e., $\pi_i(0^-) + u_i(0) > 0$. Let us denote the return as R. We also denote the value of the i^{th} stock that was sold as $V_{i_s}(0^-)$, the value of the remaining i^{th}

stock as $V_{i_r}(0^-)$ at $t = 0^-$, and $V_{i_r}(1^-)$ at $t = 1^-$. By Equation 4.4 and Equation 4.7, the return of the investor's trade at $t = 0^-$ is:

Equation 4.9

$$
\begin{aligned}
R &= W_i(1^-) - W_i(0^-) \\
&= \pi_0(1^-) - \pi_0(0^-) + \sum_{i=1}^{k} V_i(1^-) - \sum_{i=1}^{k} V_i(0^-) \\
&= \pi_0(1^-) - \pi_0(0^-) + \sum_{i=1}^{k} (V_{i_r}(1^-) - V_{i_r}(0^-)) - \sum_{i=1}^{k} V_{i_s}(0^-) \\
&= (c_i + g_i|u_i(0)|) + \sum_{i=1}^{k} V_{i_r}(1^-) - \sum_{i=1}^{k} V_{i_r}(0^-),
\end{aligned}
$$

where c_i is a fixed cost which will appear as long as $u_i(0) \neq 0$ and g_i is a proportional cost. We can see from the equation that the return is mainly determined by the difference between $V_{i_r}(1^-)$ and $V_{i_r}(0^-)$, i.e., the value of the remaining i^{th} stock at $t = 0^-$ and $t = 0^-$. Since $V_{i_r}(1^-)$ and $V_{i_r}(0^-)$, are determined by $\bar{p}_i^b(\pi_i(1^-))$ and $\bar{p}_i^b(\pi_i(0^-))$, and we cannot construct an explicit equation between $\bar{p}_i^b(\pi_i(1^-))$ and $\bar{p}_i^b(\pi_i(0^-))$, we will calculate the return based on the discrete change of $u_i(0)$. First, we consider the case when $k = 1$, i.e., the investor only has one stock in his portfolio. We omit the transaction cost during the following calculation to look for a pattern of the expected return that corresponds to different $u_i(0)$.

An investor can sell at most $\pi_i(0^-)$ shares, and at least 0 shares of the i^{th} stock. Suppose there are n prices between $p_i^b(0^+)$ and $\bar{p}_i^b(\pi_i(0^-))$, $n \in \mathbb{Z}$. We let $u_i(0) \in \{H_j\}$, where $\{H_j\}$ is defined inductively as follows,

Equation 4.10

$$\begin{cases} H_0 = \pi_i(0^-) \\ H_j = H_{j-1} - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + j - 1\right), \quad \forall j > 1 \end{cases}$$

Therefore, we have

Equation 4.11

$$\{H_n\} = \left\{-\pi_i(0^-), \mathbb{X}_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right), \mathbb{X}_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right), \ldots, 0\right\}$$

In the rest of this section, we will illustrate how to calculate the wealth process with $u_i(0) \epsilon \{H_0, H_1, H_2, H_3\}$, and denote the return in each case as R_0, R_1, R_2, R_3. We will also compare the returns to discover a pattern of return with respect to the change in trading strategy. In order to increase clarity, we only present the results in each step, and leave detailed computation and proof in the Appendix.

When $u_i(0) = H_0 = -\pi_i(0^-)$, the investor sold all of his shares at $t = 0$, and hence $\bar{p}_i^b\left(u_i(0)\right) = \bar{p}_i^b\left(\pi_i(0^-)\right)$. Since there is no remaining share for the i^{th} stock,

Equation 4.12

$$R_0 = 0$$

When $u_i(0) = H_1 = \mathbb{X}_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$, the investor sold all of his shares from the price $p_i^b(0^-)$ to $\bar{p}_i^b\left(\pi_i(0^-)\right) + 1$. Hence, $\bar{p}_i^b\left(u_i(0)\right) = \bar{p}_i^b\left(\pi_i(0^-)\right) + 1$, and Recall that we assumed in our LOB

Equation 4.13

$$\begin{cases} -X_i(1^-, p_\alpha) < -X_i\left(1^-, p_\beta\right), & p_\alpha < p_\beta \le p_i^b(0^+) \\ -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right) = -X_i\left(0^-, \bar{p}_i^b\left(u_i(0)\right) - 1\right) = \dfrac{-X_i\left(0^-, \bar{p}_i^b\left(u_i(0)\right) - 1\right)}{1 + \xi_i(p)} \le -X_i\left(1^-, \bar{p}_i^b\left(u_i(0)\right) - 1\right) & \end{cases}$$

Therefore,

Equation 4.14

$$\pi_i(0^-) + H_1 \le -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right) \le -X_i\left(1^-, \bar{p}_i^b\left(u_i(0)\right) - 1\right) \le -X_i\left(1^-, p_i^b(0^+)\right)$$

Now we can calculate the values for $V_{i_r}(0^-)$ and $V_{i_r}(1^-)$:

Equation 4.15

$$\begin{cases} V_{i_r}(1^-) = p_i^b(0^+)(\pi_i(0^-) + H_1) \\ V_{i_r}(0^-) = \bar{p}_i^b(\pi_i(0^-))(\pi_i(0^-) + H_1), \end{cases}$$

Therefore,

Equation 4.16

$$R_1 = [p_i^b(0^+) - \bar{p}_i^b(\pi_i(0^-))](\pi_i(0^-) + H_1) > 0$$

since $p_i^b(0^+) - \bar{p}_i^b(\pi_i(0^-)) > 0$ and $\pi_i(0^-) + H_1 > 0$.

We can conclude the comparison between R_0 and R_1 with the following table:

Table 4.2, Comparison Between R_0 and R_1

Subcase for R_0 \ Subcase for R_1	$H_0 \pm \pi_i(0^-) \leq -X_i(1^-, p_i^b(0^+))$
$\pi_i(0^-) + H_1 \leq -X_i(1^-, p_i^b(0^+))$	$R_1 > R_0$

When $u_i(0) = H_2 = H_1 - X_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1) = \mathbb{X}_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2)$, the investor sold all of his shares from the price $p_i^b(0^-)$ to $\bar{p}_i^b(\pi_i(0^-)) + 2$. Hence, $\bar{p}_i^b(u_i(0)) = \bar{p}_i^b(\pi_i(0^-)) + 2$, and $V_{i_r}(0^-)$ is calculated as

Equation 4.17

$$V_{i_r}(0^-) = -(\bar{p}_i^b(\pi_i(0^-)) + 1)X_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1) + \bar{p}_i^b(\pi_i(0^-))(\pi_i(0^-) + H_1)$$

With a similar argument from the case that $u_i(0) = H_1$, we have

Equation 4.18

$$\pi_i(0^-) + H_2 \leq -X_i(1^-, p_i^b(0^+)) - X_i(1^-, p_i^b(0^+) - 1)$$

Therefore, there are two sub cases for $V_{i_r}(1^-)$.

subcase 1: $\pi_i(0^-) + H_2 \leq -X_i(1^-, p_i^b(0^+))$

In this sub case, we have

Equation 4.19

$$V_{i_r}(1^-) = p_i^b(0^+)(H_1 + \pi_i(0^-) - X_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1))$$

Thus, the return is calculated as follows,

Equation 4.20

$$R_2 = [p_i^b(0^+) - \bar{p}_i^b(\pi_i(0^-))](\pi_i(0^-) + H_1) + \left(\bar{p}_i^b(\pi_i(0^-)) + 1 - p_i^b(0^+)\right) X_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1)$$

Now we compare R_2 and R_1, and we have

Equation 4.21

$$R_2 - R_1 = \left(\bar{p}_i^b(\pi_i(0^-)) + 1 - p_i^b(0^+)\right) X_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1) > 0,$$

since $\left(\bar{p}_i^b(\pi_i(0^-)) + 1 - p_i^b(0^+)\right) < 0$ and $X_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1) < 0$.

subcase 2: $\pi_i(0^-) + H_2 > -X_i(1^-, p_i^b(0^+))$

In this sub case, we have

Equation 4.22

$$V_{i_r}(1^-) = (p_i^b(0^+) - 1)\left(H_1 + \pi_i(0^-) - X_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1)\right) - X_i(1^-, p_i^b(0^+))$$

Thus, the return is calculated as follows,

Equation 4.23

$$R_2 = [p_i^b(0^+) - \bar{p}_i^b(\pi_i(0^-))](\pi_i(0^-) + H_1) - (\pi_i(0^-) + H_1) + \left(\bar{p}_i^b(\pi_i(0^-)) + 2 - p_i^b(0^+)\right) X_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1) - X_i(1^-, p_i^b(0^+))$$

Now we compare R_2 and R_1, and we have

Equation 4.24

$$R_2 - R_1 = -(\pi_i(0^-) + H_2) + \left(\bar{p}_i^b(\pi_i(0^-)) + 1 - p_i^b(0^+)\right) X_i(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1) - X_i(1^-, p_i^b(0^+))$$

We claim that

Equation 4.25

$$\pi_i(0^-) + H_2 + X_i\left(1^-, p_i^b(0^+)\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$

and the detailed proof for Equation 4.25 is provided in the Appendix. By this result, we can continue comparing R_1 and R_2.

Equation 4.26

$$R_2 - R_1 > \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) > 0$$

since $\left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) < 0$ and $X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) < 0$.

From both of the sub cases, we can conclude the comparison between R_1 and R_2 when $u_i(0) = H_2$ with the following table:

Table 4.3, Comparison Between R_1 and R_2

Subcase for R_1 / Subcase for R_2	$\pi_i(0^-) + H_1 \leq -X_i\left(1^-, p_i^b(0^+)\right)$
$\pi_i(0^-) + H_2 \leq -X_i\left(1^-, p_i^b(0^+)\right)$	$R_2 > R_1$
$\pi_i(0^-) + H_2 > -X_i\left(1^-, p_i^b(0^+)\right)$	$R_2 > R_1$

When $u_i(0) = H_3 = H_2 - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) = \mathbb{X}_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 3\right)$, the investor sold all of his shares from the price $p_i^b(0^-)$ to $\bar{p}_i^b\left(\pi_i(0^-)\right) + 3$. Hence,

Equation 4.27

$$\bar{p}_i^b\left(u_i(0)\right) = \bar{p}_i^b\left(\pi_i(0^-)\right) + 3,$$

and $V_{i_r}(0^-)$ is calculated as

Equation 4.28

$$V_{i_r}(0^-) = -\left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$

$$- \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) + \bar{p}_i^b\left(\pi_i(0^-)\right)\left(\pi_i(0^-) + H_1\right)$$

With a similar argument from the case that $u_i(0) = H_1$, we have

Equation 4.29

$$\pi_i(0^-) + H_3 \leq -X_i\left(1^-, p_i^b(0^+)\right) - X_i\left(1^-, p_i^b(0^+) - 1\right) - X_i\left(1^-, p_i^b(0^+) - 2\right)$$

Therefore, there are three sub cases for $V_{i_r}(1^-)$

subcase 1: $\pi_i(0^-) + H_3 \leq -X_i\left(1^-, p_i^b(0^+)\right)$

In this sub case, we have

Equation 4.30

$$V_{i_r}(1^-) = p_i^b(0^+)[H_1 + \pi_i(0^-) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)]$$

Thus, the return can be calculated as follows,

Equation 4.31

$$R_3 = \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right]\left(\pi_i(0^-) + H_1\right)$$

$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$

$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$

Now we can compare R_3 and R_2. Since there are two sub cases in the case that $u_i(0) = H_1$, we need to compare R_3 with the return from both sub cases.

First, we compare R_3 with R_2 in the sub case that $\pi_i(0^-) + H_2 \leq -X_i\left(1^-, p_i^b(0^+)\right)$.

Equation 4.32

$$R_3 - R_2 = \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) > 0,$$

since $\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+) < 0$ and $X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) < 0$.

Second, we compare R_3 with R_2 in the sub case that $(0^-) + H_2 > -X_i\left(1^-, p_i^b(0^+)\right)$.

Equation 4.33

$$R_3 - R_2 = \pi_i(0^-) + H_2 + X_i\big(1^-, p_i^b(0^+)\big)$$
$$+ \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) > 0$$

since $\pi_i(0^-) + H_2 > -X_i\big(1^-, p_i^b(0^+)\big)$, and $\left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) > 0$. Thus, R_3 in sub case 1 is greater than R_2 in both sub cases

subcase 2: $-X_i\big(1^-, p_i^b(0^+)\big) < \pi_i(0^-) + H_3 \le -X_i\big(1^-, p_i^b(0^+)\big) - X_i\big(1^-, p_i^b(0^+) - 1\big)$

In this sub case, we have

Equation 4.34

$$V_{i_r}(1^-) = (p_i^b(0^+) - 1)\left(H_1 + \pi_i(0^-) - X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1\big)\right) - X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big)$$
$$- X_i\big(1^-, p_i^b(0^+)\big)$$

Thus, the return can be calculated as follows,

Equation 4.35

$$R_3 = \big[p_i^b(0^+) - \bar{p}_i^b\big(\pi_i(0^-)\big) - 1\big](\pi_i(0^-) + H_1)$$
$$+ \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1\big)$$
$$+ \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 3 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) - X_i(1^-, p_i^b(0^+))$$

Now we compare R_3 and R_2. First, we compare R_3 with R_2 in the sub case that $\pi_i(0^-) + H_2 \le -X_i\big(1^-, p_i^b(0^+)\big)$.

Equation 4.36

$$R_3 - R_2 = -(\pi_i(0^-) + H_3) + \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big)$$
$$- X_i(1^-, p_i^b(0^+))$$

We claim that

Equation 4.37

$$\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) < -2X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right),$$

and the detailed proof for Equation 4.37 is provided in the Appendix. By this result, we can continue comparing R_2 and R_3.

Equation 4.38

$$R_3 - R_2 > \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) > 0$$

since $\left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+)\right) < 0$ and $X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) < 0$.

Second we compare R_3 with R_2 in the sub case that $\pi_i(0^-) + H_2 > -X_i\left(1^-, p_i^b(0^+)\right)$.

Equation 4.39

$$R_3 - R_2 = \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) > 0$$

Thus, R_3 in sub case 2 is greater than R_2 in both sub cases.

subcase 3: $-X_i\left(1^-, p_i^b(0^+)\right) - X_i\left(1^-, p_i^b(0^+) - 1\right) < \pi_i(0^-) + H_3 \leq -X_i\left(1^-, p_i^b(0^+)\right) - X_i\left(1^-, p_i^b(0^+) - 1\right) - X_i\left(1^-, p_i^b(0^+) - 2\right)$

In this sub case, we have

Equation 4.40

$$V_{i_r}(1^-) = (p_i^b(0^+) - 2)\left(H_1 + \pi_i(0^-) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$
$$- 2X_i\left(1^-, p_i^b(0^+)\right) - 2X_i\left(1^-, p_i^b(0^+) - 1\right)$$

Thus, the return can be calculated as follows,

Equation 4.41

$$R_3 = \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right) - 2\right](\pi_i(0^-) + H_1)$$

$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$

$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - 2X_i\left(1^-, p_i^b(0^+)\right)$$

$$- 2X_i\left(1^-, p_i^b(0^+) - 1\right)$$

Now we compare R_3 and R_2. First, we compare R_3 with the return in the sub case that $\pi_i(0^-) + H_2 \leq -X_i\left(1^-, p_i^b(0^+)\right)$.

Equation 4.42

$$R_3 - R_2 = -2(\pi_i(0^-) + H_3) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$

$$- 2X_i(1^-, p_i^b(0^+)) - 2X_i(1^-, p_i^b(0^+) - 1)$$

We claim that

Equation 4.43

$$\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) + X_i\left(1^-, p_i^b(0^+) - 1\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$

and the detailed proof for Equation 4.43 is provided in the Appendix. By this result, we can continue comparing R_2 and R_3.

Equation 4.44

$$R_3 - R_2 > \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) > 0$$

since $\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+) < 0$ and $X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) < 0$.

Second, we compare R_3 with the return in the sub case that $\pi_i(0^-) + H_2 > -X_i\left(1^-, p_i^b(0^+)\right)$

Equation 4.45

$$R_3 - R_2 = -(\pi_i(0^-) + H_3) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$

$$- X_i(1^-, p_i^b(0^+)) - 2X_i(1^-, p_i^b(0^+) - 1)$$

We claim that

Equation 4.46

$$\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) + 2X_i\left(1^-, p_i^b(0^+) - 1\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right),$$

and the detailed proof for Equation 4.46 is provided in the Appendix. By this result, we can continue comparing R_2 and R_3.

Equation 4.47

$$R_3 - R_2 > \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) > 0,$$

since $\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+) < 0$ and $X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) < 0$.

From all of the three sub cases, we can conclude the comparison between R_2 and $R_3$when $u_i(0) = H_3$with the following table:

Table 4.4, Comparison Between R2 and R3

Subcases for R2 / Subcases for R3	$\pi_i(0^-) + H_2 \le -X_i\left(1^-, p_i^b(0^+)\right)$	$\pi_i(0^-) + H_2 > -X_i\left(1^-, p_i^b(0^+)\right)$
$\pi_i(0^-) + H_3 \le -X_i\left(1^-, p_i^b(0^+)\right)$	$R_3 > R_2$	$R_3 > R_2$
$-X_i\left(1^-, p_i^b(0^+)\right) < \pi_i(0^-) + H_3 \le -X_i\left(1^-, p_i^b(0^+)\right) - X_i\left(1^-, p_i^b(0^+) - 1\right)$	$R_3 > R_2$	$R_3 > R_2$
$-X_i\left(1^-, p_i^b(0^+)\right) - X_i\left(1^-, p_i^b(0^+) - 1\right) < \pi_i(0^-) + H_3 \le -X_i\left(1^-, p_i^b(0^+)\right) - X_i\left(1^-, p_i^b(0^+) - 1\right) - X_i\left(1^-, p_i^b(0^+) - 2\right)$	$R_3 > R_2$	$R_3 > R_2$

5 CONCLUDING REMARKS

In this thesis, we built the LOB model and illustrated how investors can build their wealth process based on the LOB process. There are still many problems that deserve further investigations.

In section 3.4, we modeled the external limit orders that were added to LOB at time $t = 0^+$ with a linear function, based on the assumption that more orders were listed in the LOB at prices closer to the lowest ask price and highest bid price. This oversimplified model does not consider how the shape of LOB changes with respect to market movement, price volatility, or limit order investors. In fact, our model only considers how LOB changes regarding market orders. With such a model, it is difficult for us to perform trading analysis with more than one investors. Further, we cannot include limit orders in our trading strategy. In the paper "Dynamic Equilibrium Limit Order Book Model and Optimal Execution Problem" [10], the authors modeled the shape of LOB with the notion of "equilibrium density". In such an equilibrium, they assumed that every limit order would yield to the same "expected utility". Adapting this dynamic equilibrium limit order book will enable us to perform trading analysis that involves multiple investors and include limit orders to solve the optimal execution problem.

In Chapter 4, we set a restriction for the LOB that $\pi_i(t) \geq 0, \quad \forall i \in \{1,2,\dots,k\}$, i.e., an investor cannot short stocks at any time t. Taking into account the case that $\pi_i(t) < 0$ may yield to a trading strategy that generates more expected return than our current analysis with the restriction. Further, since shorting is allowed in the stock markets of U.S., Europe, Australia, and China, including shorting in our analysis can also test the accuracy of our model.

In Section 4.2, we presented how investors can build their wealth process based on a trading volume process, and found out that in the first four discrete cases, the less an investor sells, the more the return will be. This finding seems to imply that the return is monotonically decreasing with respect to the amount being sold. We also found that while the investor is selling less and less stocks, the increasing amount in the return is diminishing. In other words, while $R_3 > R_2 > R_1 > R_0$, $R_3 - R_2 < R_2 - R_1 < R_1 - R_0$. This may imply the existence of an optimal strategy. If further investigation is made, we can find out whether an investor should always hold his stocks or sell at a certain amount based on our model. Please note that the calculation of the first four discreet cases does not require the inclusion of the stochastic processes that we used to model the volume change from 0^+ to 1^-. If we continue the calculation for more cases, we may have to include the random variable ζ as defined in Section 3.5. When a random variable is included, we need to use utility functions to calculate the expected return to achieve validity for our analysis. After an investor builds a wealth process, a comparison can be made between the return with respect to each trading volume, resulting in an optimal trading strategy design.

As we mentioned in the beginning of Chapter 3, a single-period model can be served as a building block for a model in a multi-period market. With deduction and further investigation, we can extend our LOB model and trading analysis in a multi-period market. In a single-period market, we can only focus on the "optimal execution problem", but in a multi-period market, we may find out an optimal trading strategy by solving the "optimal placement problem". See for example Guo, de Larrard, and Ruan (2012) [11].

APPENDIX: CALCULATIONS AND PROOFS FOR SECTION 4.2

Equation 4.17

$$\begin{aligned}V_{i_r}(0^-) &= -\left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) \\ &\quad + \bar{p}_i^b\left(\pi_i(0^-)\right)\left(\pi_i(0^-)+H_2+X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)\right) \\ &= -\left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)+\bar{p}_i^b\left(\pi_i(0^-)\right)\left(\pi_i(0^-)+H_1\right)\end{aligned}$$

Equation 4.19

$$\begin{aligned}V_{i_r}(1^-) &= p_i^b(0^+)(H_2+\pi_i(0^-)) \\ &= p_i^b(0^+)\left(H_1+\pi_i(0^-)-X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)\right)\end{aligned}$$

Equation 4.20

$$\begin{aligned}R_2 &= p_i^b(0^+)\left(\pi_i(0^-)+H_1-X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)-\bar{p}_i^b\left(\pi_i(0^-)\right)\left(\pi_i(0^-)+H_1\right) \\ &= \left[p_i^b(0^+)-\bar{p}_i^b\left(\pi_i(0^-)\right)\right]\left(\pi_i(0^-)+H_1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1-p_i^b(0^+)\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)\end{aligned}$$

Equation 4.21

$$\begin{aligned}R_2-R_1 &= \left[p_i^b(0^+)-\bar{p}_i^b\left(\pi_i(0^-)\right)\right]\left(\pi_i(0^-)+H_1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1-p_i^b(0^+)\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) \\ &\quad - \left[p_i^b(0^+)-\bar{p}_i^b\left(\pi_i(0^-)\right)\right]\left(\pi_i(0^-)+H_1\right) \\ &= \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1-p_i^b(0^+)\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) > 0,\end{aligned}$$

Equation 4.22

$$\begin{aligned}V_{i_r}(1^-) &= -(p_i^b(0^+)X_i\left(1^-,p_i^b(0^+)\right)+(p_i^b(0^+)-1)[\pi_i(0^-)+H_2+X_i\left(1^-,p_i^b(0^+)\right)] \\ &= (p_i^b(0^+)-1)\left(H_1+\pi_i(0^-)-X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)\right)-X_i\left(1^-,p_i^b(0^+)\right)\end{aligned}$$

Equation 4.23

$$\begin{aligned}R_2 &= (p_i^b(0^+)-1)\left(H_1+\pi_i(0^-)-X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)\right)-X_i\left(1^-,p_i^b(0^+)\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)-\bar{p}_i^b\left(\pi_i(0^-)\right)\left(H_1+\pi_i(0^-)\right) \\ &= (p_i^b(0^+)-1)\left(H_1+\pi_i(0^-)\right)-\left(p_i^b(0^+)-1\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)-X_i\left(1^-,p_i^b(0^+)\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)-\bar{p}_i^b\left(\pi_i(0^-)\right)\left(H_1+\pi_i(0^-)\right) \\ &= (p_i^b(0^+)-\bar{p}_i^b\left(\pi_i(0^-)\right)-1)\left(H_1+\pi_i(0^-)\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+2-p_i^b(0^+)\right)X_i\left(0^-,\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)-X_i\left(1^-,p_i^b(0^+)\right)\end{aligned}$$

$$= \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right](\pi_i(0^-) + H_1) - (\pi_i(0^-) + H_1) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - X_i\left(1^-, p_i^b(0^+)\right)$$

Equation 4.24

$$\begin{aligned} R_2 - R_1 &= \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right](\pi_i(0^-) + H_1) - (\pi_i(0^-) + H_1) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - X_i\left(1^-, p_i^b(0^+)\right) \\ &\quad - \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right](\pi_i(0^-) + H_1) \\ &= -(\pi_i(0^-) + H_1) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) \\ &\quad - X_i\left(1^-, p_i^b(0^+)\right) \\ &= -\left(\pi_i(0^-) + H_2 + X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - X_i\left(1^-, p_i^b(0^+)\right) \\ &= -(\pi_i(0^-) + H_2) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) \\ &\quad - X_i\left(1^-, p_i^b(0^+)\right) \end{aligned}$$

Equation 4.25

We claim that

$$\pi_i(0^-) + H_2 + X_i\left(1^-, p_i^b(0^+)\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$

Proof. Since $\pi_i(0^-) + H_2 + X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) \leq -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$, there exists an $M \leq 1$ such that ,

$$\pi_i(0^-) + H_2 = -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - MX_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right)$$

With a similar argument from the case that $u_i = H_1$, we have

$$-X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) < -X_i\left(1^-, p_i^b(0^+)\right).$$

Therefore, we have the following equation:

$$\begin{aligned} &\pi_i(0^-) + H_2 + X_i\left(1^-, p_i^b(0^+)\right) \\ &\quad = -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - MX_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right) - \left(-X_i\left(1^-, p_i^b(0^+)\right)\right) \\ &< -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - MX_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right) - \left[-MX_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right)\right] \\ &= -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) > 0 \end{aligned}$$

Here we proved that

$$\pi_i(0^-) + H_2 + X_i\left(1^-, p_i^b(0^+)\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$

■

Equation 4.26

$$\begin{aligned} R_2 - R_1 &= -(\pi_i(0^-) + H_2) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) \\ &\quad - X_i\left(1^-, p_i^b(0^+)\right) \end{aligned}$$

$$> X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right)$$
$$= \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) > 0$$

Equation 4.30

$$\begin{aligned} V_{i_r}(1^-) &= p_i^b(0^+)\left(H_3 + \pi_i(0^-)\right) \\ &= p_i^b(0^+)[H_1 + \pi_i(0^-) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right)] \end{aligned}$$

Equation 4.31

$$\begin{aligned} R_3 &= p_i^b(0^+)\left[H_1 + \pi_i(0^-) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right)\right] \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) - \bar{p}_i^b\left(\pi_i(0^-)\right)\left(\pi_i(0^-) + H_1\right) \\ &= p_i^b(0^+)\left(H_1 + \pi_i(0^-)\right) - p_i^b(0^+) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) \\ &\quad - p_i^b(0^+) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) - \bar{p}_i^b\left(\pi_i(0^-)\right)\left(\pi_i(0^-) + H_1\right) \\ &= \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right]\left(\pi_i(0^-) + H_1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) \end{aligned}$$

Equation 4.32

$$\begin{aligned} R_3 - R_2 &= \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right]\left(\pi_i(0^-) + H_1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) \\ &\quad - \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right]\left(\pi_i(0^-) + H_1\right) \\ &\quad - \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) \\ &= \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) > 0 \end{aligned}$$

Equation 4.33

$$\begin{aligned} R_3 - R_2 &= \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right]\left(\pi_i(0^-) + H_1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) \\ &\quad - \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right]\left(\pi_i(0^-) + H_1\right) + \left(\pi_i(0^-) + H_1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) + X_i\left(1^-, p_i^b(0^+)\right) \\ &= \left(\pi_i(0^-) + H_1\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+1\right) \\ &\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)+2\right) + X_i\left(1^-, p_i^b(0^+)\right) \end{aligned}$$

$$= \left(\pi_i(0^-) + H_2 + X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$
$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) + X_i\left(1^-, p_i^b(0^+)\right)$$
$$= \pi_i(0^-) + H_2 + X_i\left(1^-, p_i^b(0^+)\right)$$
$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) > 0$$

Equation 4.34

$$V_{i_r}(1^-) = -(p_i^b(0^+) X_i\left(1^-, p_i^b(0^+)\right) + (p_i^b(0^+) - 1)\left(H_3 + \pi_i(0^-) - X_i\left(1^-, p_i^b(0^+) - 1\right)\right)$$
$$= (p_i^b(0^+) - 1)\left(H_1 + \pi_i(0^-) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$
$$- X_i\left(1^-, p_i^b(0^+)\right)$$

Equation 4.35

$$R_3 = (p_i^b(0^+) - 1)\left(H_1 + \pi_i(0^-) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$
$$- X_i\left(1^-, p_i^b(0^+)\right) + (\bar{p}_i^b\left(\pi_i(0^-)\right) + 2) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$
$$+ (\bar{p}_i^b\left(\pi_i(0^-)\right) + 1) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - \bar{p}_i^b\left(\pi_i(0^-)\right)(\pi_i(0^-) + H_1)$$
$$= \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right) - 1\right](\pi_i(0^-) + H_1)$$
$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$
$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - X_i(1^-, p_i^b(0^+))$$

Equation 4.36

$$R_3 - R_2 = \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right) - 1\right](\pi_i(0^-) + H_1)$$
$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$
$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - X_i\left(1^-, p_i^b(0^+)\right)$$
$$- \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right](\pi_i(0^-) + H_1)$$
$$- \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$
$$= -(\pi_i(0^-) + H_1) + X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$
$$+ \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - X_i\left(1^-, p_i^b(0^+)\right)$$
$$= -(\pi_i(0^-) + H_3) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right) X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$
$$- X_i(1^-, p_i^b(0^+))$$

Equation 4.37

We claim that

$$\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) < -2X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right).$$

proof. Since $\pi_i(0^-) + H_3 + X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1\big) + X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) \leq -X_i\left(0^-, \bar{p}_i^b(\pi_i(0^-))\right)$, there exists an $M \leq 1$, such that

$$\pi_i(0^-) + H_3 = -X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1\big) - X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) - MX_i\left(0^-, \bar{p}_i^b(\pi_i(0^-))\right)$$

With a similar argument from the case that $u_i(0) = H_1$, we have

$$-X_i\left(0^-, \bar{p}_i^b(\pi_i(0^-))\right) < -X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1\big) < -X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) < -X_i\left(1^-, p_i^b(0^+)\right).$$

Therefore, we have the following equation:

$$\begin{aligned}
\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) &= -X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) - X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1\big) \\
&\quad - MX_i\left(0^-, \bar{p}_i^b(\pi_i(0^-))\right) - \left[-X_i\left(1^-, p_i^b(0^+)\right)\right] \\
&< -X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) - X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) \\
&\quad - MX_i\left(0^-, \bar{p}_i^b(\pi_i(0^-))\right) - \left[-MX_i\left(0^-, \bar{p}_i^b(\pi_i(0^-))\right)\right] \\
&= -2X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) > 0
\end{aligned}$$

Here we proved that

$$\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) < -2X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big).$$

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Equation 4.38

$$\begin{aligned}
R_3 - R_2 &= -(\pi_i(0^-) + H_3) + \left(\bar{p}_i^b(\pi_i(0^-)) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) \\
&\quad - X_i\left(1^-, p_i^b(0^+)\right) \\
&> 2X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) + \left(\bar{p}_i^b(\pi_i(0^-)) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) \\
&= \left(\bar{p}_i^b(\pi_i(0^-)) + 4 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1\big) > 0
\end{aligned}$$

Equation 4.39

$$\begin{aligned}
R_3 - R_2 &= \left[p_i^b(0^+) - \bar{p}_i^b\big(\pi_i(0^-)\big) - 1\right](\pi_i(0^-) + H_1) \\
&\quad + \left(\bar{p}_i^b(\pi_i(0^-)) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1\big) \\
&\quad + \left(\bar{p}_i^b(\pi_i(0^-)) + 3 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) - X_i\left(1^-, p_i^b(0^+)\right) \\
&\quad - \left[p_i^b(0^+) - \bar{p}_i^b\big(\pi_i(0^-)\big) - 1\right](\pi_i(0^-) + H_1) \\
&\quad - \left(\bar{p}_i^b(\pi_i(0^-)) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 1\big) + X_i\left(1^-, p_i^b(0^+)\right) \\
&= \left(\bar{p}_i^b(\pi_i(0^-)) + 3 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b(\pi_i(0^-)) + 2\big) > 0
\end{aligned}$$

Equation 4.40

$$V_{i_r}(1^-) = -(p_i^b(0^+)X_i\left(1^-, p_i^b(0^+)\right) - (p_i^b(0^+) - 1)X_i\left(1^-, p_i^b(0^+) - 1\right) + (p_i^b(0^+) - 2)\left(H_3 + \pi_i(0^-) + X_i\left(1^-, p_i^b(0^+)\right) + X_i\left(1^-, p_i^b(0^+) - 1\right)\right)$$
$$= (p_i^b(0^+) - 2)\left(H_1 + \pi_i(0^-) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - 2X_i\left(1^-, p_i^b(0^+)\right) - 2X_i\left(1^-, p_i^b(0^+) - 1\right)$$

Equation 4.41

$$R_3 = (p_i^b(0^+) - 2)\left(H_1 + \pi_i(0^-) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - 2X_i\left(1^-, p_i^b(0^+)\right) - 2X_i\left(1^-, p_i^b(0^+) - 1\right) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - \bar{p}_i^b\left(\pi_i(0^-)\right)(\pi_i(0^-) + H_1)$$
$$= \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right) - 2\right](\pi_i(0^-) + H_1) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - 2X_i\left(1^-, p_i^b(0^+)\right) - 2X_i\left(1^-, p_i^b(0^+) - 1\right)$$

Equation 4.42

$$R_3 - R_2 = \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right) - 2\right](\pi_i(0^-) + H_1) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - 2X_i\left(1^-, p_i^b(0^+)\right) - 2X_i\left(1^-, p_i^b(0^+) - 1\right) - \left[p_i^b(0^+) - \bar{p}_i^b\left(\pi_i(0^-)\right)\right](\pi_i(0^-) + H_1) - \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 1 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)$$
$$= -2(\pi_i(0^-) + H_1) + 2X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - 2X_i\left(1^-, p_i^b(0^+)\right) - 2X_i\left(1^-, p_i^b(0^+) - 1\right)$$
$$= -2(\pi_i(0^-) + H_3) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 2 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - 2X_i(1^-, p_i^b(0^+)) - 2X_i(1^-, p_i^b(0^+) - 1)$$

Equation 4.43

We claim that

$$\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) + X_i\left(1^-, p_i^b(0^+) - 1\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$$

proof. Since $\pi_i(0^-) + H_3 + X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) + X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) \leq -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right)$, there exists an $M \leq 1$, such that

$$\pi_i(0^-) + H_3 = -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - MX_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right)$$

With a similar argument from the case that $u_i(0) = H_1$, we have

$$-X_i\left(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big)\right) < -X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1\big) < -X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) < -X_i\left(1^-, p_i^b(0^+)\right)$$

Therefore, we have the following equation:

$$\begin{aligned}
&\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) + X_i\big(1^-, p_i^b(0^+) - 1\big) \\
&\quad = -X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) - X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1\big) \\
&\quad\quad - MX_i\left(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big)\right) - \left[-X_i\left(1^-, p_i^b(0^+)\right)\right] - \left[-X_i\big(1^-, p_i^b(0^+) - 1\big)\right] \\
&< -X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) - X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1\big) - MX_i\left(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big)\right) \\
&\quad\quad - \left(-X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1\big)\right) - \left[-MX_i\left(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big)\right)\right] \\
&= -X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big)
\end{aligned}$$

Here we proved that

$$\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) + X_i\big(1^-, p_i^b(0^+) - 1\big) < -X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big)$$

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Equation 4.44

$$\begin{aligned}
R_3 - R_2 &= 2\left[-\pi_i(0^-) - H_3 - X_i\left(1^-, p_i^b(0^+)\right) - X_i\big(1^-, p_i^b(0^+) - 1\big)\right] \\
&\quad + \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) \\
&> 2X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) + + \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) \\
&= \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 4 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) > 0
\end{aligned}$$

Equation 4.45

$$\begin{aligned}
R_3 - R_2 &= \left[p_i^b(0^+) - \bar{p}_i^b\big(\pi_i(0^-)\big) - 2\right]\left(\pi_i(0^-) + H_1\right) \\
&\quad + \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 3 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1\big) \\
&\quad + \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 4 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) - 2X_i\left(1^-, p_i^b(0^+)\right) \\
&\quad - 2X_i\big(1^-, p_i^b(0^+) - 1\big) - \left[p_i^b(0^+) - \bar{p}_i^b\big(\pi_i(0^-)\big) - 1\right]\left(\pi_i(0^-) + H_1\right) \\
&\quad - \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 2 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1\big) + X_i\left(1^-, p_i^b(0^+)\right) \\
&= -\left(\pi_i(0^-) + H_1\right) + X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 1\big) \\
&\quad + \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 4 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) - X_i\left(1^-, p_i^b(0^+)\right) \\
&\quad - 2X_i\big(1^-, p_i^b(0^+) - 1\big) \\
&= -\left(\pi_i(0^-) + H_3\right) + \left(\bar{p}_i^b\big(\pi_i(0^-)\big) + 3 - p_i^b(0^+)\right) X_i\big(0^-, \bar{p}_i^b\big(\pi_i(0^-)\big) + 2\big) \\
&\quad - X_i(1^-, p_i^b(0^+)) - 2X_i(1^-, p_i^b(0^+) - 1)
\end{aligned}$$

Equation 4.46

We claim that

$$\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) + 2X_i\left(1^-, p_i^b(0^+) - 1\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right).$$

proof. Since $\pi_i(0^-) + H_3 + X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) + X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) \leq -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right)$, there exists an $M \leq 1$, such that

$$\pi_i(0^-) + H_3 = -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - MX_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right)$$

With a similar argument from the case that $u_i(0) = H_1$, we have

$$-X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) < -X_i\left(1^-, p_i^b(0^+)\right)$$

Therefore, we have the following equation:

$$\begin{aligned}
&\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) + 2X_i\left(1^-, p_i^b(0^+) - 1\right) \\
&\quad = -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) \\
&\quad\quad - MX_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right) - \left[-X_i\left(1^-, p_i^b(0^+)\right)\right] - \left[-2X_i\left(1^-, p_i^b(0^+) - 1\right)\right] \\
&< -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) - X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right) - MX_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right) \\
&\quad\quad - \left(-X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 1\right)\right) - \left[-MX_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right)\right)\right] \\
&= -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)
\end{aligned}$$

Here we proved that $\pi_i(0^-) + H_3 + X_i\left(1^-, p_i^b(0^+)\right) + 2X_i\left(1^-, p_i^b(0^+) - 1\right) < -X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right)$.

■

Equation 4.47

$$\begin{aligned}
R_3 - R_2 &= -(\pi_i(0^-) + H_3) - X_i\left(1^-, p_i^b(0^+)\right) - 2X_i\left(1^-, p_i^b(0^+) - 1\right) \\
&\quad + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) \\
&> X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) + \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 3 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) \\
&= \left(\bar{p}_i^b\left(\pi_i(0^-)\right) + 4 - p_i^b(0^+)\right)X_i\left(0^-, \bar{p}_i^b\left(\pi_i(0^-)\right) + 2\right) > 0
\end{aligned}$$

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