Gallai-Ramsey Numbers for C7 with Multiple Colors

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GALLAI-RAMSEY NUMBERS FOR C_7 WITH MULTIPLE COLORS

by

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ABSTRACT

The core idea of Ramsey theory is that complete disorder is impossible. Given a large structure, no matter how complex it is, we can always find a smaller substructure that has some sort of order. One view of this problem is in edge-colorings of complete graphs. For any graphs $G, H_1, \ldots, H_k$, we write $G \to (H_1, \ldots, H_k)$, or $G \to (H)_k$ when $H_1 = \cdots = H_k = H$, if every k-edge-coloring of G contains a monochromatic H_i in color i for some $i \in \{1, \ldots, k\}$. The Ramsey number $r_k(H_1, \ldots, H_k)$ is the minimum integer n such that $K_n \to (H_1, \ldots, H_k)$, where K_n is the complete graph on n vertices. Computing $r_k(H_1, \ldots, H_k)$ is a notoriously difficult problem in combinatorics. A weakening of this problem is to restrict ourselves to Gallai colorings, that is, edge-colorings with no rainbow triangles. From this we define the Gallai-Ramsey number $gr_k(K_3, G)$ as the minimum integer n such that either K_n contains a rainbow triangle, or $K_n \to (G)_k$. In this thesis, we determine the Gallai-Ramsey numbers for C_7 with multiple colors. We believe the method we developed can be applied to find $gr_k(K_3, C_{2n+1})$ for any integer $n \geq 2$, where C_{2n+1} denotes a cycle on $2n + 1$ vertices.

ACKNOWLEDGMENTS

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1 INTRODUCTION

We begin this thesis with an overview of basic concepts and definitions in graph theory. Let $\mathbb{N}$ be the set of natural numbers. For any $n \in \mathbb{N}$, define $[n] := \{1, 2, ..., n\}$. A **graph** G is comprised of a set $V(G)$ of vertices and a set $E(G)$ of edges that connect pairs of vertices. For notational convenience, instead of writing $\{v_1, v_2\}$ to represent an edge with endpoints v_1 and v_2, we write v_1v_2. Two vertices $x, y \in V(G)$ are **adjacent** if $xy \in E(G)$. A **complete graph** K_n is a graph with n vertices where $xy \in E(K_n)$ for all $x, y \in V(K_n)$, with $x \neq y$. A **triangle** is a K_3. A **path** P_n is a graph whose vertices can be ordered into a sequence such that consecutive vertices in the sequence are adjacent. A **cycle** C_n is a graph whose vertices can be ordered into a a sequence such that consecutive vertices in the sequence are adjacent, and the first and last vertices of the sequence are adjacent. A **k-edge-coloring** of a graph G is a function $\phi : E(G) \to [k]$ that assigns a number to each edge in G. G is **monochromatic** with respect to ϕ if $\phi(E(G)) = i \in [k]$ for some fixed $k \in \mathbb{N}$. G is **rainbow** with respect to ϕ if $\phi(e) \neq \phi(f)$ for all $e, f \in E(G)$, $e \neq f$. A **bipartite** graph is a graph whose vertices can be partitioned into two nonempty sets S_1 and S_2 such that every edge in the graph connects a vertex in S_1 with a vertex in S_2. Given a graph G, for any disjoint sets $A, B \subseteq V(G)$, A is **complete** to B if for all $a \in A$, $b \in B$, we have $ab \in E(G)$, and A is **anticomplete** to B if for all $a \in A$, $b \in B$, we have $ab \notin E(G)$. A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any $S \subseteq V(G)$, **the subgraph of G induced by** S, denoted $G[S]$, is the graph with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$. A graph H is an **induced subgraph** of a graph G if $V(H) \subseteq V(G)$ and $H = G[V(H)]$. Let $M \subseteq E(G)$. Then M is a **matching** in G if no two edges in M share a common vertex in G. M is an **induced matching** in G if M is a matching in G such that for every pair of edges $uv, u'v' \in M$, $\{u, v\}$ is anticomplete to $\{u', v'\}$ in G.

A well-known counting argument that will be used throughout this thesis is known as the Pigeonhole Principle.

Theorem 1.1 (Pigeonhole Principle). *Let $n, x_1, \ldots, x_n \in \mathbb{N}$. If $x_1 + \cdots + x_n - n + 1$ objects are distributed between n boxes, then the first box contains at least x_1 objects, or the second box contains at least x_2 objects, ..., or the n^{th} box contains at least x_n objects.*

Proof. Suppose otherwise. Then the first box contains at most $x_1 - 1$ objects, and the second box contains at most $x_2 - 1$ objects, ..., and the final box contains at most $x_n - 1$ objects, so in total we have at most $(x_1 - 1) + (x_2 - 1) + \cdots + (x_n - 1) = x_1 + x_2 + \cdots + x_n - n$ objects, a contradiction. □

Ramsey theory has its origins from the work of Frank Ramsey [13]. We introduce the following notation: For any graphs $G, H_1, \ldots, H_k$, we write $G \to (H_1, \ldots, H_k)$, or $G \to (H)_k$ when $H_1 = \cdots = H_k = H$, if every k-edge-coloring of G contains a monochromatic H_i in color i for some $i \in [k]$. The **Ramsey number** $r_k(H_1, \ldots, H_k)$ is the minimum integer n such that $K_n \to (H_1, \ldots, H_k)$, where K_n is the complete graph on n vertices. If $H_1 = \cdots = H_k = H$, then we simply write $r_k(H)$ instead of $r_k(H, \ldots, H)$.

Theorem 1.2 (Ramsey's Theorem). *For any positive integer k and any collection of graphs $H_1, \ldots, H_k$, the Ramsey number $r_k(H_1, \ldots, H_k)$ exists.*

Ramsey theory is a notoriously difficult branch of combinatorics. Many questions in the field have remained wide open for years, and some questions seem hopeless to resolve with our current knowledge. A subfield known as Gallai-Ramsey theory imposes a strengthened color condition so that a result concerning the partitioning of graphs can be utilized. In this thesis, we focus on calculating the Gallai-Ramsey numbers for cycles of length seven.

In Chapter 2, we introduce classical Ramsey theory, which is the study of Ramsey theory with no special restrictions on the edge-colorings chosen. We demonstrate some well known results of the field, and we state some special results that pertain to certain classes of graphs.

In Chapter 3, we introduce Gallai-Ramsey theory, which is the study of Ramsey theory with a strengthened color condition. We then examine the special case of Gallai-Ramsey theory on cycles, outlining our current knowledge to motivate our personal research.

In Chapter 4, we present our original research, in which we determine the Gallai-Ramsey numbers for C_7 with multiple colors.

2 CLASSICAL RAMSEY THEORY

The proof technique for finding the exact value of any Ramsey number $n = r_k(G_1, \ldots, G_k)$ is the same. First, we show that $K_{n-1} \not\rightarrow (G_1, \ldots, G_k)$, or that there exists a k-edge-coloring of K_{n-1} that does not contain a G_1 of color 1, nor a G_2 of color 2, and so forth. This shows that $r_k(G_1, \ldots, G_k) \geq n$. Then, we show that $K_n \rightarrow (G_1, \ldots, G_k)$, or that every k-edge-coloring of K_n contains a G_1 of color 1, or a G_2 of color 2, and so forth. This shows that $r_k(G_1, \ldots, G_k) \leq n$, and thus $r_k(G_1, \ldots, G_k) = n$. We first observe a simple example of this technique in practice.

Theorem 2.1. $r_2(K_3) = 6$

Proof. First, we exhibit a 2-edge-coloring of K_5 that contains neither a red nor a blue K_3. Figure 2.1 illustrates the unique coloring satisfying this.

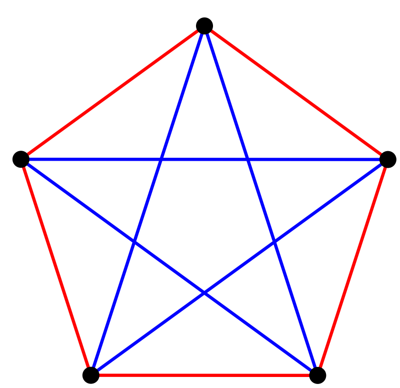

Figure 2.1: A monochromatic triangle free coloring of K_5.

	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4	9	18	25	36 - 41	49 - 61	59 - 84	73 - 115
5	14	25	43-48	58 - 87	80 - 143	101 - 216	133 - 316
6	18	36 - 41	58 - 87	102 - 165	115 - 298	134 - 495	183 - 780
7	23	49 - 61	80 - 143	115 - 298	205 - 540	217 - 1031	252 - 1713
8	28	59 - 84	101 - 216	134 - 495	217 - 1031	282 - 1870	329 - 3583
9	36	73 - 115	133 - 316	183 - 780	252 - 1713	329 - 3583	565 - 6588

Table 2.1: Various values of $r_2(K_n, K_m)$.

Let $c : E(G) \to \{\text{red}, \text{blue}\}$ be any 2-edge-coloring of $G = K_6$. We next show G contains a monochromatic triangle. Fix a vertex $v \in V(G)$. Since there are 5 edges incident to v, and each edge is one of two colors, by the pigeonhole principle at least 3 edges incident to v are of the same color, suppose blue. Let va, vb, and vc be edges colored by blue. If any of ab, ac, or bc is colored blue, then we have a blue triangle. But if none of those edges are blue, then they are all colored red, and we have a red triangle. Thus every 2-coloring of G contains a monochromatic triangle. □

Although the above proof is relatively simple, determining the values of Ramsey numbers for larger graphs quickly becomes incredibly difficult. Table 2.1, compiled by Radziszowski [12], aggregates some of the known Ramsey numbers of complete graphs when two colors are used.

We do not need to restrict ourselves only to complete graphs. Of particular interest to us are the Ramsey numbers for when both G and H are cycles of various length. For instance, we show the following.

Theorem 2.2. $r_2(C_4) = 6$.

Proof. Figure 2.1 demonstrates a valid coloring of a K_5 with no monochromatic C_4.

For the upper bound, let $G = K_6$. Let $c : E(G) \to \{\text{red}, \text{blue}\}$ be any 2-edge-coloring of G. Fix a vertex $v \in G$. Let $N(v) = \{v_1, \ldots, v_5\}$. Since $d(v) = 5$, by the Pigeonhole Principle we may assume that $vv_1, \ldots, vv_t$ are colored blue, where $t \geq 3$. Then $vv_{t+1}, \ldots, vv_5$ are red. We will divide the proof into cases based on the number of edges in the larger color class.

Case 1 ($t = 3$): Observe that if v_4 or v_5 are connected to two different vertices in $\{v_1, v_2, v_3\}$ by blue edges, then we can find a blue C_4. So each v_4 and v_5 are connected to at most one vertex in $\{v_1, v_2, v_3\}$ by blue. So both v_4 and v_5 are connected to a common vertex in $\{v_1, v_2, v_3\}$ by red, which creates a red C_4.

Case 2 ($t \geq 4$): Observe that if v_5 is connected to $\{v_1, v_2, v_3, v_4\}$ by two or more blue edges, then we easily find a blue C_4. So at least three vertices in $\{v_1, v_2, v_3, v_4\}$ are connected to v_5 by red, suppose v_1, v_2, v_3. If there are two edges of the same color between v_1, v_2, v_3 that are the same color, then we find a monochromatic C_4. But since there are three edges between them and two color classes, one color class will have at least two edges. □

Theorem 2.3. *$r_2(C_3, C_4) = 7$.*

Proof. Figure 2.2 demonstrates a valid coloring of a K_6 with no blue C_3 and no red C_4.

For the upper bound, let $G = K_7$. Let $c : E(G) \to \{\text{red}, \text{blue}\}$ be any 2-edge-coloring of G. Fix a vertex $v \in G$. Let $N(v) = \{v_1, v_2, \ldots, v_6\}$. Since $d(v) = 6$, by the Pigeonhole Principle, we may assume that $vv_1, \ldots, vv_t$ are red, where $t \geq 3$. We will divide the proof into cases based on the number of edges in each color class. In all cases, we will demonstrate that the coloring contains either a blue C_3, or a red C_4.

Case 1 ($t = 3$): Let $B = \{v_1, v_2, v_3\}$ and $R = \{v_4, v_5, v_6\}$. All edges between v_4, v_5, v_6 must

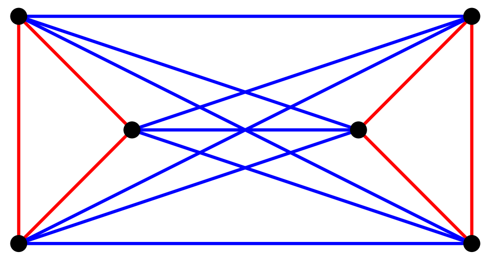

Figure 2.2: A blue C_3 and red C_4 free coloring of K_6.

be colored red, otherwise we find a blue C_3. Each vertex in R is connected to at most one vertex in B by a red edge, otherwise we find a red C_4. Additionally, each vertex in R is not connected to the same vertex in B as another vertex in R, otherwise we find a red C_4. So each vertex in R is connected to at least two vertices in B by blue edges. For any pair of vertices in R share a common vertex in B. connected by blue. So a blue edge between that pair of vertices would create a blue C_3, and if all edges between vertices in R are red, then we have a red C_4.

Case 2 ($t \geq 4$): Let $R = \{v_1, v_2, v_3, v_4\}$. If v_6 is connected to any two vertices in R by red, then we find a red C_4, so there exists at least three blue edges between v_6 and R, suppose v_1v_6, v_2v_6, v_3v_6 are blue. If any edge between v_1, v_2, v_3 is blue, then we find a blue C_3. If all of them are red, then we have a red C_4. □

Although these individual results for various pairs of cycles can be useful, a general formula is far more powerful. The following is due to Bondy and Erdős [1].

Theorem 2.4. *$r_2(C_{2n+1}) = 4n + 1$ for any integer $n \geq 2$.*

The following result by Faudree and Schelp [5] provides the two color Ramsey numbers for all pairs of cycles.

Theorem 2.5. *If* $3 \leq s \leq r$ *with* s *odd and* $(r, s) \neq (3, 3)$*, then* $r_2(C_r, C_s) = 2r - 1$.
If $4 \leq s \leq r$ *with* s *and* r *even,* $(r, s) \neq (4, 4)$*, then* $r_2(C_r, C_s) = r + \frac{1}{2}s - 1$.
If $4 \leq s \leq r$ *with* s *even and* r *odd, then* $r_2(C_r, C_s) = \max\{r + \frac{1}{2}s - 1, 2s - 1\}$.

Also of interest are what are known as multicolor Ramsey numbers, or Ramsey numbers when $k \geq 3$. However, far less is known about the multicolor situation than the two color case. Conjecture 2.1 is due to Bondy and Erdős [1]. Conjecture 2.2 is due to Erdős and Graham [3].

Conjecture 2.1. *For every integer* $n \geq 2$*,*

$$r_3(C_{2n+1}) = 8n + 1.$$

Conjecture 2.2. *For every integer* $n \geq 2$*,*

$$\lim_{k \to \infty} \frac{r_k(C_{2n+1})}{r_k(C_3)} = 0.$$

Calculating Ramsey numbers can become very difficult once multiple colors are used. One way to alter the problem is to consider a strengthened color condition so that additional structure can be used. One such strengthening is considered in the next chapter.

3 GALLAI-RAMSEY THEORY

Although classical Ramsey theory considers all colorings of a graph, a subfield known as Gallai-Ramsey theory places a restriction on the colorings considered. As stated before, a **rainbow** G is a graph G together with a color function ϕ such that $\phi(e) \neq \phi(f)$ for all $e, f \in E(G)$. Of particular interest are colorings of complete graphs that contain no rainbow triangle. Rainbow triangle free colorings are also known as **Gallai colorings**. Given the requirement of a rainbow triangle free coloring, Gallai [8] proved the following result (restated in terms of graph theory) regarding the structure of the coloring.

Theorem 3.1 (Gallai's Theorem)**.** *In any coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition (a partition into more than one part) of the vertices (known as a **Gallai partition**) such that all edges between the parts of the partition are colored at most two colors and all edges between each pair of parts are colored exactly one color.*

Definition. *The **Gallai-Ramsey number** for graphs G, H is the least integer $n = gr_k(G, H)$ such that every k-edge-coloring of K_n yields a rainbow copy of G or a monochromatic copy of H.*

The following properties of Gallai-Ramsey numbers in relation to classical Ramsey numbers can be easily obtained.

Corollary 3.1. *(a) If $|E(G)| \geq k+1$, then $gr_k(G, H) = r_k(H)$.*

(b) $gr_k(G, H) \leq r_k(H)$.

Of particular interest is the class of problems where we fix $G = K_3$, as this choice allows us to use Gallai's Theorem. We can also consider the behavior of $gr_k(K_3, G)$ by changing the structure of G. The following result is due to Gyárfás, Sárközy, Sebő, and Selkow [10].

Theorem 3.2. *If G has no isolated vertices, then $gr_k(K_3, G)$ is exponential in k if G is not bipartite and linear in k if G is bipartite and not a star.*

Definition. *Given a Gallai partition of a graph G, the **reduced graph** of G is the subgraph of G obtained by taking exactly one vertex from each part of the partition and all edges between the chosen vertices.*

By taking the reduced graph of a Gallai partition, we obtain a 2-edge-colored graph. Since far more is known about two color Ramsey numbers than multicolored Ramsey numbers, it is often a useful technique to show that the reduced graph of certain Gallai partitions will lead to known results, reducing the number of partitions that we need to consider.

One case that can be considered is when G is a cycle. At the moment, the exact number for the multicolor Gallai-Ramsey numbers for cycles is not known. The following are the best known bounds for this case, shown in [7, 11].

Theorem 3.3. *For all integers k and n with $k \geq 1$ and $n \geq 2$,*

$$(n-1)k + n + 1 \leq gr_k(K_3, C_{2n}) \leq (n-1)k + 3n,$$

$$n2^k + 1 \leq gr_k(K_3, C_{2n+1}) \leq (2^{k+3} - 1)n \log_2 n.$$

We provide a proof for the lower bound of odd cycles. The construction for this bound has been known since Erdős and Graham [3]. The proof presented here is an algorithm inspired by the original inductive proof.

Proof. Given graphs G and H, define the operation $G +_i H$ as the join of G and H where all edges connecting between G and H are colored with color i. The following algorithm constructs a k-edge-coloring of K_{n2^k} with neither a monochromatic C_{2n+1} nor a rainbow triangle.

1. Let $G = K_1$. Let $i = 0$.
2. While $i < (k-1)$:
   (a) $i \leftarrow i + 1$
   (b) $G \leftarrow G +_i G$
3. Replace each vertex of G by K_{2n}, where the edges of each K_{2n} are colored by color k.

Clearly the initial graph is rainbow triangle free. Additionally, whenever we duplicate the current graph we have, we duplicate a rainbow triangle free graph, and by Gallai's theorem, connecting the two by a single color will result in another rainbow triangle free graph, so the final graph is rainbow triangle free. Additionally, the graphs induced by each color except for k are all unions of disjoint complete bipartite graphs, and bipartite graphs are odd cycle free. The graph induced by the edges colored k is a union of disjoint K_{2n} graphs, which is C_{2n+1} free. Thus, the resulting graph G is rainbow triangle free and monochromatic C_{2n+1} free. To find $|V(G)|$, observe that we duplicate the graph with a single vertex $k-1$ times, then change every vertex to $2n$ vertices. Thus we find $|V(G)| = 2^{k-1} \cdot 2n = n2^k$. The lower bound follows. □

This algorithm provides a method of constructing a valid lower bound. For example, if we fix $k = 3$ and $n = 2$, our algorithm produces a graph as shown in Figure 3.1

However, this lower bound is not the best possible for all odd cycles. Consider $gr_k(K_3, C_3)$. The following result is known due to Chung and Graham [2].

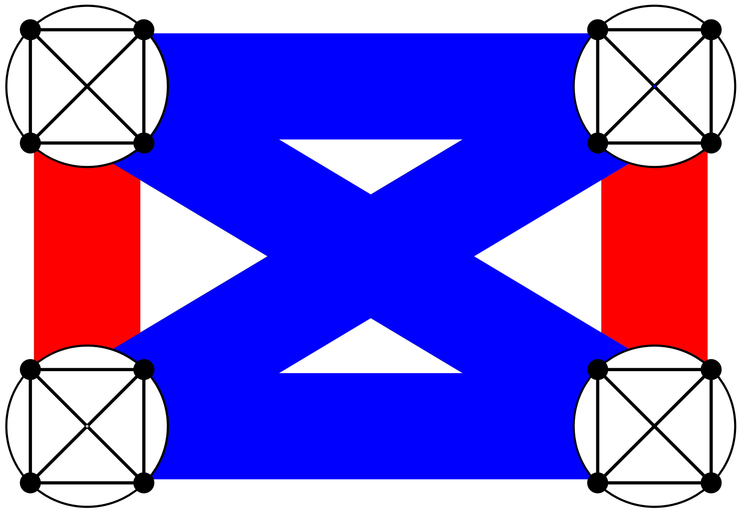

Figure 3.1: Given $k = 3$ and $n = 2$, our algorithm produces this graph. The thick lines represent that all edges between the parts are of that color.

Theorem 3.4. $gr_k(K_3, C_3) = \begin{cases} 5^{k/2} + 1 & \textit{if } k \textit{ is even} \\ 2 \cdot 5^{(k-1)/2} + 1 & \textit{if } k \textit{ is odd.} \end{cases}$

Take $k = 3$. Theorem 3.4 implies that there exists some K_{10} that contains no monochromatic or rainbow triangle. Such a coloring is shown in Figure 3.2. Observe that it is created by taking two copies of Figure 2.1 (which are rainbow and monochromatic triangle free) and connecting them by a single tertiary color, which creates a Gallai partition, so the resulting graph is rainbow and monochromatic triangle free. However, the algorithm developed in our proof of the lower bound only constructs a K_8 that is monochromatic and rainbow triangle free. Thus, in the case of a cycle of length 3, the bound constructed in the algorithm is not tight.

Figure 3.2: A coloring of K_{10} with no monochromatic or rainbow triangles.

However, there are cases in which the algorithm above does provide the best possible construction for a lower bound. The following is a recent result of Fujita and Magnant [7].

Theorem 3.5. *$gr_k(K_3, C_5) = 2^{k+1} + 1$ for every integer $k \geq 1$.*

If we fix $n = 2$ in our algorithm, then we produce a graph with 2^{k+1} vertices with no rainbow K_3 and no monochromatic C_5. Thus the lower bound in Theorem 3.3 is tight for the case of C_5.

The exact value for cycles of length 7 is currently open. Corollary 3.1 (a) shows that $gr_2(K_3, C_7) = r_2(C_7) = 13$. A result of classical Ramsey theory by Faudree, Schelten, and Schiermeyer [6] shows that $r_3(C_7) = 25$. By Theorem 3.3 and Corollary 3.1 (b), $25 \leq gr_3(K_3, C_7) \leq r_3(C_7) = 25$. Thus $gr_3(K_3, C_7) = 25$. However, $gr_k(K_3, C_7)$ is not known when $k \geq 4$.

The Gallai-Ramsey numbers for small even cycles are somewhat more solved. Theorem 3.6 is due to Faudree, Gould, Jacobson, and Magnant [4]. Theorem 3.7 is due to Fujita and Magnant [7]. Theorem 3.8 is due to Gregory and Magnant [9].

Theorem 3.6. *$gr_k(K_3, C_4) = k + 4$ for every integer $k \geq 1$.*

Theorem 3.7. $gr_k(K_3, C_6) = 2k + 4$ *for every integer* $k \geq 1$.

Theorem 3.8. $gr_k(K_3, C_8) = 3k + 5$ *for every integer* $k \geq 1$.

We determine $gr_k(K_3, C_7)$ in the next chapter.

4 GALLAI-RAMSEY NUMBERS FOR C_7 WITH MULTIPLE COLORS

In this chapter, we determine $gr_k(K_3, C_7)$ for every integer $k \geq 1$.

Theorem 4.1. *$gr_k(K_3, C_7) = 3 \cdot 2^k + 1$ for every integer $k \geq 1$.*

Proof. The lower bound follows from Theorem 3.3. We show the upper bound by induction on k. Since $r_2(C_7) = 13$, and $gr_2(K_3, C_7) = r_2(C_7)$, the statement is true for $k = 2$. Additionally, since $r_3(C_7) = 25$ by [6], we have $gr_3(K_3, C_7) \leq r_3(C_7) = 25$, so the theorem is true for $k = 3$. Let $k \geq 4$, $n = 3 \cdot 2^k + 1$, $G = K_n$, and let c be any k-edge-coloring of G that contains no rainbow triangle. Let $E_1, \ldots, E_k$ be the color classes of c. Suppose otherwise, that G contains no monochromatic C_7.

We show the following lemma, which uses the fact that $gr_k(K_3, C_5) = 2^{k+1} + 1$.

Lemma 4.1 (Matching Lemma)**.** *Let $E \subseteq E_i$ for any $i \in [k]$. If $|E| \geq 2^k + 1$, then E is not an induced matching in any $H \subseteq G$ with $E \subseteq E(H)$, that is, E is not a matching in H with $E = E_i \cap E(H)$ for any $H \subseteq G$ with $E \subseteq E(H)$.*

Proof. Suppose otherwise, that E is an induced matching in some $H \subseteq G$. Without loss of generality, let all edges in E be blue. Let $E = \{a_1b_1, a_2b_2, \cdots, a_{|E|}b_{|E|}\}$, and let $A = \{a_1, a_2, \ldots, a_{|E|}\}$. Since E is an induced matching, $G[A]$ is colored by at most $k-1$ colors. Since $gr_{k-1}(K_3, C_5) = 2^k + 1$, there exists a monochromatic C_5 in $G[A]$, say red. Let i be the index of an arbitrary vertex in the C_5. We may assume that the C_5 has vertices $a_{i-2}, a_{i-1}, a_i, a_{i+1}, a_{i+2}$ in order. For any $S = \{a_ib_i, a_{i+1}b_{i+1}\}$, since a_ia_{i+1} is red and a_ib_i is blue, b_ia_{i+1} must be colored either red or blue, since G is rainbow triangle free. But

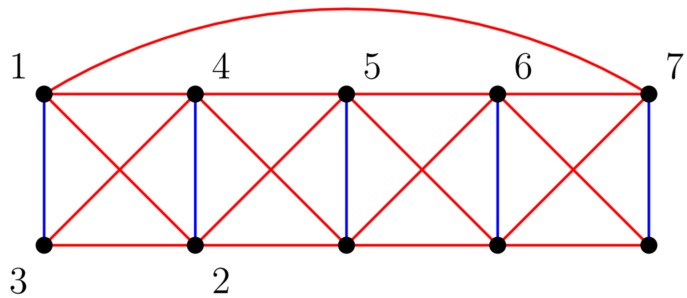

Figure 4.1: Finding a red C_7 from the original existence of a red C_5 (which is represented by the cycle 1,4,5,6,7).

since E is an induced matching, $b_i a_{i+1}$ must be red. By similar arguments, $a_i b_{i+1}$ and $b_i b_{i+1}$ are colored red. We obtain a red C_7 with vertices $a_{i-2}, a_{i-1}, a_i, b_{i+1}, b_i, a_{i+1}, a_{i+2}$ in order, a contradiction. □

Let $X = \{x_1, \ldots, x_m\} \subseteq V(G)$ be a a maximum sequence of vertices such that for all $j \in [m]$, all edges between x_j and $V(G) \setminus \{x_1, \ldots, x_j\}$ are of the same color. Let $c(x_i)$ be the color of all edges between x_i and $V(G) \setminus \{x_1, \ldots, x_i\}$. Then we have the following.

Lemma 4.2. *For any $u \neq v \in X$, $c(u) \neq c(v)$.*

Proof. Suppose otherwise, that $c(x_i) = c(x_j)$, with $i < j$ and $c(x_j)$ being the first repeated color. Then $j \leq k+1$. Take $A = \{x_1, x_2, ..., x_j\}$ and $B = G \setminus A$. Let $c(x_i)$ and $c(x_j)$ be blue.

If there is a blue $P_3 = p_1p_2p_3 \subseteq B$, then if there exists a blue edge $ab \in E(B \setminus P_3)$, there exists a blue C_7 through $x_i ab x_j p_1 p_2 p_3$. Suppose $B \setminus P_3$ has no blue edge; then it is colored by $k-1$ colors. So $|B \setminus P_3| = |G| - |A| - 3 \geq 3 \cdot 2^k + 1 - (k+1) - 3 = 3 \cdot 2^{k-1} + 1 + (3 \cdot 2^{k-1} - (k+1) - 3) \geq 3 \cdot 2^{k-1} + 1$. By induction, $B \setminus P_3$ contains a monochromatic C_7, a contradiction.

The remaining case is when the blue edges in B induce a matching M. By the Matching

Lemma, $|M| \leq 2^k$. Let A be a set of vertices obtained by taking exactly one vertex incident to each edge in M. So $|B \backslash A| \geq 3 \cdot 2^k + 1 - (k+1) - 2^k = 3 \cdot 2^{k-1} + 1 + (3 \cdot 2^{k-1} - (k+1) - 2 \cdot 2^{k-1}) = 3 \cdot 2^{k-1} + 1 + (2^{k-1} - k - 1) \geq 3 \cdot 2^{k-1} + 1$, so by induction there exists a monochromatic C_7 in $B \setminus M$, a contradiction. □

By Lemma 4.2, $|X| \leq k$. Let $H := G \setminus X$. Consider a Gallai partition of H, where blue and red are the colors between parts. Let $|A_1| \leq |A_2| \leq ... \leq |A_s| \leq 2 < |A_{s+1}| \leq ... \leq |A_\ell|$. Since $r_2(C_7) = 13$, it follows that $\ell \leq 12$. Denote $\mathcal{R}(H)$ as the reduced graph of H. Take ℓ to be as small as possible. We will now show the following.

Remark 1. *Assume vertex sets X and Y are disjoint with $3 \leq |X| \leq |Y|$ and all possible edges between X and Y exist and are colored a single color (assume blue).*
i) If $|Y| \geq 4$, then there does not exist a blue edge in Y.
ii) If $|X| \geq 4$, then there does not exist a blue edge in neither X nor Y.

Proof. In both cases, if there was a blue edge ab in Y, a blue P_7 could be found by alternating vertices between X and Y with starting vertex a and ending vertex b, creating a blue C_7. Similarly, if $|X| \geq 4$, then if there was a blue edge ab in X, then we can again construct a blue P_7 with starting vertex a and ending vertex b by alternating between X and Y. □

Lemma 4.3. *If $a_{i_1}, a_{i_2}, a_{i_3}$ forms a monochromatic, say blue, triangle in $\mathcal{R}(H)$, then $i_1 < i_2 < i_3 \leq s$.*

Proof. Suppose $i_3 > s$. If $i_2 > s$, then we can easily find a blue C_7, a contradiction. Thus $i_1 < i_2 \leq s < i_3$. Let $A = A_{i_3}$. Let A_{i_4} be the set of all vertices in $H \setminus (A_{i_1} \cup A_{i_2} \cup A)$ connected to A by blue. If $|A_{i_4}| \geq 2$, then we easily find a blue C_7, so $|A_{i_4}| \leq 1$. Let $B = H \setminus (A_{i_1} \cup A_{i_2} \cup A \cup A_{i_4})$. So A is completely connected to B by red.

Suppose that $|A| = 3$. Then $|B| \geq |G| - 8 - k = 3 \cdot 2^{k-1} + 1 + (3 \cdot 2^{k-1} - 8 - k) \geq 3 \cdot 2^{k-1} + 1 \geq 4$. By Remark 1, there is no red edge in B, so B is colored by $k-1$ colors, and by induction has a monochromatic C_7, a contradiction. So $|A| \geq 4$. By Remark 1 A has no red edge.

If $|B| \leq 3$, then $|A| \geq 3 \cdot 2^k + 1 - 6 - k \geq 3 \cdot 2^{k-1} + 1$, so by induction A contains a monochromatic C_7, a contradiction. So $|B| \geq 4$, and by Remark 1 B has no red edge.

If $|B| \geq 3 \cdot 2^{k-1} + 1$, then by induction it has a monochromatic C_7, a contradiction. So $|B| \leq 3 \cdot 2^{k-1}$, and thus $|A| \geq |G| - |B| - 3 - k = 3 \cdot 2^{k-1} + 1 - 1 - 3 - k = 2^k + 1 + (2^{k-1} - k - 4) \geq 2^k + 1$.

If there exists a blue P_3 in A, then $A \setminus P_3$ cannot contain a blue edge, or we find a blue C_7 through A_{i_1} and A_{i_2}, and thus $A \setminus P_3$ is colored by $k-2$ colors. But since $|A \setminus P_3| \geq 2^k + 1 - 3 = 3 \cdot 2^{k-2} + 1 + (2^{k-2} - 3) \geq 3 \cdot 2^{k-2} + 1$, it contains a monochromatic C_7, a contradiction. So blue induces a matching M in A. By the Matching Lemma, $|M| \leq 2^{k-1}$. Let C be a set of vertices obtained by taking exactly one vertex incident to each edge in M. If $|A \setminus C| \geq 3 \cdot 2^{k-2} + 1$, by induction $A \setminus C$ contains a monochromatic C_7, a contradiction.

Since $G = A_{i_1} \cup A_{i_2} \cup A \cup A_{i_4} \cup B \cup X$, we have $|A \setminus C| + |B| \geq |G| - |C| - |X| - 4 \geq 3 \cdot 2^k + 1 - 2^{k-1} - k - 4 = 3 \cdot 2^{k-1} + 3 \cdot 2^{k-1} + 1 - k - 4 = 3 \cdot 2^{k-1} + 3 \cdot 2^{k-2} + (3 \cdot 2^{k-2} - k - 4) + 1 \geq 3 \cdot 2^{k-1} + 3 \cdot 2^{k-2} + 1$. Thus by the Pigeonhole Principle, either $|A \setminus C| \geq 3 \cdot 2^{k-2} + 1$ or $|B| \geq 3 \cdot 2^{k-1} + 1$, so either $A \setminus C$ or B contains a monochromatic C_7, a contradiction. $\square$

We will complete the proof by considering whether H contains a monochromatic triangle T.

Case 1 (H contains a monochromatic T): Define $Y = A_{s-1} \cup A_s \cup A_{s+1} \cup \cdots \cup A_\ell$. Let $\mathcal{R}(Y)$ be the reduced graph of Y. Since $\mathcal{R}(Y)$ contains no monochromatic triangle, $|\mathcal{R}(Y)| \leq 5$. Since any monochromatic C_5 in $\mathcal{R}(Y)$ would have two connected vertices whose corresponding parts in Y have more than two vertices, we could construct a monochromatic C_7 from any monochromatic C_5. So $|\mathcal{R}(Y)| = \ell - s + 2 \leq 4$, thus $\ell - s \leq 2$. Since $\ell > s$, we have either

$\ell - s = 1$ or $\ell - s = 2$.

Subcase 1 ($\ell - s = 1$): Let $A = A_{s+1}$. Since $\mathcal{R}(H) \leq 12$, $|A| \geq 3 \cdot 2^k + 1 - 22 - k \geq 4$. Let B be the subgraph of the partition that is complete to A by blue, and R be the subgraph of the partition that is complete to A by red. If $|V(B)| \geq 3$ ($|V(R)| \geq 3$), then A contains no blue (red) edge. Suppose at least one of B (R) has at least three vertices, then A does not contain a blue (red) edge. Additionally, there exists no $x \in X$ such that $c(x)$ is blue (red), otherwise we find a monochromatic C_7. So $|A \cup X| \geq 3 \cdot 2^k + 1 - 22 = 3 \cdot 2^{k-1} + 1 + (3 \cdot 2^{k-1} - 22) \geq 3 \cdot 2^{k-1} + 1$, and since $A \cup X$ is colored by $k-1$ colors, it contains a monochromatic C_7, a contradiction. So $|V(B)| \leq 2$ and $|V(R)| \leq 2$. By the maximality of X, at least one of B or R is of order 2. We may assume that $|V(B)| = 2$. If a blue $P_3 \subseteq A$, then $A \setminus P_3$ cannot have a blue edge, so it is colored by $k-1$ colors, and $|A \setminus P_3| \geq 3 \cdot 2^k + 1 - 7 - k \geq 3 \cdot 2^{k-1} + 1$, so by induction $A \setminus P_3$ contains a monochromatic C_7. So blue induces a matching M in A. Let C be a set of vertices obtained by taking exactly one vertex incident to each edge in M. By the Matching Lemma, $|C| \leq 2^k$. So $|A \setminus C| \geq 3 \cdot 2^k + 1 - 4 - 2^k = 3 \cdot 2^{k-1} + 1 + (2^{k-1} - 4) \geq 3 \cdot 2^{k-1} + 1$, so $A \setminus M$ contains a monochromatic C_7 by induction, a contradiction.

Subcase 2 ($\ell - s = 2$): Suppose without loss of generality A_{s+1} and A_{s+2} are connected by blue. Since $|A_{s+2}| \geq 4$, A_{s+2} contains no blue edge. Additionally, there exists no vertex $x \in X$ such that $c(x)$ is blue, otherwise we find a blue C_7. If $|A_{s+1}| = 3$, then we find that $|A_{s+2} \cup X| \geq 3 \cdot 2^k + 1 - 23 \geq 3 \cdot 2^{k-1} + 1 + (3 \cdot 2^{k-1} - 23) \geq 3 \cdot 2^{k-1} + 1$, and since $A_{s+2} \cup X$ is colored by $k-1$ colors, $A_{s+2} \cup X$ contains a blue C_7, a contradiction. So $|A_{s+1}| \geq 4$, and A_{s+1} contains no blue edge. Since A_{s+2} is colored by $k-1$ colors, it follows that $|A_{s+2}| \leq 3 \cdot 2^{k-1}$. Since if any A_i, $i \in [s]$, is connected to both A_{s+1} and A_{s+2} by blue, we find a blue C_7, it follows that all A_i, $i \in [s]$, are connected to at least one of A_{s+1}, A_{s+2} by red. Let B_1 be the subgraph of parts A_i, $i \in [s]$, such that A_i is connected to A_{s+1} by blue and A_{s+2} by red, B_2 be the subgraph of parts A_i, $i \in [s]$, such that A_i is connected to A_{s+2} by blue and A_{s+1} by

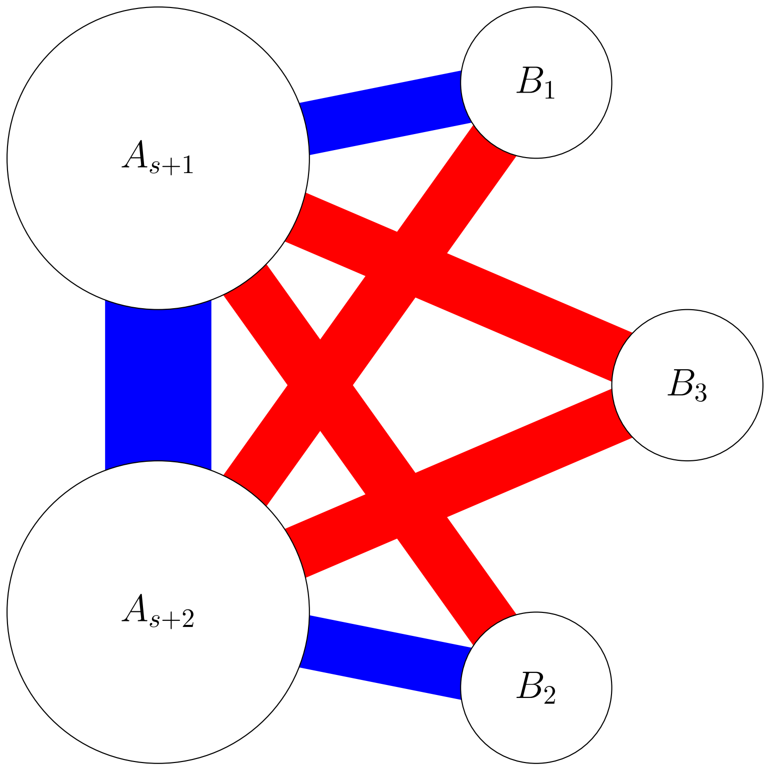

Figure 4.2: An illustration of how the A_i, $i \in [s]$, parts are connected to A_{s+1} and A_{s+2}.

red, and B_3 be the subgraph of parts connected to both A_{s+1} and A_{s+2} by red, as illustrated in Figure 4.2. If $|V(B_3)| \geq 3$, then neither A_{s+1} nor A_{s+2} has a red edge and there exists no $x \in X$ such that $c(x)$ is red, so $|A_{s+1} \cup A_{s+2} \cup X| \geq 3 \cdot 2^k + 1 - 20 \geq 3 \cdot 2^{k-1} + 1$, and since $A_{s+1} \cup A_{s+2} \cup X$ is colored by $k-1$ colors it has a monochromatic C_7, a contradiction. So $|V(B_3)| \leq 2$. If $|V(B_1)| \geq 4$ ($|V(B_2)| \geq 4$), then by connecting to A_{s+1} (A_{s+2}), B_1 (B_2) contains no blue by Remark 1, and by connecting to A_{s+2} (A_{s+1}), B_1 (B_2) contains no red by Remark 1, so vertices in B_1 (B_2) are connected by neither red nor blue, and thus cannot be in different parts of the partition, so B_1 (B_2) consists of a single part of the partition, a contradiction. So both $|V(B_1)| \leq 3$ and $|V(B_2)| \leq 3$. If there is a blue edge in either B_1 or B_2, then we can find a blue C_7, a contradiction, so neither B_1 nor B_2 have blue edges. We will complete this case by considering if T is blue or red. If T is blue, then at most one part can be in B_1 and at most one part can be in B_2. If one part each is in B_1, B_2, and B_3, then we find a blue C_7 through the blue C_5 formed in the reduced graph. So the remaining case is when two parts are in B_3 and one part is in either B_1 or B_2, suppose without loss of generality B_1. If there is either a red edge in A_{s+1} or A_{s+2}, then we can construct a red C_7, so $A_{s+1} \cup A_{s+2}$ is colored by $k-1$ colors, and $|A_{s+1} \cup A_{s+2}| \geq 3 \cdot 2^k + 1 - k - 8 \geq 3 \cdot 2^{k-1} + 1$, so by induction $A_{s+1} \cup A_{s+2}$ contains a monochromatic C_7, a contradiction. So T must be red. If T is contained entirely in B_1 and B_2, then B_3 must be empty, or we could find a red C_7. Additionally, since neither B_1 nor B_2 have a blue edge, by the Pigeonhole Principle it follows that either $|A_{s+1} \cup B_2 \cup X| \geq 3 \cdot 2^{k-1} + 1$, or $|A_{s+2} \cup B_1 \cup X| \geq 3 \cdot 2^{k-1} + 1$, and since both $A_{s+1} \cup B_2 \cup X$ and $A_{s+2} \cup B_1 \cup X$ are colored by $k-1$ colors (no blue), the larger set will contain a monochromatic C_7, a contradiction. So at least one part of T is in B_3. Since $|B_3| \leq 2$, if any part of T is in B_3, then $T \cap \mathcal{R}(B_1) \neq \emptyset$ or $T \cap \mathcal{R}(B_2) \neq \emptyset$, assume without loss of generality $T \cap \mathcal{R}(B_1) \neq \emptyset$. Let b_1 be a part of T contained in B_1 and b_3 be a part of T contained in B_3. Then $b_1 b_3 a_{s+2}$ forms a red triangle in $\mathcal{R}(H)$, a contradiction.

Case 2 (H does not contain a monochromatic T): Because $r_2(K_3) = 6$, it follows that $\ell \leq 5$. Suppose $\ell = 5$, then $\mathcal{R}(H)$ has the unique monochromatic triangle free coloring on 5 vertices. If any two parts of the partition of H contain two or more vertices, then we can find a monochromatic C_7 in the same color that connects the two parts, a contradiction. So all parts but one have exactly one vertex; call the large part A. If there is a blue P_3 in A, then we can find a blue C_7, a contradiction. So blue induces a matching M in A. By the Matching Lemma, $|M| \leq 2^k$. Let B be a set of vertices obtained by taking exactly one vertex incident to each edge in M. So $|A \setminus B| \geq 3 \cdot 2^k + 1 - 4 - k - 2^k = 3 \cdot 2^{k-1} + 1 + (3 \cdot 2^{k-1} - 4 - k - 2 \cdot 2^{k-1}) =$

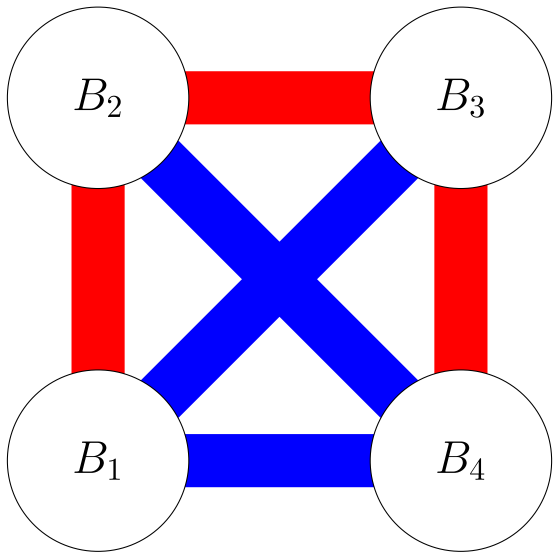

Figure 4.3: When $\ell = 4$, both blue and red induce P_4's in H.

$3 \cdot 2^{k-1} + 1 + (2^{k-1} - 4 - k) \geq 3 \cdot 2^{k-1} + 1$. By induction, $A \setminus B$ contains a monochromatic C_7, a contradiction. So it follows that $\ell \leq 4$.

If the subgraph of $\mathcal{R}(H)$ induced on either red or blue is disconnected, then because we take ℓ to be as small as possible, either $\ell = 2$ or $\ell = 4$ with both red and blue subgraphs of $\mathcal{R}(H)$ being P_4's. We will check these two cases.

Subcase 1 ($\ell = 4$): In this case, $\mathcal{R}(H)$ has blue and red P_4's. Each vertex of $\mathcal{R}(H)$ is either an endpoint of a red (blue) P_4 and an internal vertex of a blue (red) P_4. Let B_i, $i \in [4]$ be a permutation of the A_j's, $j \in [4]$. We may assume that $|B_4| \geq \max\{|B_1|, |B_2|, |B_3|\}$ and $b_1 b_2 b_3 b_4$ is a red P_4 as depicted in Figure 4.3, where $b_i \in B_i$ for all $i \in [4]$. Then $b_3 b_1 b_4 b_2$ is the blue P_4. By the Pigeonhole Principle, $|B_4| \geq \frac{3 \cdot 2^k + 1 - k}{4}$. There are four cases for the sizes

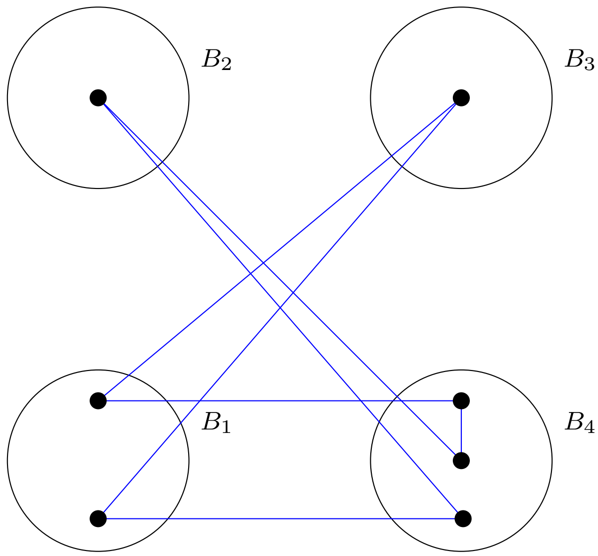

Figure 4.4: Finding a blue C_7 if $|B_1| = 2$.

of the other three parts.

Subcase i ($|B_i| \leq 2$, $i \in [3]$): If B_4 has no blue edges, it is colored by $k-1$ colors, and $|B_4| \geq 3 \cdot 2^k + 1 - 6 - k \geq 3 \cdot 2^{k-1} + 1$, so by induction we find a monochromatic C_7. If B_4 has a blue P_3 as a subgraph, then any blue edge in $B_4 \setminus P_3$ creates a monochromatic C_7, so $B_4 \setminus P_3$ is colored by $k-1$ colors, and we have $|B_4 \setminus P_3| \geq 3 \cdot 2^k + 1 - 9 - k \geq 3 \cdot 2^{k-1} + 1$, so by induction we find a monochromatic C_7. So blue induces a matching M in B_4. If $|B_1| = 2$ or $|B_2| = 2$, then we can find blue C_7's as in Figures 4.4 and 4.5 respectively. If the size of the matching M is at least $2^k + 1$, then by the Matching Lemma we find a monochromatic C_7. Let A be a set of vertices obtained by taking exactly one vertex incident to each edge in M. Then we have $|B_4 \setminus A| \geq 3 \cdot 2^k + 1 - 4 - k - 2^k = 3 \cdot 2^{k-1} + 1 + (2^{k-1} - k - 4) \geq 3 \cdot 2^{k-1} + 1$, and by induction we find a monochromatic C_7.

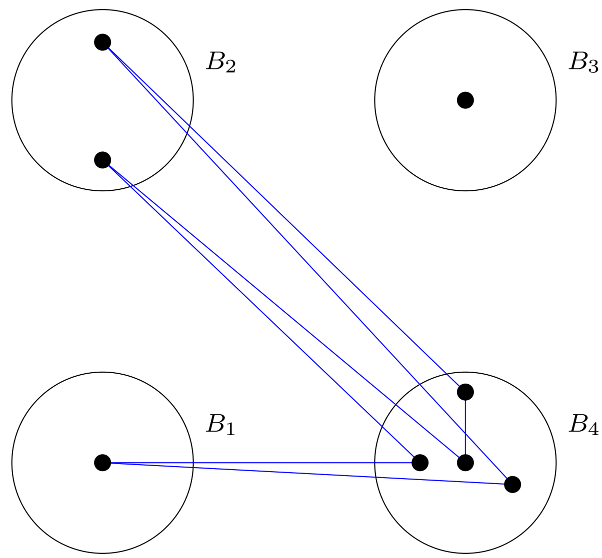

Figure 4.5: Finding a blue C_7 if $|B_2| = 2$.

Subcase ii ($|B_1| \geq 3$): By Remark 1, B_4 has no blue edges. Additionally, if there is a blue edge in B_1, we can find a blue C_7 as in Figure 4.6. Additionally, if either B_2 or B_3 have a blue edge, then we can find a blue C_7 as in Figure 4.7. Also, if $c(x)$ is blue for any $x \in X$, then we can find a blue C_7. So it follows that $B_1 \cup B_2 \cup X$ and $B_3 \cup B_4 \cup X$ have no blue edges, so they are colored by $k-1$ colors, and thus by the pigeonhole principle, $\max\{|B_1 \cup B_2 \cup X|, |B_3 \cup B_4 \cup X|\} \geq \lceil \frac{1}{2}(3 \cdot 2^k + 1 - |X|) \rceil + |X| \geq 3 \cdot 2^{k-1} + 1$, so by induction there exists a monochromatic C_7.

Subcase iii ($|B_1| \leq 2$, $|B_2| \geq 3$): By Remark 1, B_4 contains no blue edges. Additionally, if there is a blue edge in B_1 or B_2, we can find a blue C_7. If B_3 has no blue edges, then we can find a monochromatic C_7 in the same fashion as subcase ii, so B_3 must have a blue edge. So $|B_3| \geq 2$. Thus, neither B_2 nor B_4 contains no red edge, or we can construct a red C_7.

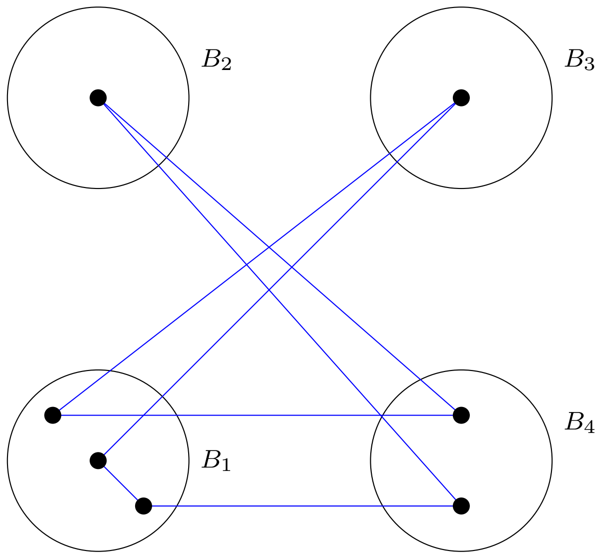

Figure 4.6: Finding a blue C_7 if B_1 has a blue edge.

So $B_2 \cup B_4$ is not colored by red, and $|B_2 \cup B_4| \geq 3 \cdot 2^k + 1 - k - 4 \geq 3 \cdot 2^{k-1} + 1$, and by induction $B_2 \cup B_4$ contains a monochromatic C_7.

Subcase iv ($|B_1| \leq 2$, $|B_2| \leq 2$, $|B_3| \geq 3$): By Remark 1, B_4 contains no red edges. If $|B_3| = 3$, then $|B_4| \geq 3 \cdot 2^k + 1 - 7 - k = 3 \cdot 2^{k-1} + 1 + (3 \cdot 2^{k-1} - 7 - k) \geq 3 \cdot 2^{k-1} + 1$, so by induction B_4 contains a monochromatic C_7. So $|B_3| \geq 4$, and by Remark 1 B_3 has no red edges. If $|B_1| = 2$, then neither B_3 nor B_4 can have a blue edge as in the vertical mirror of Figure 4.7 and Figure 4.6 respectively, so $B_3 \cup B_4$ is colored by $k-1$ colors and $|B_3 \cup B_4| \geq 3 \cdot 2^k + 1 - 4 - k \geq 3 \cdot 2^{k-1} + 1$, so by induction it has a monochromatic C_7. So $|B_1| = 1$, so B_1 contains no red edge. If B_2 contains a red edge, then it follows that a red C_7 can be constructed as in Figure 4.8, a contradiction. So no B_i, $i \in [4]$, contains a red edge. Additionally, there can be no $x \in X$ such that $c(x)$ is red, otherwise we can construct a red

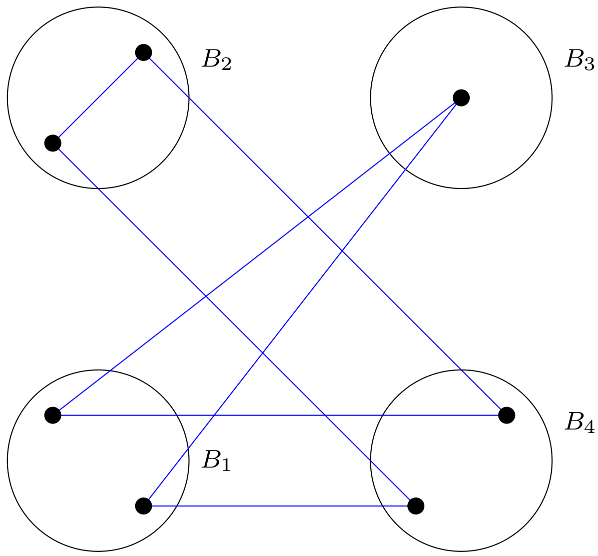

Figure 4.7: Finding a blue C_7 if B_2 has a blue edge. By vertically mirroring the image, we find the same C_7 if B_3 has a blue edge.

C_7. Thus, $B_1 \cup B_3 \cup X$ and $B_2 \cup B_4 \cup X$ contain no red edges, and so by the pigeonhole principle, $\max\{|B_1 \cup B_3 \cup X|, |B_2 \cup B_4 \cup X|\} \geq \lceil \frac{1}{2}(3 \cdot 2^k + 1 - |X|) \rceil + |X| \geq 3 \cdot 2^{k-1} + 1$, so by induction there exists a monochromatic C_7.

Subcase 2 ($\ell = 2$): In this case, we may assume that a_1a_2 is colored blue in $\mathcal{R}(H)$. By the construction of X, $|A_1| \geq 2$. If $|A_1| \geq 3$, then by Remark 1, A_2 contains no blue edge. Additionally, there exists no $x \in X$ such that $c(x)$ is blue, otherwise a blue C_7 could be constructed from a C_6 between A_1 and A_2. Then $|A_2 \cup X| \geq \lceil \frac{1}{2}(3 \cdot 2^k + 1 - |X|) \rceil + |X| \geq 3 \cdot 2^{k-1} + 1$, and $A_2 \cup X$ is colored on $k-1$ colors, so it contains a monochromatic C_7 by induction. So $|A_2| = 2$. If A_1 contains a blue P_3, then $A_1 \setminus P_3$ cannot contain a blue edge, so it is colored by $k-1$ colors. Then $|A_1 \setminus P_3| \geq 3 \cdot 2^k + 1 - k - 5 \geq 3 \cdot 2^{k-1} + 1$, so by induction

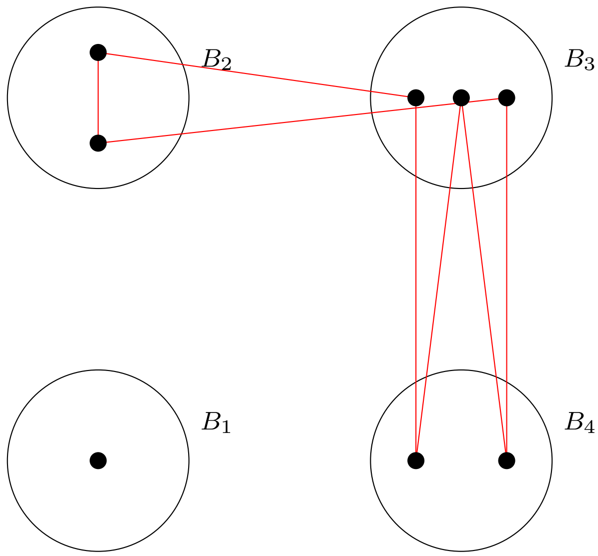

Figure 4.8: Constructing a red C_7 if B_2 contains a red edge.

it contains a monochromatic C_7. So blue induces a matching M in A_1. By the Matching Lemma, $|M| \leq 2^k$. Let B be a set of vertices obtained by taking exactly one vertex incident to each edge in M. Thus $|A_1 \backslash B| \geq 3 \cdot 2^k + 1 - 2^k - 2 - k = 3 \cdot 2^{k-1} + 1 + (2^{k-1} - 2 - k) \geq 3 \cdot 2^{k-1} + 1$, and by induction contains a monochromatic C_7. □

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