The Use of Filters in Topology

By

ABDELLATIF DASSER
B.S. University of Central Florida, 2002

A thesis submitted in partial fulfillment of the requirements for the degree of
Master of Science
in the Department of Mathematics
in the College of Arts and Sciences
at the University of Central Florida
Orlando, Florida

Fall Term
2004

ABSTRACT

Sequences are sufficient to describe topological properties in metric spaces or, more generally, topological spaces having a countable base for the topology. However, filters or nets are needed in more abstract spaces. Nets are more natural extension of sequences but are generally less friendly to work with since quite often two nets have distinct directed sets for domains. Operations involving filters are set theoretic and generally certain to filters on the same set. The concept of a filter was introduced by H. Cartan in 1937 and an excellent treatment of the subject can be found in N. Bourbaki (1940).

ACKNOWLEDGEMENTS

I would like to express my gratitude to my supervisor, Dr. Richardson, for many insightful conversations during the development of the ideas in this thesis, his understanding, endless patience, and encouragement.

I would also like to think my defense committee members, Dr. Mohapatra and Dr. Han for asking me excellent questions .

Finally, but not least, I would like to think Ahmed Ameur for the friendship and comic relief when things started to get difficult.

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CHAPTER 1

INTRODUCTION AND EXAMPLES

The study of filters is a very natural way to describe convergence in general topological space. Filters were introduced in 1937 by Cartan (1937 a,b). Bourbaki (1940) employed filters in order to prove several results in their text . In the same year Tukey (1940) studied sets, filters, and various modifications of the two concepts. A complete reliance on filters for the development of topology can be found in Kowalsky (1961). There are traces of the concept of filters as early as 1914 in Root's article. More recently, filters play a fundamental role in the development of fuzzy spaces which have applications in computer science and engineering. Filters are also an important tool used by researchers describing non-topological convergence notions in functional analysis. (e.g see Beattie and Butzmann,2002). Moreover, Preuss (2002) has applied filters throughout his book on categorical topology.

The purpose of this paper is to provide thorough discussion of filters and their applications. Filters are used in general topology to characterize such important concepts as continuity, initial and final structures, compactness, etc.

The following examples are given to show that sequences are not sufficient to characterize points of closure, continuity, and compactness.

Example 1.1 let X to be an uncountable set and fix $x_0 \in X$.

Define $\tau = \{A \subseteq X : (x_0 \notin A) or (x_0 \in A \, and \, A^c \; is \, countable)\}$. Then τ is a topology for X.

(a) $\phi, X \in \tau$

(b) A, B $\in \tau$ implies that A $\cap$ B $\in \tau$

(c) $A_\alpha \in \tau, \alpha \in J$ implies that $\bigcup_\alpha A_\alpha \in \tau$. The latter holds since if $x_0 \in A_{\alpha_0}$ for some α_0,

then $\left(\bigcup_\alpha A_\alpha\right)^c = \bigcap_\alpha A_\alpha^c \subseteq A_{\alpha_0}^c$ which is countable. Thus, $\bigcup_\alpha A_\alpha \in \tau$.

We claim that $x_n \xrightarrow{\tau} x$ if and only if $x_n = x$ eventually (ie $x_n = x \ \forall \ n \geq N$).

(a) Suppose that $x \neq x_0$. Since $\{x\} \in \tau$, $x_n \xrightarrow{\tau} x$ if and only if $x_n = x$ eventually.

(b) Suppose that $x = x_0$ and $x_n \neq x_0$ for infinitely many n. Define F=$\{x_n : x_n \neq x_0\}$. Then $F^c \in \tau$ and $x_n \in F^c$ eventually fails to hold. Hence $x_n \neq x_0$ infinitely often. Therefore, x_n does not converge to x_0. Conversely, if x_n does not converge to x_0, then there exist $O \in \tau$, $x_0 \in O$ such that $x_n \notin O$ infinitely often. That is, $x_n \neq x_0$ infinitely often. Hence, $x_n \xrightarrow{\tau} x_0$ if and only if $x_n = x_0$ eventually.

A: $x_0 \in \overline{\{x_0\}^c}$ but it does not exist a sequence $\{x_n\}$ in $\{x_0\}^c$ such that $x_n \xrightarrow{\tau} x_0$.

If $x_n \in \{x_0\}^c$, then $x_n \neq x_0$ for all $n \geq 1$ and by above results, x_n does not converge to x_0. Hence, there is no sequence contained in $\{x_0\}^c$ that converges to x_0. However, $x_0 \in \overline{\{x_0\}^c}$ since $O \cap \{x_0\}^c \neq \phi$ for each $O \in \tau$, $x_0 \in O$. And therefore, sequences do not characterize points of closure.

B: Let σ be the discrete topology for X, ie σ is the set of all subsets of X. One can see that $x_n \xrightarrow{\sigma} x_0$ if and only if $x_n = x_0$ eventually. Hence, σ and τ have the same convergent sequences. Let the Id: $(X, \tau) \to (X, \sigma)$ denote the identity function. The function Id is sequentially continuous since τ and σ have the same convergent sequences. However, since $\tau \subset \sigma$, the above function is not continuous. Hence, sequences do not characterize continuity.

Example1.2 Let (X, d) be a metric space that is not compact. And let (X^*, τ^*) be the Stone-Cech compactification of (X, d). Since (X, d) is not sequentially compact, there exist $\{x_n\}$ contained in X which has no convergent subsequence in (X, d). It is known that no sequence contained in X converges to a point in $X^* - X$. Hence, (X^*, τ^*) is not sequentially compact. Therefore, sequences do not characterize compactness.

CHAPTER 2

FILTERS

Definition 2.1 Consider an arbitrary set *X*. A set τ of subsets of *X* satisfying the conditions:

(a) $\phi \in \tau$ and $X \in \tau$

(b) $U \cap V \in \tau$ whenever $u \in \tau$ and $V \in \tau$

(c) The union of the members of an arbitrary subset of τ belongs to τ

is called a *topology on X*. A *topological space* is a pair (X, τ) where τ is a topology on X.

The members of τ are called *open sets.*

Definition 2.2 Consider a set $X \neq \phi$. A *filter* F on *X* is a set of subsets of *X* satisfying the conditions:

(a) $\mathsf{F} \neq \phi$ and $\phi \notin \mathsf{F}$

(b) If $A, B \in \mathsf{F}$ then $A \cap B \in \mathsf{F}$

(c) If $A \in \mathsf{F}$ and $A \subseteq B \subseteq X$ then $B \in \mathsf{F}$

A subset $\beta \subseteq \mathsf{F}$ is called a *base for the filter* F if every member of F contains some member of β. The definition of a filter base for some filter is as follows:

Definition 2.3: β is called a *base* for a filter on X if and only if β is a set of subsets of X satisfying the conditions:

(a) $\beta \neq \phi$, $\phi \notin \beta$

(b) $B_1,\ B_2 \in \beta \Rightarrow \exists B_3 \in \beta$ such that $B_3 \subseteq B_1 \cap B_2$.

Example 2.4 Let $X \neq \phi$ be an arbitrary set. Fix $x_0 \in X$ and then

$\dot{x}_0 = \{A : A \subseteq X \text{ and } x_0 \in A\}$ is a filter on X.

Note that $\{x_0\} \in \dot{x}_0$.

Example 2.5 Fix a set $\phi \not\subset A_0 \subseteq X$, then $\dot{A}_0 = \{B \subseteq X : B \supseteq A_0\}$ is a filter on *X*. In particular, if $A_0 = X$, $\dot{X} = \{X\}$ is the "smallest" possible filter on *X*.

Example 2.6 If *X* is any nonempty set and $\{x_n\}$ is a sequence in *X*. Define $B_n = \{x_k : k \geq n \geq 1\}$. Then, $\mathsf{F} = \{A \subseteq X : A \supseteq B_n\ \exists n \geq 1\}$ is a filter on *X* and is called the *elementary filter* determined by $\{x_n\}$.

Example 2.7 If *X* is an infinite set then $\mathsf{F} = \{F \subseteq X : F^c \textit{ is finite}\}$ *a filter on X* and is called the *cofinite* or *Fréchet filter.*

Example 2.8 If *X* is a topological space and $x \in X$, then the family U(x) of all neighborhoods of *x* is a filter and is called the *neighborhood filter of x.*

Example 2.9 The family of all 'tails' of the sequence $\{x_n\}$ on X is a base for the corresponding elementary filter; a tail is the set of the form $B_N = \{x_n : n \geq N\}$.

Example 2.10 The family $\{\{x\}\}$ is a base for the filter $\dot{x}$ on *X.*

Let *F (X)* denote the set of all filters on a set *X* and $\mathsf{F}, \mathsf{G} \in F(X)$. We call a filter G *finer* than the filter F if $\mathsf{F} \subseteq \mathsf{G}$, we also call F *coarser that* G. Note that $\mathsf{F} = \{X\}$ is the coarsest member in *F (X).* It is easy to verify that $(F(X), \leq)$ is a poset.

Also, $F(X, \leq)$ is not linearly ordered since $\dot{x} \not\subset \dot{y}$ or $\dot{y} \not\subset \dot{x}$. Before discussing filter and convergence, one wants to prove and define various things about filters. *DeMorgan's law* states that if $X \neq \phi$ and $\{ A_\alpha : \alpha \in I \}$ is a collection of subsets of *X*. Then:

(a) $\left(\bigcup_{\alpha \in I} A_i \right)^c = \bigcap_{\alpha \in I} A_\alpha^c$

(b) $\left(\bigcap_{\alpha \in I} A_i \right)^c = \bigcup_{\alpha \in I} A_\alpha^c$

Proposition 2.11 Assume that $X \neq \phi$ and $\mathcal{F}_\alpha \in F(X)$, $\alpha \in I$ (index set) then $\cap \mathcal{F}_\alpha \in F(X)$ is the finest filter on *X* which is coarser than each $\mathcal{F}_\alpha$, $\alpha \in I$.

Proof (a): Note that $\phi \notin \mathcal{F}_\alpha$, $\alpha \in I$, implies that $\phi \notin \bigcap_{\alpha \in I} \mathcal{F}\alpha$ and also X belongs to each $\mathcal{F}_\alpha$, $\alpha \in I$.

Thus X$\in \bigcap_{\partial \in I} \mathcal{F}_\alpha$ and it follows that $\bigcap_{\partial \in I} \mathcal{F}_\alpha \neq \phi$.

(b): Let A, B $\in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$; then A and B belong to each $\mathcal{F}_\alpha$. Therefore, A $\cap$ B belongs to each $\mathcal{F}_\alpha, \alpha \in I$, and thus A $\cap$ B$\in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$.

(c): Let A$\in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$; then A $\in$ each $\mathcal{F}_\alpha$, $\alpha \in I$. Let B $\supseteq$ A thus B belongs to each $\mathcal{F}_\alpha$ since B is an over set of A. It follows that B$\in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. Since (a), (b) and (c) are satisfied, $\bigcap_{\alpha \in I} \mathcal{F}_\alpha$ is a filter. Clearly $\bigcap_{\alpha \in I} \mathcal{F}_\alpha \subseteq \mathcal{F}_\alpha$, for each $\alpha \in$ I.

Next, let $\mathcal{G} \subseteq \mathcal{F}_\alpha$ for each $\alpha \in I$ and let us prove that $\mathcal{G} \subseteq \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. Let A $\in \mathcal{G}$; thus A belongs to each $\mathcal{F}_\alpha$, $\alpha \in I$, and then A $\in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. It follows that $\mathcal{G} \subseteq \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. ▫

In general, the union of two filters may or may not be a filter. For example, if $\mathcal{F}$ and $\mathcal{G}$ may contain disjoint members.

Proposition 2.12 Let $\mathcal{F}_\alpha$, $\alpha \in I$, be filters on *X*. Then

$$\bigcap_{\alpha \in I} \mathcal{F}_\alpha = \{ A \subseteq X : A = \bigcup_{\alpha \in I} F_\alpha \ \textit{for some} \ F_\alpha \in \mathcal{F}_\alpha \}$$

Proof Let B $\in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$; then B belongs to each $\mathcal{F}_\alpha$ and thus B = $\bigcup_{\alpha \in I} F_\alpha$ by choosing each F_α = B . Conversely, Let B $\in \{ A \subseteq X : A = \bigcup_{\alpha \in I} F_\alpha \ \textit{for some} \ F_\alpha \in \mathcal{F}_\alpha \}$, thus B = $\bigcup_{\alpha \in I} F_\alpha$ for some $F_\alpha \in \mathcal{F}_\alpha$ and thus B belongs to each $\mathcal{F}_\alpha$. Hence B $\in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$ ▫

Let $\phi \subseteq X \subseteq Y$. If $\mathcal{F} \in F(X)$, then $\mathcal{F}$ is a *filter base on Y.* That is $\{A \subseteq Y : F \subseteq A \exists F \in \mathcal{F}\}$ is a filter on *Y* generated by $\mathcal{F}$. The generated filter is denoted by [$\mathcal{F}$]. Conversely, if $\mathcal{G}$ is a filter on *Y* and $G \cap X \neq \phi$ for each G $\in \mathcal{G}$, then $\mathcal{F} = \{G \cap X : G \in \mathcal{G}\} \in$ F(X). This filter is called the *induced filter on X,* or the *trace of* $\mathcal{G}$ *on X.*

Example 2.13 Let $X = [0,1], Y = R$ and let $\mathcal{G}$ be the filter on *Y* whose base is $\{(-\varepsilon, \varepsilon) : \varepsilon > 0\}$. Then the trace of $\mathcal{G}$ on *X* is the filter on *X* having a base $\{[0, \varepsilon] : \varepsilon > 0\}$.

CHAPTER 3

ULTRAFILTERS

Definition 3.1 An ultrafilter is a maximal filter in the poset $\left(F\left(X\right),\leq\right)$,where the ordering $\mathcal{F} \leq \mathcal{G}$ means that $\mathcal{F} \subseteq \mathcal{G}$. That is a filter $\mathcal{U}$ on X is an ultrafilter provided $\mathcal{U} \subseteq \mathcal{G}$ implies that $\mathcal{U} = \mathcal{G}$.

Proposition 3.2 (Zorn's lemma) If X is partially ordered set in which every linearly ordered subset (any two elements are comparable) has an upper bound , then X has a maximal element. That is, there exists $x \in X$ such that there is no $y \neq x$ with $x \leq y$.

Proposition 3.3 Let X be a set and $\mathcal{F}$ a filter on X. Then there exists an ultrafilter $\mathcal{U}$ on X that is finer than $\mathcal{F}$. .

Proof: Consider the family P= $\{\mathcal{G} \in F(X) : \mathcal{G}\, \textit{is a filter that finer than } \mathcal{F}\}$. The family P is partially ordered by $\subseteq$. Suppose that $\mathcal{C} = \{\mathcal{G}_\alpha : \alpha \in I\}$ is a chain in P. That is $\mathcal{C}$ is linearly ordered subset of P for each $\mathcal{G} \in P$.

Denote H= $\cup\{\mathcal{G}_\alpha : \alpha \in I\} = \{ A : A \in \mathcal{G}_\alpha\ \exists \alpha \in I \}$

(a) $\mathcal{F} \in P$ by construction, thus $H \neq \phi$

(b) $\phi \notin H$ *since* $\phi \notin \mathcal{G}$, *for each* $\mathcal{G} \in P$

(c) Let A, B $\in H$; *then* $A \in \mathcal{G}_\beta$, $B \in \mathcal{G}_\beta$ for some α, β . Now, either $\mathcal{G}_\beta \subseteq \mathcal{G}_\alpha$ or $\mathcal{G}_\beta \subseteq \mathcal{G}_\alpha$ holds. Then *A, B* $\in \mathcal{G}_\beta$ and thus $A \cap B \in \mathcal{G}_\beta$ since $\mathcal{G}_\beta$ is a filter. Then, $A \cap B \in H$.

(d) If $A \in H$ then $A \in \mathcal{G}_\alpha$ for some $\mathcal{G}_\alpha \in \mathcal{C}$. If $B \supseteq A$ then $B \in \mathcal{G}_\alpha$ and thus B $\in H$.

Therefore, H is a filter and hence an upper bound for $\mathcal{C}$ *in P* The partially ordered set $\mathcal{P}$ satisfies the assumptions of Zorn's lemma; hence there is a maximal element $\mathcal{U} \in P$. Therefore, $\mathcal{U}$ is an ultrafilter containing $\mathcal{F}$. ▫

Proposition 3.4 Let $\mathcal{F}$ be a filter on a set X; then, the following are equivalent:

(a) $\mathcal{F}$ is an ultrafilter.

(b) For any two subsets A and B of X we have: If A U B $\in \mathcal{F}$ then A$\in \mathcal{F}$ or B$\in \mathcal{F}$.

c) For every subset A of X either A$\in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Proof (a) ⇒ (b): Assume $A \; U B \in \mathcal{F}$ and $A \notin \mathcal{F}$ and B$\notin \mathcal{F}$.

Define $\mathcal{G} = \{C \subseteq X : A \cup C \in \mathcal{F}\}$*; then* $\mathcal{G} \in$ F(X) .Further, $\mathcal{F} \subseteq \mathcal{G}$, $B \in \mathcal{G}$ and thus $\mathcal{F} \neq \mathcal{G}$. But $\mathcal{F}$ is an ultrafilter. Thus, there is a contradiction. Therefore, A$\in \mathcal{F}$ or B $\in \mathcal{F}$.

(b) ⇒ (c): Clearly $A \cup A^c = X \in \mathcal{F}$; therefore A $\in \mathcal{F}$ or $A^c \in \mathcal{F}$ using(b).

(c) ⇒ (a): Let $\mathcal{G}$ be a filter that is finer than $\mathcal{F}$ and let A$\in \mathcal{G}$ be arbitrary. Then $A^c \notin \mathcal{F}$ because A has nonempty intersection with every element of $\mathcal{F}$. It follows that A$\in \mathcal{F}$. Thus, $\mathcal{G} = \mathcal{F}$. ▫

Proposition 3.5 Let $\mathcal{F} \in$ F(X). Then:

$$\mathcal{F} = \cap \{ \mathcal{U} : \mathcal{U} \supseteq \mathcal{F}, \mathcal{U} \text{ is an } \textit{ultrafilter} \text{ on X} \}$$

Proof Clearly $\mathcal{F} \subseteq \cap \{ \mathcal{U} : \mathcal{U} \supseteq \mathcal{F}, \mathcal{U} \text{ is an } \textit{ultrafilter} \text{ on X} \}$. Assume that there exists an A$\in \cap \{ \mathcal{U} : \mathcal{U} \supseteq \mathcal{F}, \mathcal{U} \text{ is an } \textit{ultrafilter} \text{ on X} \}$ such that A$\notin \mathcal{F}$. Then $F \cap A^c \neq \phi$ for each F$\in \mathcal{F}$ and thus $\beta = \{F \cap A^c : F \in \mathcal{F}\}$ is a base for a filter $\mathcal{G}$. Note that $\mathcal{F} \subseteq \mathcal{G}$.

Let $\mathcal{U}_G$ be an ultrafilter containing $\mathcal{G}$. Since, $\mathcal{F} \subseteq \mathcal{U}_G$, $A \in \mathcal{U}_G$. However, $A^c \in \mathcal{G}$ $\subseteq \mathcal{U}_G$ and thus $A^c \cap A \in \mathcal{U}_G$, a contradiction. Therefore,

$\mathcal{F} = \cap \{ \mathcal{U} : \mathcal{U} \supseteq \mathcal{F}, \mathcal{U}$ is an *ultrafilter* on X $\}$.

Proposition 3.6 Let $\mathcal{U}$ be an ultrafilter on the set *X*. If $A_1 A_2 ..., A_n$ are subsets of X such that $\bigcup_{i=1}^{n} A_i$ belongs to $\mathcal{U}$, then at least one of the sets A_i belongs to $\mathcal{U}$.

Proof If no A_i belongs to $\mathcal{U}$, then A_i^c belongs to $\mathcal{U}$ for $i = 1,2,....,n$ by Proposition 2.3, and hence $\cap A_i^c = (\cup A_i)^c$ belongs to $\mathcal{U}$, which is impossible since $\bigcup_{i=1}^{n} A_i$ is given to belong to $\mathcal{U}$..

Proposition 3.7 let $\mathcal{U}$ be an ultrafilter on *X* and $A \subseteq X$ such that

$U \cap A \neq \phi$ for all $U \in \mathcal{U}$. Then $A \in \mathcal{U}$.

Proof Assume that $A \notin \mathcal{U}$ and define $\beta = \{ U \cap A : U \in \mathcal{U} \}$. Note that

$\phi \notin \beta$ and $\beta \neq \phi$ since $U \cap A \neq \phi$ for all $U \in \mathcal{U}$. Also,

$(U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A \in \beta$ since $(U_1 \cap U_2) \in \mathcal{U}$. Thus β is a base filter for some filter $\mathcal{G}$. Note that $A \in \mathcal{G}$ because $A \supseteq U \cap A$ and $A \in \mathcal{U}$.

Therefore, $\mathcal{U} \subset \mathcal{G}$, a contradiction and hence $A \in \mathcal{U}$.

Proposition 3.8 Let $f : X \to Y$ be a function and $\mathcal{U}$ is an ultrafilter on *X*. Then $f(\mathcal{U})$ is an ultrafilter on *Y*.

Proof Assume that $A \notin f(\mathcal{U})$; then for each $U \in \mathcal{U}$, $A^c \cap f(U) \neq \phi$.

Hence, $f^{-1}\left(A^c\right) \cap U \neq \phi$ for each $U \in \mathcal{U}$ and thus by Proposition 2.6, $f^{-1}\left(A^c\right) \in \mathcal{U}$.

Therefore, $f\left(f^{-1}\left(A^c\right)\right) \in f(\mathcal{U})$ and thus $A^c \in f(\mathcal{U})$ since $f(f^{-1}(A^c)) \subseteq A^c$. Hence, $f(\mathcal{U})$ is an ultrafilter on *Y.* ▫

CHAPTER 4

CONVERGENCE AND FILTERS

Definition 4.1 Let (X, τ) be a topological space and let *U (x)* denote the neighborhood filter at *x*. A filter F *on X converges* to *x* if U (x) $\subseteq$ F.

Proposition 4.2 Let A be a subset of a topological space *X*. Then, for $x \in X$,

$x \in \bar{A}$ if and only if there exists a filter on X which contains A and converges to *x*.

Proof Assume that $x \in \bar{A}$. Then any neighborhood of x has a nonempty intersection with *A*. Now all the sets $A \cap U$, where *U* is a neighborhood of *x,* form a filter base, and the corresponding filter converges to *x*. Conversely, assume that F is a filter containing *A* and converging to *x* . Choose any neighborhood *U* of *x*. Then U $\in$ F , and thus

$U \cap A \neq \phi$ since A $\in$ F. This proves that $x \in \bar{A}$.

Definition 4.3 A topological space (X, τ) is Hausdorff or T_2 provided if $x \neq y$, then there exist sets $O_x, O_y \in \tau$ such that $x \in O_x, y \in O_y$ and $O_x \cap O_y = \phi$.

Proposition 4.4 (X, τ) is T_2 if and only if each filter converges to at most one point, i.e.

F $\xrightarrow{\tau} x, y$ implies *x=y.*

Proof Suppose that (X, τ) is Hausdorff and suppose F $\xrightarrow{\tau} x, y$ where $x \neq y$.

Then there exist $O_x, O_y \in \tau$ such that $x \in O_x, y \in O_y$ and $O_x \cap O_y = \phi$.However,

F $\xrightarrow{\tau} x, y$ implies that $O_x \in$ F, $O_y \in$ F, and

$O_x \cap O_y = \phi \in$ F which is a contradiction. Therefore, each filter converges to at most one point. Conversely, suppose that $x \neq y$ and assume that $O_x \cap O_y \neq \phi$ for each

$O_x, O_y \in \tau$, $x \in O_x, y \in O_y$

We claim that $\beta = \{O_x \cap O_y : \forall O_x \in \tau, O_y \in \tau, x \in O_x, y \in O_y\}$ is a base for some filter F.Observe that $(O_x \cap O_y) \cap (G_x \cap G_y) = (O_x \cap G_x) \cap (O_y \cap G_y) \in \beta$ since $(O_x \cap G_x)$ is an open set containing *x* and $(O_y \cap G_y)$ is an open set containing *y*. Thus β is a base for some filter F since $O_x \supseteq (O_x \cap O_y)$ implies that each $O_x \in$ F,

F converges to x. Likewise, $O_y \supseteq (O_x \cap O_y)$ implies that $O_y \in$ F, and thus F converges to y, a contradiction. Therefore, there doesn't exist an O_x and O_y such that $O_x \cap O_y \neq \phi$ where $x \in O_x$ and $y \in O_y$. Hence (X, τ) is Hausdorff.

The following result shows that continuity of maps between topological spaces can be characterized in terms of convergence of filters. Recall that a mapping

g: X$\rightarrow$Y between two topological spaces *is continuous at x* provided that for each neighborhood V of g (x), there exist a neighborhood U of x such that

g(U) $\subseteq$ V.

Proposition 4.5 Let *X, Y* be topological spaces with x$\in X$ and

g: $X \rightarrow Y$. Then g is continuous at *x* if and only if whenever F is a filter such that F $\rightarrow$

x, g(F) $\rightarrow$*g(x)*.

Proof Suppose g is continuous at *x* and F $\rightarrow$ *x*. Let *V* be a neighborhood of g*(x)*. By continuity there is a neighborhood *U* of *x* such that g*(U)* $\subseteq$ *V*. *S*ince *U*$\in$ F,

$g(U) \in g(\mathcal{F})$. And thus $V \in g(\mathcal{F})$. Hence $g(\mathcal{F}) \to g(x)$. Conversely, suppose that whenever $\mathcal{F} \to x$, $g(\mathcal{F}) \to g(x)$. Then $g(U(x)) \to g(x)$ by hypothesis. Then, for each neighborhood V of $g(x)$, $V \in g(U(x))$. Then there exists a $U \in$ U(x) such that $g(U) \subseteq V$ and thus g is continuous at x.

CHAPTER 5

COMPACTNESS AND FILTERS

Recall that a topological space (X,τ) is compact provided each open covering of X has a finite subcovering . It is shown below that compactness can be characterized in terms of convergence of ultrafilters.

Proposition 5.1 Let (X,τ) be a topological space. Then the following statements are equivalent:

(a) (X,τ) is compact

(b) Each ultrafilter on X converges

(c) $\cap\{\overline{A} : A \in \mathcal{F}\} \neq \phi$ for each filter on X

Proof (a)$\Rightarrow$ (b): Suppose (X,τ) is compact and that there exist an ultrafilter $\mathcal{F}$ that doesn't converge. Then for each $x \in X$, there exists $V_x \notin \mathcal{F}$ V and thus $C = \{V_x : x \in X\}$ is an open cover of X. Hence $\bigcup_{i=1}^{n} V_{x_i}$. =X. Since $\mathcal{F}$ is an ultrafilter, implies that $V_x^c \in \mathcal{F}$ for each $x \in X$ and therefore $X^c = \bigcap_{i=1}^{n} V_{x_i}^c = \phi \in \mathcal{F}$, which is a contradiction. Thus, each ultrafilter on X converges.

(b)$\Rightarrow$(c): Given any filter $\mathcal{F}$ on X; let $\mathcal{G}$ be an ultrafilter containing $\mathcal{F}$. Then $\mathcal{G}$ converges to x in (X,τ), for some $x \in X$. Given any neighborhood V of x and $A \in \mathcal{F}$; then $A \in \mathcal{G}$ and V$\in \mathcal{G}$. Hence, $A \cap V \in \mathcal{G}$ and thus $A \cap V \neq \phi$. Therefore, $x \in \overline{A}$.

(c)$\Rightarrow$(a): Suppose (X,τ) is not compact. Let C $=\{O_\alpha : \alpha \in J\}$ be an open cover of X with no finite subcover. Then, $\bigcup_{i=1}^{n} O_{\alpha_i} \neq X$, for each n. Let F be the filter on X whose base $\left\{\bigcap_{i=1}^{n} O_{\alpha_i}^{c} : n \geq 1, O_\alpha \in C\right\}$ .However, $\cap\{\overline{A} : A \in F\} \subseteq \cap O_\alpha = (\bigcup_{\alpha\in J} O_\alpha)^c = X^c = \phi$, which is a contradiction.

Proposition 5.2 Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a continuous function and onto. If (X,τ) is compact then (Y,σ) is compact.

Proof Let U be an ultrafilter on Y, F = $f^{-1}(U)$ and by Proposition2.2 there exist an ultrafilter on X. G$\supseteq$F. Then: $G \xrightarrow{\tau} x$, for some $x \in X$ since (X,τ) is compact.

The continuity assumption of f implies that $f(G) \rightarrow f(x)$ according to Proposition 4.6.

Since G$\supseteq$F then $f(G) \supseteq f(F) = f(f^{-1}(U)) \supseteq U$. Also f is onto and thus

$f(f^{-1}(U)) = U$. Hence, $f(G) \supseteq U$ and since U is an ultrafilter $f(G) = U$.

Consequently, $f(G) \rightarrow f(x)$ implies that $U \rightarrow f(x)$. Therefore, by Proposition 4.1, (Y,σ) is compact.

CHAPTER 6

INITIAL STRUCTURES

Proposition 6.1 Consider the source $X \xrightarrow{f_\alpha} (Y_\alpha, \tau_\alpha)$, $\alpha \in J$ where J, is an index class.

Then:

(a) There exists a coarsest (smallest) topology τ_I on X for which each $f_\alpha : (X, \tau_I) \to (Y_\alpha, \tau_\alpha)$ is continuous, $\alpha \in J$.

(b) Each g: $(Y, \sigma) \to (X, \tau_I)$ is continuous if and only if $f_\alpha o\, g : (Y, \sigma) \to (Y_\alpha, \tau_\alpha)$ is continuous, for each $\alpha \in J$.

(c) τ_I is the unique topology for X which obeys (b)

(d) Given $\mathsf{F} \in F(X)$, $\mathsf{F} \xrightarrow{\tau_I} x_i$ if and only if $f_\alpha(\mathsf{F}) \xrightarrow{\tau_I} f(x_\alpha)$

for each $\alpha \in J$.

Proof A subbase for τ_I is $\mathsf{S} = \{f_\alpha^{-1}(O_\alpha) : O_{\alpha \in \tau_\alpha}\}$.

(a) It easily follows that τ_I is the coarsest such topology such that each f_α is continuous.

(b) The composition of two continuous functions in continuous. Conversely, assume that $f\, o\, g$ is continuous, for each $\alpha \in$ J. Let $f_\alpha^{-1}(O_\alpha) \in \mathsf{S}$ then, $g^{-1}(f_\alpha^{-1}(O_\alpha)) = (f_\alpha o\, g)^{-1}(O_\alpha) \in \sigma$ and thus, $g^{-1}\left(f_\alpha^{-1}(O_\alpha)\right) \in \sigma$

Since continuity of g: $(Y, \sigma) \to (X, \tau_I)$ is determined by the subbase S for τ_I, it follows that g: $(Y, \sigma) \to (X, \tau_I)$ is continuous.

(c) Let τ_X be another topology for X obeying (b). Since id: $(X,\tau_I) \to (X,\tau_X)$ is continuous, $f_\alpha : (X,\tau_X) \to (Y_\alpha,\tau_\alpha)$ is continuous, for each $\alpha \in J$. Hence by (a), $\tau_I \subseteq \tau_X$.

Moreover, consider id: $(X,\tau_X) \to (X,\tau_I)$.

Since $f_\alpha = f_\alpha o\ id : (X,\tau_I) \to (Y_\alpha,\tau_\alpha)$ is continuous for each $\alpha \in J$, the hypothesis implies that id: $(X,\tau_I) \to (X,\tau_X)$ is continuous. Hence, $\tau_X \subseteq \tau_I$ and thus $\tau_X = \tau_I$.

(d) Since each $f_\alpha : (X,\tau_I) \to (Y_\alpha,\tau_\alpha)$ is continuous, $\mathsf{F} \xrightarrow{\tau_I} x$ implies that . $f_\alpha(\mathcal{F}) \xrightarrow{\tau_\alpha} f_\alpha(x)$. Conversely, assume that $f_\alpha(\mathcal{F}) \xrightarrow{\tau_\alpha} f_\alpha(x)$ for each $\alpha \in J$. Now, $f_\alpha^{-1}(O_\alpha) \in \mathcal{S}$ and if $x \in f_\alpha^{-1}(O_\alpha)$, $f_\alpha(x) \in O_\alpha \in f_\alpha(\mathcal{F})$ and thus $f_\alpha(F_\alpha) \subseteq O_\alpha$ for some $F_\alpha \in \mathcal{F}$. Hence $f_\alpha^{-1}(O_\alpha) \supseteq F_\alpha$ and thus $f_\alpha^{-1}(O_\alpha) \in \mathcal{F}$. Since $\bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(O_{\alpha_i})$ is a base member for τ_I, it follows that base members containing x belong to $\mathcal{F}$. Hence $\mathcal{F} \xrightarrow{\tau_I} x$.

In particular, when $X = \Pi X_\alpha$ is a product set and $f_\alpha = P_\alpha$ are projection maps , τ_I is called the *product topology* for *X. According to Proposition 5.1* τ_I is the coarsest topology for X such that each projection map is continuous.

CONCLUSION

A discussion of filters and their applications to the theory of general topology has been presented. These ideas have been primarily developed by European mathematicians, beginning with the work of H. Cartan and recorded in the various texts written under the pseudoname “ N. Bourbaki.” The Moore- Smith “ net convergence” has been more widely taught in American universities. Both are used to characterize topological concepts in more abstract spaces.

LIST OF REFERENCES

Beattie, R. and Butzmann, H.-P. , Convergence structure and applications to Functional analysis, klwwer acod. Pub. , Dordrecht, 2002.

Bourbaki, N.(1940), Topologie generale. Chapitre 1 et 2. Actualites Sci. Ind., 858

Cartan, H.(1937a), Theorie des filters, compt. Rend., 205, 595-598.

Cartan, H.(1937b), Filtres et ultrafiltres, Compt. Rend., 205, 777-779.

Kowalsky, H. J.(1961), Topologische Raume, Basel-Stuttgart (Topological Spaces, Academic Press, New York, 1964).

Preuess, Gerhard, Foundations of Topology, kluwer acod. Pub. , Dordrecht, 2002.

Root, R.(1914), Iterated limits in general analysis, Am. J. Math., 36, 79-134.