Frames in Hilbert C^*-modules

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Abstract

Since the discovery in the early 1950's, frames have emerged as an important tool in signal processing, image processing, data compression and sampling theory etc. Today, powerful tools from operator theory and Banach space theory are being introduced to the study of frames producing deep results in frame theory.

In recent years, many mathematicians generalized the frame theory from Hilbert spaces to Hilbert C^*-modules and got significant results which enrich the theory of frames. Also there is growing evidence that Hilbert C^*-modules theory and the theory of wavelets and frames are tightly related to each other in many aspects. Both research fields can benefit from achievements of the other field. Our purpose of this dissertation is to work on several basic problems on frames for Hilbert C^*-modules.

We first give a very useful characterization of modular frames which is easy to be applied. Using this characterization we investigate the modular frames from the operator theory point of view. A condition under which the removal of element from a frame in Hilbert C^*-modules leaves a frame or a non-frame set is also given. In contrast to the Hilbert space situation, Riesz bases of Hilbert C^*-modules may possess infinitely many alternative duals due to the existence of zero-divisors and not every dual of a Riesz basis is again a Riesz basis. We will present several such examples showing that the duals of Riesz bases in Hilbert C^*-modules are much different and more complicated than the Hilbert space cases. A complete characterization of all the dual sequences for a Riesz basis, and a necessary and sufficient condition for a dual sequence of a Riesz basis to be a Riesz basis are also given. In the case that the underlying C^*-algebra is a commutative W^*-algebra, we prove that the set of the Parseval frame generators for a unitary group can be parameterized by the set of all the unitary operators in the double commutant of the unitary group. Similar result holds for the set of all the general frame generators where the unitary operators are replaced by invertible and adjointable operators. Consequently, the set of all the Parseval frame generators is path-connected. We also prove the existence and uniqueness of the best Parseval multi-frame approximations for multi-frame generators of unitary groups on Hilbert C^*-modules when the underlying C^*-algebra is commutative. For the dilation results of frames we show that a complete Parseval frame vector for a unitary group on Hilbert C^*-module can be dilated to a complete wandering vector. For any dual frame pair in Hilbert C^*-modules, we prove that the pair are orthogonal compressions of a Riesz basis and its canonical dual basis for some larger Hilbert C^*-module. For the perturbation of frames and Riesz bases in Hilbert C^*-modules we prove that the Casazza-Christensen general perturbation theorem for frames in Hilbert spaces remains valid in Hilbert C^*-modules. In the Hilbert space setting, under the same perturbation condition, the perturbation of any Riesz basis remains a Riesz basis. However, this no longer holds for Riesz bases in Hilbert C^*-modules. We also give a complete characterization on all the Riesz bases for Hilbert C^*-modules such that the perturbation (under Casazza-Christensen's perturbation condition) of a Riesz basis still remains a Riesz basis.

To the memory of my mother

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CHAPTER 1

INTRODUCTION

In the study of vector spaces one of the most important concepts is that of a basis, allowing each element in the space to be written as a linear combination of the elements in the basis. However, the conditions to a basis are very restrictive: linear independence between the elements. This makes it hard or even impossible to find bases satisfying extra conditions, and this is the reason that one might look for a more flexible substitute.

Frames are such tools. A frame for a vector space equipped with an inner product also allows each element in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required.

Frames for Hilbert space were formally defined by Duffin and Schaeffer ([22]) in 1952. They used frames as a tool in the study of nonharmonic Fourier series, i.e., sequence of the type $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$, where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a family of real or complex numbers. Apparently, the idea of Duffin and Schaeffer did not seem to generate much interest outside of nonharmonic Fourier series, and the importance of the concept was not realized by the mathematical community; at least it took 30 years before the next treatment appeared in print. In 1980 Young wrote his book [56], which contains the basic facts about frames. Frames were presented in the abstract setting, and again used in the context of nonharmonic Fourier series. Then, in 1985, as the wavelet era began, Daubechies, Grossmann and Mayer ([20]) observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are very similar to the expansions using orthonormal bases. This was probably the time when many mathematicians started to see the potential of the topic. Since then, the theory of frames has been more widely studied.

Frames have been used in signal processing, image processing, data compression and sampling theory. Today, ever more use are being found for the theory such as optics, signal detection, as well as the study of Besov spaces in Banach space theory etc. In the other direction, powerful tools from operator theory and Banach space theory are being introduced to the study of frames producing deep results in frame theory. At this very moment, the theory is beginning to grow rapidly with the host of new people entering the area.

One of the nice things about frame theory is the fact that big portions are still underdeveloped. Also, many of the extensively developed areas, such as Weyl-Heisenberg frames and exponential frames, still have many fundamental open questions to challenge anyone, such as the complete classification of Weyl-Heisenberg frames or the classification of exponential frame. Another interesting feature of the area is the broad spectrum of people working in different parts of it including biologists, engineers, mathematicians, etc. Although each group has it own interests, there is an opportunity here to interact with a broad spectrum of researchers.

Recent research also shows that frame theory has strong connections with some famous results in other aspects of mathematics, for example, the Kadison-Singer Conjecture in C^*-algebra and Naimark Dilation Theorem in operator-valued measure theory. In frame theory the Feichtinger Conjecture states that every bounded frame can be written as a finite union of Riesz basic sequences. Much work has been done on this conjecture in just the last few years. This is because the conjecture is not just interesting and important for frame theory but also is connected to the famous Kadison-Singer Conjecture [41], which is known to be equivalent to the paving conjecture. Recall that the Kadison-Singer Conjecture, which is still open, states that whether every pure state on $\mathbb{D}$, the C^*-algebra of the diagonal operators on l^2, admits a unique extension to a (pure) state on $B(l^2)$, the C^*-algebra of all bounded linear operators on l^2. In [12], it was shown that the Kadison-Singer Conjecture implies the Feichtinger Conjecture. It is unknown whether these two problems are equivalent, but the result in [12] indicates that they are certainly very close. In particular, it is proved in [12] that the Feichtinger Conjecture is equivalent to the conjectured generalization of the Bourgain-Tzafriri Restricted-Invertibility Theorem.

In recent years, many mathematicians generalized the frame theory in Hilbert spaces to frame theory in Hilbert C^*-modules and got significant results which enrich the theory of frames. Also there is growing evidence that Hilbert C^*-modules theory and the theory of wavelets and frames are tightly related to each other in many aspects. Both research fields can benefit from achievements of the other field.

Beside Kasparov's Stabilization Theorem the inner structure of self-dual Hilbert W^*-modules as described by Paschke in [49] has been another source of inspiration for frames of Hilbert C^*-modules. Rephrasing his description in the context of frames it reads as the proof of the general existence of orthogonal normalized tight frames $\{x_j\}_{j\in\mathbb{J}}$ for self-dual Hilbert W^*-modules, where additionally the values $\{\langle x_j, x_j\rangle : j \in \mathbb{J}\}$ are projections. This point of view was already realized by Denizeau and Havet ([21]) in 1994. They went one step further by taking a topologically weak reconstruction formula for normalized tight frames as a cornerstone to characterize the concept of "quasi-bases" for Hilbert W^*-modules. These "quasi-bases" are a special example of module frames in Hilbert C^*-modules. The special frames appearing from Paschke's result are called "orthogonal bases" by these authors. The two concepts were investigated by them to the extent of tenser product properties of quasi-bases for C^*-correspondences of W^*-algebras (cf. [21]).

Frank and Larson ([23]) defined the standard frames in Hilbert C^*-modules in 1998 and got a series of result for standard frames in finitely or countably generated Hilbert C^*-modules over unital C^*-algebras. Note that the frames exist in abundance in finitely or countably generated Hilbert C^*-modules over unital C^*-algebra A as well as in the C^*-algebras itself (see [26]). This fact allows us to rely on standard decompositions for elements of Hilbert C^*-modules despite the general absence of orthogonal and orthonormal Riesz bases in them.

Meanwhile, the case of Hilbert C^*-modules over non-unital C^*-algebras has been investigated by Raeburn and Thompson ([50]), as well as by Bakić and Guljaš ([4]) discovering standard frames even for this class of countably generated Hilbert C^*-modules in a well-defined larger multiplier module.

However, many problems about frames in Hilbert C^*-modules still have to be solved. For example, the well-known open problem: Does every Hilbert C^*-module admit a modular frame? These problems are attracting more and more people to enter this field.

The areas of applications indicate a large potential of problems for the investigation of which the results of frames in Hilbert C^*-modules could be applied. From the point of view of applied frame theory, the advantage of the generalized setting of Hilbert C^*-modules may consist in the additional degree of freedom coming from the C^*-algebra A of coefficients and its special inner structure, together with the handling of the basic features of the generalized theory in almost the same manner as for Hilbert spaces.

The aim of this manuscript is to continue the study of frames in Hilbert C^*-modules. The considerations follow the line of the geometrical and operator-theoretical approach worked by Han and Larson ([37]) in the main. However, proofs that generalize from the Hilbert space cases, when attainable, are usually considerably more difficult for the module case for reasons that do not occur in the simpler Hilbert space cases.

Let's describe the chapters in more details. Chapter 2 contains the basic results of frames in Hilbert spaces and the basic properties of Hilbert C^*-modules.

In Chapter 3 we introduce the concept of frames in Hilbert C^*-modules. The basic properties of modular frames are given in Section 3.1. Note that from the definition of modular frames, it is clear that we need to compare positive elements in the underlying C^*-algebra in order to test whether a sequence is a frame or not. This usually is not a trivial task. An equivalent definition was established in Section 3.2., which is much easier to be applied. Another advantage of this equivalent definition is that it allows us to characterize modular frames from the operator theory point of view which is the goal of Section 3.3. It is very interesting that if we remove an element from a basis, then we must get a set which is not a basis. But for frame this is not the case. Due to the redundancy of frame if we remove an element from a frame we may get a new frame. In Section 3.4 we will give a characterization of the removal of an element from a modular frame.

The aim of Chapter 4 is to characterize the modular Riesz bases and their duals. We first give a characterization of Riesz bases in Hilbert C^*-modules in Section 4.1. It is well-known that in Hilbert spaces every Riesz basis has a unique dual which is also a Riesz basis. But in Hilbert C^*-modules, due to the zero-divisors, not all Riesz bases have unique duals and not every dual is a Riesz basis. We will present several such examples showing that the duals of Riesz bases in Hilbert C^*-modules are much different and more complicated than the Hilbert space cases. For example, a dual sequence of a Riesz basis can even not be a Bessel sequence, and a dual Bessel sequence of a Riesz basis may not be a Riesz basis. Several examples are provided in Section 4.2 to show the complexity of duals of modular Riesz bases. We also characterize all the dual sequences for a Riesz basis. And a necessary and sufficient condition for a dual sequence of a Riesz basis to be a Riesz basis is given in Section 4.2.

The main purpose of Chapter 5 is to initiate the study of structured frames for Hilbert C^*-modules. In Hilbert space frame theory, structured frames are the ones that have attracted the most attentions. Typical examples include wavelet frames, Gabor frames and frames induced by group representations. These frames are the ones that have been the main focuses in the research of frame theory. In [19], Dai and Larson introduced one class of structure frames: frames associated with a system of unitary operators. The systematic study of this kind of structured frames can be found in the two memoirs papers [19, 37]. In this chapter, we will focus our attention on the frames induced by a group of unitary operators. More precisely, we work on two closely related issues: frame vector parameterizations and Parseval frame approximations.

In [19] the set of all wandering vectors for a unitary system was parameterized by the set of unitary operators in the so-called local commutant of the system at a particular fixed wandering vector. However, unlike the wandering vector case, it was shown in [37] that the set of all the Parseval frame vectors for a unitary group can not be parameterized by the set of all the unitary operators in the commutant of the unitary group. This means that the Parseval frame vectors for a representation of a countable group are not necessarily unitarily equivalent. However, this set can be parameterized by the set of all the unitary operators in the von Neumann algebra generated by the representation ([32, 37]). This turns out to be a very useful result in Gabor analysis (cf. [32, 29]). Although it remains a question whether this result is still valid in the Hilbert C^*-module setting, in Section 5.1 we will prove that this result holds in Hilbert C^*-modules when the underlying C^*-algebra is a commutative W^*-algebra.

In the Hilbert space frame setting, the original work on symmetric orthogonalization was done by Löwidin ([47]) in the late 1970's. The concept of symmetric approximation of frames by Parseval frame was introduced in [27] to extend the symmetric orthogonalization of bases by orthogonal bases in Hilbert spaces. The existence and the uniqueness results for the symmetric approximation of frames by Parseval frames were obtained in [27]. Following their definition, a Parseval frame $\{y_j\}_{j=1}^{\infty}$ is said to be a *symmetric approximation* of frame $\{x_j\}_{j=1}^{\infty}$ in Hilbert space H if it is similar to $\{x_j\}_{j=1}^{\infty}$ and

$$\sum_{j=1}^{n} \|z_j - x_j\|^2 \geq \sum_{j=1}^{n} \|y_j - x_j\|^2 \tag{1.1}$$

is valid for all Parseval frames $\{z_j\}_{j=1}^{\infty}$ of H that are similar to $\{x_j\}_{j=1}^{\infty}$.

Note that in some situations the symmetric approximation fails to work when the underlying Hilbert space is infinite dimensional since if we restrict ourselves to the frames induced by a unitary system then the summation in (1.1) is always infinite when the given frame is not Parseval. Instead of using the symmetric approximations to consider the frames generated by a collection of unitary transformations and some window functions, it was proposed to approximate the frame generator by Parseval frame generators. The existence and uniqueness results for such a best approximation were obtained in [32, 33]. In Section 5.2 we will prove that this result still holds for Hilbert C^*-module frames when the underlying C^*-algebra is commutative. It remains open whether this is true when the underlying C^*-algebra is non-commutative.

In Chapter 6 we will investigate the dilation of modular frames. It is well-known that every frame in Hilbert space is a direct summand of Riesz basis, in other words, each frame is a compression of a Riesz basis of a larger space. In [26] it was shown that this is still true for the modular frames. In particular, it was prove in ([26]) that each Parseval frame of Hilbert C^*-modules can be dilated to an orthonormal basis. It is natural to ask whether a complete Parseval frame vector for a unitary group on Hilbert C^*-module can be dilated to a complete wandering vector. We will answer this question affirmatively in Section 6.1. More generally, a dual frame pair in Hilbert space can be dilated to a Riesz basis and its dual Riesz basis (see [13]). In Section 6.2 we will see that this remains true for Hilbert C^*-module frames. We want to mention here that the proof of this result for Hilbert space frames used in [13] can not be directly applied to Hilbert C^*-module frames since the adjointablity of operators is always an issue in dealing with operators in Hilbert C^*-modules. In Section 6.3 we will discuss the projective frames for future study.

Let $\{f_j\}_{j=1}^{\infty}$ be a basis of a Banach space X, and $\{g_j\}_{j=1}^{\infty}$ a sequence of vectors in X. If there exists a constant $\lambda \in [0, 1)$ such that

$$\| \sum c_j(f_j - g_j)\| \leq \lambda \| \sum c_j f_j \|$$

for all finite sequence $\{c_j\}$ of scalars, then $\{g_j\}_{j=1}^{\infty}$ is also a basis for X. This result is the well-known classical Paley-Wiener Theorem on perturbation of bases ([48]).

In the last decade, many attentions have been paid to generalize the Paley-Wiener perturbation result to the perturbation of frames in Hilbert spaces (see [5], [11], [15] and [16]). The most general result was obtained by Casazza and Christensen ([11]):

Theorem 1.1. *Let* $\{x_j\}_{j\in\mathbb{J}}$ *be a frame for a Hilbert space* H *with frame bounds* C *and* D*. Suppose that* $\{y_j\}_{j\in\mathbb{J}}$ *is a sequence of* H *and there exist* $\lambda_1, \lambda_2, \mu \geq 0$ *such that* $\max\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\} < 1$*. If one of the following conditions is fulfilled for any finite scalar sequence* $\{c_j\}$ *and all* $x \in H$*, then* $\{y_j\}_{j\in\mathbb{J}}$ *is also a frame for* H*:*

(1) $(\sum_{j\in\mathbb{J}} |\langle x, x_j - y_j\rangle|^2)^{\frac{1}{2}} \leq \lambda_1(\sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2)^{\frac{1}{2}} + \lambda_2(\sum_{j\in\mathbb{J}} |\langle x, y_j\rangle|^2)^{\frac{1}{2}} + \mu\|x\|$*;*

(2) $\|\sum_{j=1}^{n} c_j(x_j - y_j)\| \leq \lambda_1\|\sum_{j=1}^{n} c_j x_j\| + \lambda_2\|\sum_{j=1}^{n} c_j y_j\| + \mu(\sum_{j=1}^{n} |c|^2)^{\frac{1}{2}}$*.*

Moreover, if $\{x_j\}_{j\in\mathbb{J}}$ *is a Riesz basis for* H *and* $\{y_j\}_{j\in\mathbb{J}}$ *satisfies* (2)*, then* $\{y_j\}_{j\in\mathbb{J}}$ *is also Riesz basis for* H*.*

The purpose of Chapter 7 is to investigate whether the above perturbation result remains valid for Hilbert C^*-modular frames. We prove that the Casazza-Christensen general perturbation theorem for frames in Hilbert spaces remains valid in Hilbert C^*-modules. In the Hilbert space setting, under the same perturbation condition, the perturbation of any Riesz basis remains a Riesz basis. However, this no longer holds for Riesz bases in Hilbert C^*-modules. We give a complete characterization on all the Riesz bases for Hilbert C^*-modules such that the perturbation (under Casazza-Christensen's perturbation condition) of a Riesz basis still remains a Riesz basis.

CHAPTER 2

PRELIMINARIES

2.1 Frames in Hilbert Spaces

2.1.1 Frames in Hilbert Spaces

The main feature of a basis $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space H is that every $f \in H$ can be represented as an (infinite) linear combination of the elements f_k in the basis:

$$f = \sum_{k=1}^{\infty} c_k(f) f_k. \tag{2.1}$$

The coefficients $c_k(f)$ are unique. We now introduce the concept of frames. A frame is also a sequence of elements $\{f_k\}_{k=1}^{\infty}$ in H, which allows every $f \in H$ to be written as in (2.1). However, the corresponding coefficients are not necessarily unique. Thus a frame might not be a basis.

We now give the definition of frames.

Definition 2.1. A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in Hilbert space H is a *frame* for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{2.2}$$

The numbers A, B are called *frame bounds*. They are not unique.

If $A = B$, then $\{f_k\}_{k=1}^{\infty}$ is called a *tight frame*, and a *Parseval frame* if $A = B = 1$.

Particularly, if the right inequality

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H,$$

holds true, we call $\{f_k\}_{k=1}^{\infty}$ a *Bessel sequence.*

We now give a few more examples of frames. They might appear quite constructed, but are useful for the theoretical understanding of frames.

Example 2.2. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for a Hilbert space H.

(1) By repeating each element of $\{e_k\}_{k=1}^{\infty}$ twice we have

$$\{f_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, \dots\}$$

which is a tight frame with frame bound $A = B = 2$.

If only e_1 is repeated we get

$$\{f_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_3, \dots\}$$

which is a frame with bounds $A = 1$ and $B = 2$.

(2) Let

$$\{f_k\}_{k=1}^{\infty} = \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \dots\}.$$

Note that for each $f \in H$, we have

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 = \sum_{k=1}^{\infty} k|\langle f, \frac{1}{\sqrt{k}}e_k \rangle|^2 = \|f\|^2.$$

So $\{f_k\}_{k=1}^{\infty}$ is a Parseval frame.

Example 2.3. Let H and K be Hilbert spaces with $H \subset K$, and let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for K. Let P denote the orthogonal projection from K onto H, and let $x_j = Pe_j$ for all j. If $x \in H$ is arbitrary, then

$$\|x\|^2 = \sum_j |\langle x, e_j \rangle|^2 \tag{2.3}$$

and

$$x = \sum_j \langle x, e_j \rangle e_j. \tag{2.4}$$

Since $x = Px$ and $x_j = Pe_j$ we have $\langle x, e_j \rangle = \langle x, x_j \rangle$, so (2.3) becomes

$$\|x\|^2 = \sum_j |\langle x, x_j \rangle|^2$$

and hence $\{x_j\}_{j=1}^{\infty}$ is a Parseval frame for H. Moreover, applying P to (2.4) then yields

$$x = \sum_j \langle x, x_j \rangle x_j \tag{2.5}$$

for all $x \in H$. The formula (2.5) is called the *reconstruction formula* for $\{x_j\}$.

Let $\{f_k\}_{k=1}^{\infty}$ be a frame of Hilbert space H, then we have the corresponding pre-frame operator, analysis operator and frame operator as follows.

The operator $T : H \to l^2$ defined by

$$Tf = \{\langle f, f_k \rangle\}_{k=1}^{\infty},$$

is called the *analysis operator*. The adjoint operator $T^* : l^2 \to H$ is given by

$$T^* \{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k.$$

T^* is called *pre-frame operator* or the *synthesis operator*. By composing T and T^*, we obtain the *frame operator* $S : H \to H$:

$$Sf = T^*Tf = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k.$$

Let state some of the important properties of S:

Proposition 2.4. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S and frame bounds A, B. Then the following holds:*

(1) S is bounded, invertible, self-adjoint, and positive.

(2) $\{S^{-1} f_k\}_{k=1}^{\infty}$ is a frame with bounds B^{-1}, A^{-1}. The frame operator for $\{S^{-1} f_k\}_{k=1}^{\infty}$ is S^{-1}.

The most important frame result is the following reconstruction formula. It shows that if $\{f_k\}_{k=1}^{\infty}$ is a frame for Hilbert space H, then every element in H has a representation as an infinite linear combination of the frame elements. Thus it is natural to view a frame as some kind of "generalized basis".

Theorem 2.5. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S for Hilbert space H. Then*

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k, \quad \forall f \in H. \tag{2.6}$$

It should be mentioned here that to every frame we can associate a *canonical Parseval frame*:

Proposition 2.6. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for Hilbert space H with frame operator S. Denote the positive square root of S^{-1} by $S^{-\frac{1}{2}}$. Then $\{S^{-\frac{1}{2}} f_k\}_{k=1}^{\infty}$ is a Parseval frame, and*

$$f = \sum_{k=1}^{\infty} \langle f, S^{-\frac{1}{2}} f_k \rangle S^{-\frac{1}{2}} f_k, \quad \forall f \in H.$$

We now introduce the definition of dual frames.

Definition 2.7. Let $\{f_k\}_{k=1}^{\infty}$ be a frame of a Hilbert space H. We call a sequence $\{g_k\}_{k=1}^{\infty} \subseteq H$ a *dual frame* of $\{f_k\}_{k=1}^{\infty}$ if

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k$$

holds true for every $f \in H$.

In particular, $\{S^{-1} f_k\}_{k=1}^{\infty}$ is called the *canonical dual* (or. *standard dual*) of $\{f_k\}_{k=1}^{\infty}$, where S is the frame operator of $\{f_k\}_{k=1}^{\infty}$.

Note that the roles of a frame and its duals can be interchanged in the following sense.

Proposition 2.8. *Assume that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences in Hilbert space H. Then the following are equivalent:*

(1) $f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \quad \forall f \in H.$

(2) $f = \sum_{k=1}^{\infty} \langle f, f_k \rangle g_k, \quad \forall f \in H.$

(3) $\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, f_k \rangle \langle g_k, f \rangle, \quad \forall f, g \in H.$

In case the equivalent conditions are satisfied, $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are dual frames for* H*.*

We now list some characterizations of frames in Hilbert spaces from the operator theory point of view.

Theorem 2.9. *A sequence* $\{f_k\}_{k=1}^{\infty}$ *in Hilbert space* H *is a frame for* H *if and only if*

$$T : \{c_k\}_{k=1}^{\infty} \to \sum_{k=1}^{\infty} c_k f_k$$

is a well-defined mapping of l^2 *onto* H*.*

Note that the question of existence of an upper and lower frame bound, via Theorem 2.9, is replaced by an investigation of infinite series: we need to check that $\sum_{k=1}^{\infty} c_k f_k$ converges for all $\{c_k\}_{k=1}^{\infty} \in l^2$ and that each $f \in H$ can be represented via such an infinite series. The following characterization of frames involves the information about the frame bounds.

Theorem 2.10. *A sequence* $\{f_k\}_{k=1}^{\infty}$ *in Hilbert space* H *is a frame for* H *with bounds* A, B *if and only if the following conditions are satisfied:*

(1) $\overline{span\{f_k\}_{k=1}^{\infty}} = H;$

(2) The pre-frame operator T *is well defined on* l^2 *and*

$$A \sum_{k=1}^{\infty} |c_k|^2 \leq \|T\{c_k\}_{k=1}^{\infty}\|^2 \leq B \sum_{k=1}^{\infty} |c_k|^2, \quad \forall \{c_k\}_{k=1}^{\infty} \in (KerT)^{\perp}.$$

2.1.2 Riesz Bases in Hilbert Spaces

There are many equivalent definitions for Riesz bases in Hilbert spaces. Here we adopt the following:

Definition 2.11. A *Riesz basis* for a Hilbert space H is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H and $U : H \to H$ is a bounded bijective operator.

A Riesz basis is actually a basis. In fact, one can characterize Riesz bases in terms of bases satisfying extra conditions:

Proposition 2.12. *A sequence $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis for Hilbert space H if it is an unconditional basis for H and*

$$0 < \inf_k \|f_k\| \leq \sup_k \|f_k\| < \infty.$$

The dual basis associated to a Riesz basis is also a Riesz basis and is unique:

Proposition 2.13. *If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis for Hilbert space H, there exists a unique sequence $\{g_k\}_{k=1}^{\infty}$ in H satisfying*

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \quad \forall f \in H.$$

$\{g_k\}_{k=1}^{\infty}$ is also a Riesz basis, and $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are biorthogonal.

Note that a Riesz basis is a frame:

Proposition 2.14. *If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis for Hilbert space H, then there exist constants A, B such that*

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H.$$

For a frame to be a Riesz basis, we have the following proposition:

Proposition 2.15. *Let $\{f_j\}_{j=1}^{\infty}$ be a frame for Hilbert space H. Then the following are equivalent:*

(1) $\{f_j\}_{j=1}^{\infty}$ is a Riesz basis for H.

(2) $\{f_j\}_{j=1}^{\infty}$ is an exact frame, i.e. it ceases to be a frame when an arbitrary element is removed.

(3) $\{f_j\}_{j=1}^{\infty}$ *is minimal, i.e.* $f_j \notin \overline{span\{j_k : k \neq j\}}$ *for any* j.

(4) $\{f_j\}_{j=1}^{\infty}$ *and* $\{S^{-1}f_j\}_{j=1}^{\infty}$ *are biorthogonal, where* S *is the frame operator of* $\{f_j\}_{j=1}^{\infty}$.

(5) If $\sum_{j=1}^{\infty} c_j f_j = 0$ *for some* $\{c_j\}_{j=1}^{\infty} \in l^2$, *then* $c_j = 0$ *for all* j.

(6) $\{f_j\}_{j=1}^{\infty}$ *is a basis.*

We now list a characterization of Riesz bases for Hilbert spaces.

Proposition 2.16. *For a sequence* $\{f_k\}_{k=1}^{\infty}$ *in Hilbert space* H, *the following conditions are equivalent:*

(1) $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis for* H.

(2) $\overline{span\{f_k\}_{k=1}^{\infty}} = H$, *and there exist constants* $A, B > 0$ *such that for any finite sequence* $\{c_k\}$ *one has*

$$A \sum |c_k|^2 \leq \| \sum c_k f_k \|^2 \leq B \sum |c_k|^2.$$

Let's summarize the relations between orthonormal bases, Riesz bases and frames in Hilbert spaces as follows.

Theorem 2.17. *Let* $\{e_k\}_{k=1}^{\infty}$ *be an arbitrary orthonormal basis for Hilbert space* H. *Then*

(1) The orthonormal bases of H *are the families* $\{Ue_k\}_{k=1}^{\infty}$, *where linear operator* $U : H \to H$ *is unitary.*

(2) The Riesz bases of H *are the families* $\{Ue_k\}_{k=1}^{\infty}$, *where linear operator* $U : H \to H$ *is bounded and bijective.*

(3) The frames of H *are the families* $\{Ue_k\}_{k=1}^{\infty}$, *where linear operator* $U : H \to H$ *is bounded and surjective.*

2.1.3 Dilation Results of Frames

It turns out that Example 2.3 is a generic and serves as a model for arbitrary Parseval frames. One can always dilate such a frame to an orthonormal basis. One immediate consequence of the dilation is that the reconstruction formula (2.5) always holds for a Parseval frame. We have the following dilation result.

Proposition 2.18. *([37]) Let $\mathbb{J}$ be a countable (or finite) index set. Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a Parseval frame for Hilbert space H. Then there exist a Hilbert space $K \supseteq H$ and an orthonormal basis $\{e_j\}_{j\in\mathbb{J}}$ for K such that $x_j = Pe_j$, where P is the orthogonal projection from K onto H.*

Also we can summarize the dilation results in the following way.

Proposition 2.19. *Frames are precisely the inner direct summands of Riesz bases. Parseval frames are precisely the inner direct summands of orthonormal bases.*

Example 2.20. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for a 3-dimensional Hilbert space K. Another orthonormal basis for K is then

$$\{\frac{1}{\sqrt{3}}(e_1+e_2+e_3), \frac{1}{\sqrt{6}}(e_1-2e_2+e_3), \frac{1}{\sqrt{2}}(e_1-e_3)\}.$$

Note that

$$\{\frac{1}{\sqrt{3}}(e_1+e_2), \frac{1}{\sqrt{6}}(e_1-2e_2), \frac{1}{\sqrt{2}}e_1\}$$

is a Parseval frame for $H = span\{e_1, e_2\}$.

Suppose that $\{u_j\}$ is a Riesz basis for a Hilbert space $K \supset H$ with its unique dual $\{u_j^*\}$. If P is the orthogonal projection from K onto H then $\{Pu_j\}$ is a frame for H with an alternate dual $\{Pu_j^*\}$. In general $\{Pu_j^*\}$ is not the canonical dual for $\{Pu_j\}$ unless P commutes with the frame operator of $\{u_j\}$ (see [37]). So it is natural to ask whether a given frame $\{x_j\}$ and one of its alternate duals $\{y_j\}$ can be dilated to a Riesz basis $\{u_j\}$ for some larger Hilbert space K so that $x_j = Pu_j$ and $y_j = Pu_j^*$. It was affirmatively answered in [13] as follows.

Proposition 2.21. *Suppose that $\{x_j\}$ and $\{y_j\}$ are alternate dual frames in a Hilbert space H. Then there is a Hilbert space $K \supset H$ and a Riesz basis $\{u_j\}$ for K with $Pu_j = x_j$ and $Pu_j^* = y_j$, where $\{u_j^*\}$ is the (unique) dual of $\{u_j\}$ and P is the orthogonal projection from K onto H.*

2.1.4 Structured Frames in Hilbert Spaces

In applications the most important and practical frames are the ones that are generated by a single vector in a Hilbert space under the action of a suitable collection of unitary operators. Wavelet frames and Gabor frames are typical examples. A *unitary system* $\mathcal{U}$ is a countable set of unitary operators acting on a separable Hilbert space H that contains the identity operator. We say that a vector $\phi \in H$ is a *complete frame vector* (resp. *complete Parseval frame vector*) for $\mathcal{U}$ if $\mathcal{U}\phi := \{U\phi : U \in \mathcal{U}\}$ is a frame (resp. Parseval frame) for H. When $\mathcal{U}\phi$ is an orthonormal basis for H, ϕ is called a *complete wandering vector* for $\mathcal{U}$, the set of all complete wandering vectors for $\mathcal{U}$ is denoted by $\mathcal{W}(\mathcal{U})$.

If $\mathcal{U}$ is a unitary system and $\phi \in \mathcal{W}(\mathcal{U})$, the *local commutant* $C_\phi(\mathcal{U})$ at ϕ is defined by $\{T \in B(H) : (TU - UT)\phi = 0, U \in \mathcal{U}\}$. Clearly $C_\phi(\mathcal{U})$ contains the commutant $\mathcal{U}'$ of $\mathcal{U}$. When $\mathcal{U}$ is a unitary group, it is actually the commutant of $\mathcal{U}$.

For the characterization of frame vectors for unitary systems we have the following result.

Proposition 2.22. *([19], [32]) Suppose that ϕ is a complete wandering vector for a unitary system $\mathcal{U}$. Then*

(1) a vector $\xi \in H$ is a complete wandering vector for $\mathcal{U}$ if and only if there is a (unique) unitary operator $A \in \mathcal{U}'$ such that $\xi = A\phi$.

(2) a vector $\eta \in H$ is a complete Parseval frame vector for $\mathcal{U}$ if and only if there is a (unique) co-isometry $A \in C_\phi(\mathcal{U})$ such that $\eta = A\phi$.

Recall that a unitary system $\mathcal{U}$ is said to be *group-like* if

$$group(\mathcal{U}) \subset \mathbb{T}\mathcal{U} := \{tU : t \in \mathbb{T}, U \in \mathcal{U}\}$$

and if different U and V in $\mathcal{U}$ are always linearly independent, where $group(\mathcal{U})$ denotes the group generated by $\mathcal{U}$ with respect to multiplication and $\mathbb{T}$ denotes the unit circle.

We have the following parameterization of frame vectors at a fixed complete Parseval frame vector for group-like unitary systems.

Proposition 2.23. *([32]) Let η be a complete Parseval frame vector for a group-like unitary system $\mathcal{U}$ and $w^*(\mathcal{U})$ be the von Neumann algebra generated by $\mathcal{U}$. Suppose that $\xi \in H$. Then*

(1) ξ is a complete Parseval frame vector for $\mathcal{U}$ if and only if there exists a unitary operator $A \in w^*(\mathcal{U})$ such that $A\eta = \xi$.

(2) ξ is a complete frame vector for $\mathcal{U}$ if and only if there exists an invertible operator $A \in w^*(\mathcal{U})$ such that $A\eta = \xi$.

Another kind of important structured frames is multi-generated frames which are generated by some (usually finite number of) vectors under the action of a collection of unitary operators. For example, Gabor frames and wavelet frames are of this kind. Recall that $\Phi = (\phi_1, \ldots, \phi_N)$ with $\phi_j \in H$ is called a *multi-frame generator* (resp. *Parseval multi-frame generator*) for a unitary system $\mathcal{U}$ if $\{U\phi_j : U \in \mathcal{U}, 1 \leq j \leq N\}$ is a frame (resp. *Parseval frame*) for H.

We now define the best Parseval multi-frame approximation for a multi-frame generator.

Definition 2.24. Let $\Phi = (\phi_1, \ldots, \phi_N)$ be a multi-frame generator for a unitary system $\mathcal{U}$. Then a Parseval multi-frame generator $\Psi = (\psi_1, \ldots, \psi_N)$ for $\mathcal{U}$ is called a *best Parseval multi-frame approximation* for Φ if the inequality

$$\sum_{j=1}^{N} \|\phi_j - \psi_j\|^2 \leq \sum_{j=1}^{N} \|\phi_j - \xi_j\|^2$$

is valid for all the Parseval multi-frame generator $\Xi = (\xi_1, \ldots, \xi_N)$ for $\mathcal{U}$.

The following result was proved in [33].

Proposition 2.25. *Let $\mathcal{U}$ be a group-like unitary system acting on a Hilbert space H and let $\Phi = (\phi_1, \ldots, \phi_N)$ be a multi-frame generator for $\mathcal{U}$. Then $S^{-1/2}\Phi$ is the unique best Parseval multi-frame approximation for Φ, where S is the frame operator for the multi-frame $\{U\phi_j : U \in \mathcal{U}, j = 1, \ldots, N\}$.*

2.2 Hilbert C^*-modules

The aim of this section is to cover the basic results of Hilbert C^*-modules. To introduce the concept of Hilbert C^*-modules, we first introduce the definition of C^*-algebras.

2.2.1 C^*-algebras and W^*-algebras

Let's begin with

Definition 2.26. Let $\mathcal{A}$ be an associative algebra over the complex numbers. The algebra $\mathcal{A}$ is called a *normed algebra* if there is associated to each element x a real number $\|x\|$, called the *norm* of x, with the properties:

(1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|x + y\| \leq \|x\| + \|y\|$.

(3) $\|\lambda x\| = |\lambda|\|x\|$, λ is a complex number.

(4) $\|xy\| \leq \|x\|\|y\|$

If $\mathcal{A}$ is complete with respect to the norm (i.e. if $\mathcal{A}$ is also a Banach space), then it is called a *Banach algebra*.

Note that any closed subalgebra of a Banach algebra is a Banach algebra.

Example 2.27. (1) Let X be a Banach space, denote by $B(X)$ the set of all bounded linear operators on X. Then $B(X)$ is a Banach algebra with the pointwise-defined operations for addition and scalar multiplication, multiplication given by $(AB)(x) = A(B(x))$, and the operator norm.

(2) The algebra $M_n(\mathbb{C})$ of $n \times n$-matrices with entries in $\mathbb{C}$ is identified with $B(\mathbb{C}^n)$. It is therefore a Banach algebra.

Definition 2.28. Let $\mathcal{A}$ be a unital Banach algebra and a an element of $\mathcal{A}$. Then a is *invertible* if there is an element $b \in \mathcal{A}$ such that

$$ab = ba = 1.$$

The set

$$Inv(\mathcal{A}) = \{a \in \mathcal{A} : a \text{ is invertible}\}$$

is a group under multiplication.

We define the *spectrum* of an element a to be the set

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin Inv(\mathcal{A})\}.$$

Example 2.29. (1) Let $\mathcal{A} = C(X)$, where X is a compact Hausdorff space. Then $\sigma(f) = f(X)$ for all $f \in \mathcal{A}$.

(2) Let $\mathcal{A} = M_n(\mathbb{C})$, the algebra of all complex $n \times n$-matrices. Then for each $A \in \mathcal{A}$, $\sigma(A)$ is the set of eigenvalues of A.

Definition 2.30. Let $\mathcal{A}$ be a Banach algebra. A mapping $x \mapsto x^*$ of $\mathcal{A}$ into itself is called an involution if for all $x, y \in \mathcal{A}$ and any scalar $\lambda \in \mathbb{C}$ the following conditions are satisfied:

(1) $(x^*)^* = x$.

(2) $(x + y)^* = x^* + y^*$.

(3) $(xy)^* = y^* x^*$.

(4) $(\lambda x)^* = \overline{\lambda} x^*$.

Then $\mathcal{A}$ is called a *Banach $*$-algebra*.

Definition 2.31. A Banach $*$-algebra $\mathcal{A}$ is called a C^**-algebra* if it satisfies

$$\|x^*x\| = \|x\|^2 \tag{2.7}$$

for all $x \in \mathcal{A}$.

Here are a few examples.

Example 2.32. (1) $B(H)$, the algebra of bounded linear operator on a Hilbert space H, is a C^*-algebra, where for each operator A, A^* is the adjoint of A.

(2) Any closed $*$-subalgebra of $B(H)$ is a C^*-algebra.

(3) $C(X)$, the algebra of continuous functions on a compact space X, is an abelian C^*-algebra, where $f^*(x) \equiv \overline{f(x)}$.

(4) $C_0(X)$, the algebra of continuous functions on a locally compact space X that vanish at infinity, is an abelian C^*-algebra, where $f^*(x) \equiv \overline{f(x)}$.

Definition 2.33. Let $\mathcal{A}$ be a unital C^*-algebra and $a, p, u \in \mathcal{A}$. Then

(1) a is *normal* if $a^*a = aa^*$.

(2) p is a *projection* if $p = p^* = p^2$.

(3) u is a *unitary* if $u^*u = uu^* = 1$.

(4) u is an *isometry* if $u^*u = 1$.

(5) u is a *co-isometry* if $uu^* = 1$.

For the spectrum of normal elements in C^*-algebras we have the following famous Spectral Mapping Theorem.

Theorem 2.34. *Let a be a normal element of a unital C^*-algebra $\mathcal{A}$, and $f \in C(\sigma(a))$. Then*

$$\sigma(f(a)) = f(\sigma(a)).$$

Definition 2.35. An element a of a C^*-algebra $\mathcal{A}$ is *positive* if $a^* = a$ and $\sigma(a) \subseteq \mathbb{R}^+$. We write $a \geq 0$ to mean that a is positive.

We have the following characterizations on positive elements.

Proposition 2.36. *Let $\mathcal{A}$ be a C^*-algebra. The following statements are equivalent:*

(1) a is positive.

(2) $a = b^2$ for some positive element $b \in \mathcal{A}$.

(3) $a = x^*x$ for some $x \in \mathcal{A}$.

(4) $a^* = a$ and $\|t - a\| \leq t$ for all $t \geq \|a\|$.

(5) $a^* = a$ and $\|t - a\| \leq t$ for some $t \geq \|a\|$.

We summarize some elementary factors about positive elements in the following proposition.

Proposition 2.37. *Let $\mathcal{A}$ be a C^*-algebra.*

(1) If $0 \leq a$ and $0 \leq b$, then $0 \leq a + b$.

(2) If $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.

(3) If $\mathcal{A}$ is unital and a, b are positive invertible elements in $\mathcal{A}$, then $a \leq b$ implies $0 \leq b^{-1} \leq a^{-1}$.

(4) If a, b are positive elements in $\mathcal{A}$, then $a \leq b$ implies that $a^t \leq b^t$ for any $t \in (0, 1)$.

(5) If $a^2 \leq b^2$ for all $a, b \in \mathcal{A}$ with $0 \leq a \leq b$, the $\mathcal{A}$ must be commutative.

(6) If $a, b \in \mathcal{A}$, then $abb^*a^* \leq \|b\|^2 aa^*$ and $abb^*a^* \leq \|a\|^2 bb^*$.

We now give the definition of W^*-algebras as follows.

Definition 2.38. A C^*-algebra $\mathcal{M}$ is called a *W^*-algebra* if it is a dual space as a Banach space, i.e., if there exists a Banach space $\mathcal{M}_*$ such that $(\mathcal{M}_*)^* = \mathcal{M}$, where $(\mathcal{M}_*)^*$ is the dual Banach space of $\mathcal{M}_*$.

Definition 2.39. Let p, q be two projections of a W^*-algebra $\mathcal{M}$. If there exists a partial isometry u in $\mathcal{M}$ such that $u^*u = p$ and $uu^* = q$, then p is said to be *equivalent* to q and denote this by $p \sim q$. If there exists a projection $q_1(\leq q)$ equivalent to p, write this by $p \prec q$ or $q \succ p$.

For the equivalence of projections we have the famous Comparability Theorem.

Theorem 2.40. *Let p and q be projections of a W^*-algebra $\mathcal{M}$. Then there exists a central projection $z \in \mathcal{M}$ such that*

$$pz \succ qz \quad \text{and} \quad pz' \prec qz'$$

where $z' = 1 - z$.

Definition 2.41. Let p be a projection of a W^*-algebra $\mathcal{M}$. p is said to be *finite* if for a projection p_1 in $\mathcal{M}$, $p_1 \leq p$ and $p_1 \sim p$ imply $p_1 = p$.

Definition 2.42. A W^*-algebra is said to be *finite* if its identity is finite.

Note that a W^*-algebra $\mathcal{M}$ is finite if and only if every isometry in $\mathcal{M}$ is unitary.

For the equivalence of projections in finite W^*-algebras we have the following result.

Proposition 2.43. *let $\mathcal{M}$ be a finite W^*-algebra, and let p, p_1, q and q_1 be projections in $\mathcal{M}$ satisfying the following conditions:*

$$p_1 \leq p, \quad q_1 \leq q, \quad p_1 \sim q_1 \text{ and } p \sim q.$$

Then $p - p_1 \sim q - q_1$.

Particularly we have

Corollary 2.44. *Let $\mathcal{M}$ be a finite W^*-algebra, and p, q be two equivalent projections in $\mathcal{M}$. Then $1 - p$ and $1 - q$ are equivalent.*

2.2.2 Hilbert C^*-modules

Hilbert C^*-modules form a category in between Banach spaces and Hilbert spaces. The basic idea was to consider module over C^*-algebra instead of linear space and to allow the inner product to take values in a more general C^*-algebra than $\mathbb{C}$. The structure was first used by Kaplansky [42] in 1952 and more carefully investigated by Rieffel [51] and Paschke [49] later in 1972/73.

We give only a brief introduction to the theory of Hilbert C^*-modules to make our explanations self-contained. For comprehensive accounts we refer to the lecture note of Lance [46] and the book of Wegge-Olsen [55].

We now give the definition of Hilbert C^*-modules.

Definition 2.45. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$ be a (left) $\mathcal{A}$-module. Suppose that the linear structures given on A and $\mathcal{H}$ are compatible, i.e. $\lambda(ax) = a(\lambda x)$ for every $\lambda \in \mathbb{C}, a \in \mathcal{A}$ and $x \in \mathcal{H}$. If there exists a mapping $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ with the properties

(1) $\langle x, x \rangle \geq 0$ for every $x \in \mathcal{H}$,

(2) $\langle x, x \rangle = 0$ if and only if $x = 0$,

(3) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in \mathcal{H}$,

(4) $\langle ax, y \rangle = a\langle x, y \rangle$ for every $a \in \mathcal{A}$, and every $x, y \in \mathcal{H}$,

(5) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for every $x, y, z \in \mathcal{H}$.

Then the pair $\{\mathcal{H}, \langle \cdot, \cdot \rangle\}$ is called a (left-) pre-Hilbert $\mathcal{A}$-module. The map $\langle \cdot, \cdot \rangle$ is said to be an $\mathcal{A}$-valued inner product. If the pre-Hilbert $\mathcal{A}$-module $\{\mathcal{H}, \langle \cdot, \cdot \rangle\}$ is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ then it is called a *Hilbert $\mathcal{A}$-module.*

Here are some important examples.

Example 2.46. The C^*-algebra $\mathcal{A}$ itself can be reorganized to become a Hilbert $\mathcal{A}$-module if we define the inner product

$$\langle a, b \rangle = ab^*, \quad \forall a, b \in \mathcal{A}.$$

The corresponding norm is just the norm on $\mathcal{A}$ because of the C^*-equation (2.7).

Example 2.47. If $\{\mathcal{H}_i\}_{k=1}^n$ be a finite set of Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$, then one can define the direct sum $\oplus_{k=1}^n \mathcal{H}_k$. The inner product on $\oplus_{k=1}^n \mathcal{H}_k$ is given by the formula

$$\langle x, y \rangle := \sum_{k=1}^{n} \langle x_k, y_k \rangle_{\mathcal{H}_k},$$

where $x = x_1 \oplus x_2 \oplus \cdots \oplus x_n, y = y_1 \oplus y_2 \oplus \cdots \oplus y_n \in \oplus_{k=1}^{n} \mathcal{H}_k$. Then $\oplus_{k=1}^{n} \mathcal{H}_k$ is a Hilbert $\mathcal{A}$-module.

We denote the direct sum of n copies of a Hilbert C^*-module $\mathcal{H}$ by $\mathcal{H}^n$.

Example 2.48. If $\{\mathcal{H}_k\}, k \in \mathbb{N}$ is a countable set of Hilbert $\mathcal{A}$-modules over C^*-algebra $\mathcal{A}$, then one can define their direct sum $\oplus_{k\in\mathbb{N}} \mathcal{H}_k$. On the $\mathcal{A}$-module $\oplus_{k\in\mathbb{N}} \mathcal{H}_k$ of all sequences $x = (x_1, x_2, \dots), x_k \in \mathcal{H}_k$, such that the series $\sum_{k\in\mathbb{N}} \langle x_k, x_k \rangle$ is norm-convergent in the C^*-algebra $\mathcal{A}$, we define the inner product by

$$\langle x, y \rangle := \sum_{k\in\mathbb{N}} \langle x_k, y_k \rangle_{\mathcal{H}_k}$$

for $x, y \in \oplus_{k\in\mathbb{N}} \mathcal{H}_k$.

Then $\oplus_{k\in\mathbb{N}} \mathcal{H}_k$ is a Hilbert $\mathcal{A}$-module.

The direct sum of a countable number of copies of a Hilbert C^*-module $\mathcal{H}$ is denoted by $l^2(\mathcal{H})$.

Note that in Hilbert C^*-modules the Cauchy-Schwartz Inequality is valid.

Proposition 2.49. *Let $\mathcal{H}$ be a Hilbert C^*-module, and $x, y \in \mathcal{H}$, then*

$$\|\langle x, y \rangle\|^2 \leq \|\langle x, x \rangle\| \cdot \|\langle y, y \rangle\|.$$

We are especially interested in finitely and countably generated Hilbert C^*-modules over unital C^*-algebra $\mathcal{A}$. A Hilbert $\mathcal{A}$-module $\mathcal{H}$ is *(algebraically) finitely generated* if there exists a finite set $\{x_1, \cdots, x_n\} \subseteq \mathcal{H}$ such that every element $x \in \mathcal{H}$ can be expressed as an $\mathcal{A}$-linear combination $x = \sum_{i=1}^{n} a_i x_i, a_i \in \mathcal{A}$. A Hilbert $\mathcal{A}$-module $\mathcal{H}$ is *countably generated* if there exists a countable set of generators.

Note that algebraically finitely generated Hilbert $\mathcal{A}$-module over unital C^*-algebra $\mathcal{A}$ are precisely the finitely generated projective $\mathcal{A}$-modules in a pure algebraic sense.

Theorem 2.50. *Let $\mathcal{A}$ be a unital C^*-algebra. Every algebraically finitely generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ is an orthogonal summand of some free Hilbert $\mathcal{A}$-module $\mathcal{A}^n$ for a finite number n.*

Let $\mathcal{A}$ be a C^*-algebra. The Hilbert $\mathcal{A}$-module $l_2(\mathcal{A})$ serves as an universal environment for countably generated Hilbert $\mathcal{A}$-module that can be described as (special) orthogonal summands. This result was given by Kasparov ([43]) in 1980.

Theorem 2.51. *Let $\mathcal{A}$ be a unital C^*-algebra. Every countably generated Hilbert A-module $\mathcal{H}$ posses an embedding into $l_2(\mathcal{A})$ as an orthogonal summand in such a way that the orthogonal complement of it is isometrically isomorphic to $l_2(\mathcal{A})$ again, i.e. $\mathcal{H} \oplus l_2(\mathcal{A}) = l_2(\mathcal{A})$.*

Remark 2.52. Note that not every Hilbert C^*-module has an orthonormal basis. Though any countably generated Hilbert C^*-module admits a standard frame, there are countably generated Hilbert C^*-modules that contain no orthonormal basis even orthogonal Riesz basis (see Example 3.4 in [26]).

We now list some properties of operators on Hilbert C^*-modules.

Definition 2.53. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$. A map $T : \mathcal{H} \to \mathcal{H}$ (a priori neither linear nor bounded) is said to be *adjointable* if there exists a map $T^* : \mathcal{H} \to \mathcal{H}$ satisfying

$$\langle x, Ty \rangle = \langle T^*x, y \rangle$$

for all $x, y \in \mathcal{H}$. Such a map T^* is called the *adjoint* of T.

By $End^*_{\mathcal{A}}(\mathcal{H})$ we denote the set of all adjointable maps on $\mathcal{H}$.

It is surprising that every adjointable operator is automatically linear and bounded.

Proposition 2.54. *Suppose that T, S are two adjointable operators on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a C^*-algebra $\mathcal{A}$, then*

(1) The adjoint of T is unique and adjointable with $T^{**} = T$.

(2) ST is adjointable with $(ST)^* = T^*S^*$.

(3) T is a $\mathbb{C}$-linear module map which is bounded with respect to the operator norm.

(4) $T^*S = 0$ if and only if $T(\mathcal{H}) \perp S(\mathcal{H})$.

(5) $KerT = Ker|T|, KerT^* = T(\mathcal{H})^{\perp}$ and $(KerT^*)^{\perp} = T(\mathcal{H})^{\perp\perp} \supset \overline{T(\mathcal{H})}$.

Proposition 2.55. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$. Then $End^*_{\mathcal{A}}(\mathcal{H})$ is a C^*-algebra equipped with the operator norm*

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}.$$

It should be mentioned here that, unlike $B(H)$, $End^*_{\mathcal{A}}(\mathcal{H})$ is not a von Neumann algebra in general.

For adjointable operators we have the following Polar Decomposition Theorem.

Theorem 2.56. *Let $\mathcal{H}$ be a Hilbert C^*-module and $T \in End^*(\mathcal{H})$, then the following are equivalent:*

(1) T has a polar decomposition $T = V|T|$, where $V \in End^*(\mathcal{H})$ is a partial isometry for which

$$KerV = KerT, \quad KerV^* = KerT^*,$$

$$V(\mathcal{H}) = \overline{T(\mathcal{H})}, \quad V^*(\mathcal{H}) = \overline{T(\mathcal{H})}.$$

(2) $\mathcal{H} = Ker|T| \oplus \overline{|T|(\mathcal{H})}$ and $\mathcal{H} = KerT^* \oplus \overline{T(\mathcal{H})}$.

(3) Both $\overline{T(\mathcal{H})}$ and $\overline{|T|(\mathcal{H})}$ are complementable in $\mathcal{H}$.

The following result will be frequently used in this manuscript.

Theorem 2.57. *Suppose that $\mathcal{H}$ is a Hilbert C^*-module and T an adjoint operator on $\mathcal{H}$ with closed range, then T^* and $|T|$ have closed ranges and*

$$\mathcal{H} = Ker|T| \oplus |T|(\mathcal{H}) = KerT^* \oplus T(\mathcal{H}) = KerT \oplus T^*(\mathcal{H}).$$

Definition 2.58. Let $\mathcal{H}$ be a Hilbert C^*-module and $\mathcal{M} \subseteq \mathcal{H}$ a submodule. Then $\mathcal{M}$ is said to be *complementable* if $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ for some submodule $\mathcal{N} \subseteq \mathcal{H}$.

Note that submodules in Hilbert C^*-modules need not be complementable in general. We have the following example.

Example 2.59. Let $\mathcal{A} = C[0,1]$ be the set of all continuous functions on $[0,1]$ with the norm closed ideal $\mathcal{M} = C_0[0,1]$, where $C_0[0,1] = \{f \in C[0,1] : f(0) = f(1) = 0\}$. In this case $\mathcal{M}$ is a Hilbert $\mathcal{A}$-submodule with $\mathcal{M}^\perp = \{0\}$, so $\mathcal{H} \neq \mathcal{M} \oplus \mathcal{M}^\perp$ and $\mathcal{M} \neq \mathcal{M}^{\perp\perp} = \mathcal{H}$.

Since it is more convenient to work with orthogonal decompositions, we would like to describe situations where such a decomposition exists.

Proposition 2.60. *Suppose that $\mathcal{H}$ is a Hilbert C^*-module and T is an adjointable operator on $\mathcal{H}$ with closed range, then both the range and kernel of T are complementable in $\mathcal{H}$.*

Corollary 2.61. *Let $\mathcal{H}$ be a Hilbert C^*-module. If $P \in End^*(\mathcal{H})$ is an idempotent, then its range is an orthogonally complementable submodule in $\mathcal{H}$.*

We also have

Proposition 2.62. *A closed submodule of a Hilbert C^*-module is complementable precisely when it is the range of an adjointable operator.*

For a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a C^*-algebra $\mathcal{A}$, let us denote by $\mathcal{H}'$ the set of all bounded $\mathcal{A}$-linear maps from $\mathcal{H}$ to $\mathcal{A}$.

Definition 2.63. A Hilbert C^*-module $\mathcal{H}$ is called *self-dual* if $\mathcal{H} = \mathcal{H}'$.

The condition of self-duality is very strong. Below we shall see that there are quite a few self-dual modules: any Hilbert module over a C^*-algebra $\mathcal{A}$ is self-dual if and only if $\mathcal{A}$ is finite dimensional. If $\mathcal{A}$ is a unital C^*-algebra, then the Hilbert module $\mathcal{A}^n$ is obviously self-dual. Self-dual Hilbert C^*-modules behave quite like Hilbert spaces. In the same way as in the case of Hilbert spaces, the following statements can be easily checked.

Proposition 2.64. *Let $\mathcal{H}$ be a self-dual Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$, $\mathcal{K}$ an arbitrary Hilbert $\mathcal{A}$-module and $T : \mathcal{H} \to \mathcal{K}$ a bounded $\mathcal{A}$-linear operator. Then there exists an operator $T^* : \mathcal{K} \to \mathcal{H}$ such that the equality*

$$\langle Tx, y\rangle = \langle x, T^* y\rangle$$

holds for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$, i.e. T is adjointable.

Proposition 2.65. *Let $\mathcal{H}$ be a self-dual Hilbert C^*-module and let $\mathcal{H} \subset \mathcal{K}$. Then $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$.*

We complete this chapter by the following remark.

Remark 2.66. It should mention here that by no means all results of Hilbert space theory can be simply generalized to the situation of Hilbert C^*-modules. For example,

(1) The analogue of the Riesz representation theorem for bounded $\mathcal{A}$-linear mapping is not valid for $\mathcal{H}$.

(2) Since in general a Hilbert C^*-module $\mathcal{H}$ need not be self-dual, the bounded A-linear operator on $\mathcal{H}$ may not have an adjoint operator.

(3) Since a Hilbert C^*-submodule $\mathcal{M}$ of the Hilbert C^*-module $\mathcal{H}$ may not be complementable in general, the corresponding projection may not be orthogonal.

CHAPTER 3

HILBERT C^*-MODULE FRAMES

We first introduce the definition of modular frames and list some basic and important properties of modular frames. According to this definition we need to compare positive elements in the underlying C^*-algebra in order to test whether a sequence is a frame or not. This is not a trivial task. In Section 3.2 we give an equivalent definition of frames in Hilbert C^*-modules which is much easier to be applied. Based on this equivalent definition, we characterize modular frames from the operator theory point of view in Section 3.3.

3.1 Frames in Hilbert C^*-modules

Definition 3.1. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathbb{J}$ be a finite or countable index set. A sequence $\{x_j\}_{j\in\mathbb{J}}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a *frame* if there exist two constants $C, D > 0$ such that

$$C \cdot \langle x, x\rangle \leq \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \leq D \cdot \langle x, x\rangle \tag{3.1}$$

for every $x \in \mathcal{H}$. The optimal constants (i.e. maximal for C and minimal for D) are called frame bounds.

The frame $\{x_j\}_{j\in\mathbb{J}}$ is said to be *tight frame* if $C = D$, and said to be *Parseval* if $C = D = 1$.

Likewise, $\{x_j\}_{j\in\mathbb{J}}$ is called a *Bessel sequence* with bound D if there exists $D > 0$ such that

$$\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \leq D\langle x, x\rangle \tag{3.2}$$

for every $x \in \mathcal{H}$.

A sequence $\{x_j\}_{j\in\mathbb{J}}$ is said to be a *Riesz basis* of $\mathcal{H}$ if it is a frame and a generating set with the additional property that $\mathcal{A}$-linear combinations $\sum_{j\in S} a_j x_j$ with coefficients $\{a_j : j \in S\} \subseteq A$ and $S \in \mathbb{J}$ are equal to zero if and only if in particular every summand $a_j x_j$ equals zero for $j \in S$.

We consider standard (normalized tight) frames, standard Bessel sequences and standard Riesz bases in the main for which the sums in the inequalities (3.1) and (3.2) always converges in norm.

It should be remarkable that following Theorem 2.50 and Theorem 2.51, it was proved in [26] that every finitely generated or countably generated Hilbert C^*-module admits a (standard) frame.

Note that we can also define the analysis operator, synthesis operator and frame operator for modular frames as follows.

Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a frame of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$.

The operator $T : \mathcal{H} \to l^2(\mathcal{A})$ defined by

$$Tx = \{\langle x, x_j\rangle\}_{j\in\mathbb{J}},$$

is called the *analysis operator*. The adjoint operator $T^* : l^2(\mathcal{A}) \to \mathcal{H}$ is given by

$$T^*\{c_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_j x_j.$$

T^* is called *pre-frame operator* or the *synthesis operator*. By composing T and T^*, we obtain the *frame operator* $S : \mathcal{H} \to \mathcal{H}$:

$$Sx = T^*Tx = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle x_j. \tag{3.3}$$

The frame $\{S^{-1}x_j\}_{j\in\mathbb{J}}$ is said to be the *canonical dual frame* of $\{x_j\}_{j\in\mathbb{J}}$.

The main property of frames for Hilbert spaces is the existence of the reconstruction formula that allows a simple standard decomposition of every element of the spaces with respect to the frame. For the frames in Hilbert C^*-modules, we have the following results.

Theorem 3.2. *([23]) Let A be a unital C^*-algebra, $\mathcal{H}$ be a finitely or countably generated Hilbert A-module and $\{x_j\}_{j\in\mathbb{J}}$ be a Parseval frame (not necessarily standard) of $\mathcal{H}$. Then the reconstruction formula*

$$x = \sum_{j\in\mathbb{J}} \langle x, x_j \rangle x_j \tag{3.4}$$

holds for every $x \in \mathcal{H}$ in the sense of convergence with respect to the topology that is induced by the set of semi-norms $\{|f(\langle\cdot,\cdot\rangle)|^{1/2} : f \in A^*\}$. The sum converges always in norm if and only if the frame $\{x_j\}_{j\in\mathbb{J}}$ is standard.

Also from equation (3.3) we see that

$$x = \sum_{j\in\mathbb{J}} \langle x, S^{-1}x_j \rangle x_j$$

is valid for every $x \in \mathcal{H}$.

More generally, we have an existence and uniqueness result that provides us with a reconstruction formula for standard frame. Also this result guarantees the existence of a dual for any Hilbert C^*-module frame.

Theorem 3.3. *([26]) Let $\{x_j\}_{j\in\mathbb{J}}$ be a standard frame in a finitely or countably generated Hilbert A-module $\mathcal{H}$ over a unital $\mathcal{A}$-algebra $\mathcal{A}$. Then there exists a unique operator $S \in End^*_{\mathcal{A}}(\mathcal{H})$ such that*

$$x = \sum_{j\in\mathbb{J}} \langle x, S(x_j) \rangle x_j \tag{3.5}$$

for every $x \in \mathcal{H}$. The operator can be explicitly given by the formula $S = G^*G$ for any adjointable invertible bounded operator G mapping $\mathcal{H}$ onto some other Hilbert $\mathcal{A}$-module $\mathcal{K}$ and realizing $\{G(x_j) : j \in \mathbb{J}\}$ to be a standard normalized tight frame in $\mathcal{K}$.

Similar to the case of Hilbert space frames, we also have the following dilation result for modular frames.

Proposition 3.4. *([26]) Modular frames are precisely the inner direct summands of standard Riesz bases of $\mathcal{A}^n$ or $l^2(\mathcal{A})$, where $\mathcal{A}$ is a C^*-algebra.*

3.2 An Equivalent Definition of Modular Frames

Our first observation shows that the analysis operator of Bessel sequence is adjointable.

Lemma 3.5. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a Bessel sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then the analysis operator $T : \mathcal{H} \to l^2(\mathcal{A})$ defined by*

$$Tx = \sum_{j\in\mathbb{J}} \langle x, x_j \rangle e_j$$

is adjointable and fulfills $T^* e_j = x_j$ for all j.

Proof. It follows directly from the proofs of Theorem 4.1 and Theorem 4.4 in [26]. □

We need the following lemma to prove our results.

Lemma 3.6. *([49]) Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$ and let $T : \mathcal{M} \to \mathcal{N}$ be a linear map. Then the following conditions are equivalent:*

(1) the operator T is bounded and $\mathcal{A}$-linear;

(2) there exists a constant $K \geq 0$ such that the inequality $\langle Tx, Tx \rangle \leq K\langle x, x \rangle$ holds in $\mathcal{A}$ for all $x \in \mathcal{M}$.

We have the following equivalent definition for Bessel sequences in Hilbert C^*-modules.

Lemma 3.7. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with bound D if and only if*

$$\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| \leq D\|x\|^2$$

holds for all $x \in \mathcal{H}$.

Proof. "$\Rightarrow$" Obvious.

"$\Leftarrow$" Define a linear operator $T : \mathcal{H} \to l^2(\mathcal{A})$ by

$$Tx = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle e_j, \quad \forall x \in \mathcal{H}.$$

Then

$$\|Tx\|^2 = \|\langle Tx, Tx\rangle\| = \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| \leq D\|x\|^2,$$

which implies that $\|Tx\| \leq \sqrt{D}\|x\|$. Hence T is bounded.

It is obvious that T is $\mathcal{A}$-linear. Then by Lemma 3.6, we have

$$\langle Tx, Tx\rangle \leq D\langle x, x\rangle,$$

equivalently, $\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \leq D\langle x, x\rangle$, as desired.

□

With the same argument we obtain the following equivalent definition of frames in Hilbert C^*-modules.

Proposition 3.8. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ and $\{x_j\}_{j\in\mathbb{J}} \subseteq \mathcal{H}$ a sequence. Then $\{x_j\}_{j\in\mathbb{J}}$ is a frame of $\mathcal{H}$ with bounds C and D if and only if*

$$C\|x\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| \leq D\|x\|^2$$

for all $x \in \mathcal{H}$.

One of the advantages of this equivalent definition is that it is much easier to compare the norms of two elements than to compare two elements in C^*-algebras.

Using the above equivalent definition of frames we can easily prove the following result which will be used in the proofs of Theorem 4.9, Theorem 4.13 and Theorem 6.2.

Proposition 3.9. *Suppose that $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ be two Bessel sequences in $\mathcal{H}$. If $x = \sum_{j\in\mathbb{J}}\langle x, y_j\rangle x_j$ holds for any $x \in \mathcal{H}$, then both $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ are frames of $\mathcal{H}$ and $x = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle y_j$ holds for all $x \in \mathcal{H}$.*

Proof. Let's denote the Bessel bound of $\{y_j\}_{j\in\mathbb{J}}$ by D_Y. For all $x \in \mathcal{H}$ we have

$$\begin{aligned} \|x\|^4 &= \|\langle\sum_{j\in\mathbb{J}}\langle x, y_j\rangle x_j, x\rangle\|^2 = \|\sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle x_j, x\rangle\|^2 \\ &\leq \|\sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle y_j, x\rangle\| \cdot \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| \\ &\leq D_Y\|x\|^2 \cdot \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\|. \end{aligned}$$

It follows that

$$D_Y^{-1}\|x\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\|.$$

Similarly, we can show that $\{y_j\}_{j\in\mathbb{J}}$ is also a frame of $\mathcal{H}$.

It follows directly from Proposition 6.3 in [26] that

$$x = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle y_j$$

holds true for all $x \in \mathcal{H}$. □

3.3 Characterizations of Frames and Bessel Sequences in Hilbert C^*-modules

The aim of this section is to give some characterizations of Bessel sequences and frames in Hilbert C^*-modules from the operator-theoretic point of view. These results will be used to prove our main results in Chapter 7.

The following lemma is due to Heuser ([38]). Heuser only considered the $l^2(\mathbb{C})$-sequence case, but his proof works in more general setting. We include the proof here for the sake of completeness.

Lemma 3.10. *Let $\mathcal{A}$ be a C^*-algebra and $\{c_j\}_{j\in\mathbb{J}}$ a sequence in $\mathcal{A}$. If $\sum_{j\in\mathbb{J}} c_j\xi_j^*$ converges for all $\{\xi_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$, then $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$.*

Proof. We define a sequence of operators F_n and an operator F by

$$F_n(\{\xi_j\}) = \sum_{j=1}^{n} c_j\xi_j^* \quad \text{and} \quad F(\{\xi_j\}) = \sum_{j=1}^{\infty} c_j\xi_j^*, \qquad \forall\{\xi_j\} \in l^2(\mathcal{A}).$$

Observe that

$$\|F_n(\{\xi_j\})\|^2 = \|\sum_{j=1}^{n} c_j\xi_j^*\|^2 \leq \|\sum_{j=1}^{n} c_jc_j^*\| \cdot \|\sum_{j=1}^{n} \xi_j\xi_j^*\| \leq \|\{c_j\}\|^2 \cdot \|\sum_{j=1}^{n} \xi_j\xi_j^*\|.$$

It follows that F_n is bounded for each n.

Clearly, $F_n \to F$ pointwise as $n \to \infty$, so F is bounded by the Uniform Boundedness Theorem. Therefore $\|F(\{\xi_j\})\| \leq \|F\|\cdot\|\{\xi_j\}\|$ for each $\{\xi_j\} \in l^2(\mathcal{A})$.

Now fix n, and let

$$\xi_j = \begin{cases} c_j^*, & \text{if } 1 \leq j \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{\xi_j\} \in l^2(\mathcal{A})$.

We compute

$$
\begin{aligned}
\|\sum_{j=1}^{n} c_j c_j^*\| &= \|\sum_{j=1}^{n} c_j \xi_j\| \leq \|F\| \cdot \|\{\xi_j\}\| \\
&= \|F\| \cdot \|\sum_{j=1}^{\infty} \xi_j \xi_j^*\|^{\frac{1}{2}} = \|F\| \cdot \|\sum_{j=1}^{n} \xi_j \xi_j^*\|^{\frac{1}{2}} \\
&= \|F\| \cdot \|\sum_{j=1}^{n} c_j c_j^*\|^{\frac{1}{2}}.
\end{aligned}
$$

Therefore $\|\sum_{j=1}^{n} c_j c_j^*\|^{\frac{1}{2}} \leq \|F\|$. It follows that $\|\sum_{j=1}^{\infty} c_j c_j^*\|^{\frac{1}{2}} \leq \|F\|$, and hence $\{c_j\} \in l^2(\mathcal{A})$. □

We first give a characterization of Bessel sequences in terms of operators in Hilbert C^*-modules.

Proposition 3.11. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with Bessel bound D if and only if the operator $U : l^2(\mathcal{A}) \to \mathcal{H}$ defined by*

$$U\{c_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_j x_j$$

is a well-defined bounded operator from $l^2(\mathcal{A})$ into $\mathcal{H}$ with $\|U\| \leq \sqrt{D}$.

Proof. "$\Rightarrow$". Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with bound D. We first show that U is well-defined.

For arbitrary $n > m$, we have

$$\begin{aligned}
\|\sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{m} c_j x_j\|^2 &= \|\sum_{j=m+1}^{n} c_j x_j\|^2 \\
&= \sup_{\|x\|=1} \|\langle \sum_{j=m+1}^{n} c_j x_j, x\rangle\|^2 \\
&= \sup_{\|x\|=1} \|\sum_{j=m+1}^{n} c_j \langle x_j, x\rangle\|^2 \\
&\leq \sup_{\|x\|=1} \|\sum_{j=m+1}^{n} \langle x, x_j\rangle\langle x_j, x\rangle\| \cdot \|\sum_{j=m+1}^{n} c_j c_j^*\| \\
&\leq D\|\sum_{j=m+1}^{n} c_j c_j^*\|,
\end{aligned}$$

which implies that $\sum_{j\in\mathbb{J}} c_j x_j$ converges. Therefore U is well-defined.

To see the boundedness of U, we consider

$$\begin{aligned}
\|U\{c_j\}\|^2 &= \sup_{\|x\|=1} \|\langle U\{c_j\}, x\rangle\|^2 \\
&= \sup_{\|x\|=1} \|\sum_{j\in\mathbb{J}} c_j \langle x_j, x\rangle\|^2 \\
&\leq \sup_{\|x\|=1} \|\sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle\| \cdot \|\sum_{j\in\mathbb{J}} c_j c_j^*\| \\
&\leq D\|\sum_{j\in\mathbb{J}} c_j c_j^*\| = D\|\{c_j\}\|^2.
\end{aligned}$$

This yields that $\|U\| \leq \sqrt{D}$.

"$\Leftarrow$". For arbitrary $x \in \mathcal{H}$ and $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$, we have

$$\langle x, U\{c_j\}\rangle = \langle x, \sum_{j\in\mathbb{J}} c_j x_j\rangle = \sum_{j\in\mathbb{J}} \langle x, x_j\rangle c_j^*. \tag{3.6}$$

By Lemma 3.10, we see that $\{\langle x, x_j\rangle\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$.

From (3.6), we get

$$\langle x, U\{c_j\}\rangle = \langle \{\langle x, x_j\rangle\}, \{c_j\}\rangle,$$

which implies that U is adjointable, with $U^*x = \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}$, and hence U is bounded.

Note that

$$\|U^*x\|^2 = \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| \leq D\|\langle x, x\rangle\| = D\|x\|^2.$$

Consequently, $\|U\| = \|U^*\| \leq \sqrt{D}$, as desired. □

For the case of frames in Hilbert C^*-modules we have the following two characterizations.

Proposition 3.12. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a sequence of $\mathcal{H}$. Then $\{x_j\}_{j\in\mathbb{J}}$ is a frame of $\mathcal{H}$ if and only if the operator $U : l^2(\mathcal{A}) \to \mathcal{H}$ defined by*

$$U\{c_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_j x_j$$

is a well-defined bounded operator from $l^2(\mathcal{A})$ onto $\mathcal{H}$.

Proof. "$\Rightarrow$". Obvious.

"$\Leftarrow$". By Proposition 3.11, $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence. Let D be the Bessel bound of $\{x_j\}_{j\in\mathbb{J}}$.

Note that for each $x \in \mathcal{H}$, we have

$$x = UU^*(UU^*)^{-1}x = \sum_{j\in\mathbb{J}}\langle (UU^*)^{-1}x, x_j\rangle x_j.$$

Now

$$
\begin{aligned}
\|x\|^4 &= \|\langle x,x\rangle\|^2 = \|\sum_{j\in\mathbb{J}}\langle (UU^*)^{-1}x, x_j\rangle\langle x_j, x\rangle\|^2 \\
&\leq \|\sum_{j\in\mathbb{J}}\langle (UU^*)^{-1}x, x_j\rangle\langle x_j, (UU^*)^{-1}x\rangle\| \cdot \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| \\
&\leq D\|\langle (UU^*)^{-1}x, (UU^*)^{-1}x\rangle\| \cdot \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| \\
&= D\|(UU^*)^{-1}x\|^2 \cdot \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| \\
&\leq D\|(UU^*)^{-1}\|^2 \cdot \|x\|^2 \cdot \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\|,
\end{aligned}
$$

which leads to the lower bound inequality of frame, that is

$$
\frac{1}{D\|(UU^*)^{-1}\|^2}\|x\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\|.
$$

□

Proposition 3.13. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ be a sequence of $\mathcal{H}$, then $\{x_j\}_{j\in\mathbb{J}}$ is a frame of $\mathcal{H}$ with bounds C and D if and only if*

(1) $\overline{span\{x_j : j \in \mathbb{J}\}} = \mathcal{H}$;

(2) the operator $U : l^2(\mathcal{A}) \to \mathcal{H}$ defined by

$$
U\{c_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_j x_j
$$

is a well-defined bounded operator from $l^2(\mathcal{A})$ into $\mathcal{H}$ and satisfies

$$
\sqrt{C}\|\{c_j\}\| \leq \|U\{c_j\}\| \leq \sqrt{D}\|\{c_j\}\|, \quad \forall \{c_j\} \in (KerU)^{\perp}. \tag{3.7}
$$

Proof. "⇒". Suppose first that $\{x_j\}_{j\in\mathbb{J}}$ is a frame. Let S be the frame operator of $\{x_j\}_{j\in\mathbb{J}}$. Then we have $S = UU^*$.

By Proposition 3.12, it is enough to show that

$$
\sqrt{C}\|\{c_j\}\| \leq \|U\{c_j\}\|
$$

holds for all $\{c_j\} \in (KerU)^{\perp}$.

Since $\{x_j\}_{j\in\mathbb{J}}$ is a frame, it follows that $Rang(U^*)$ is closed. Therefore we have

$$(KerU)^{\perp} = \overline{Rang(U^*)} = Rang(U^*).$$

As a sequence, $(KerU)^{\perp} = \{\{\langle x, x_j\rangle\}_{j\in\mathbb{J}} : x \in \mathcal{H}\}$.

Now for any $x \in \mathcal{H}$, we see that

$$\begin{aligned} \|\sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle\|^2 &= \|\langle Sx, x\rangle\|^2 \leq \|Sx\|^2 \cdot \|x\|^2 \\ &\leq \|Sx\|^2 \cdot \frac{1}{C}\|\sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle\|. \end{aligned}$$

Therefore $C\|\sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle\| \leq \|Sx\|^2 = \|UU^*x\| = \|U\{\langle x, x_j\rangle\}\|^2$, as desired.

"$\Leftarrow$". To show that $\{x_j\}_{j\in\mathbb{J}}$ is a frame, by Proposition 3.12, it suffices to show that $Rang(U) = \mathcal{H}$.

Since $span\{x_j : j \in \mathbb{J}\} \subseteq Rang(U)$, it only needs to prove that $Rang(U)$ is closed.

Suppose that $\{u_n\} \subseteq Rang(U)$ and $u_n \to u$ as $n \to \infty$. Then we can find $\{v_n\} \subseteq (KerU)^{\perp}$ such that $Uv_n = u_n$.

It follows from (3.7) that $\{v_n\}$ is a Cauchy sequence. Suppose that $v_n \to v$ as $n \to \infty$. Therefore $u_n = Uv_n \to Uv = u$ as $n \to \infty$. This completes the proof. □

3.4 Removal of Elements from Frames

It is obvious that if we remove an element from a basis, then we must get a set which is not a basis. But for frame this is not the case. Due to the redundancy of frame if we remove an element from a frame we may get a new frame. For the removal of elements from frames in Hilbert spaces Christensen ([17]) gave the following characterization.

Theorem 3.14. *The removal of a vector f_j from a frame $\{f_k\}_{k=1}^{\infty}$ for Hilbert space H leaves either a frame or an incomplete set. More precisely,*

(1) If $\langle f_j, S^{-1}f_j\rangle \neq 1$, then $\{f_k\}_{k\neq j}$ is a frame for H;

(2) If $\langle f_j, S^{-1}f_j\rangle = 1$, then $\{f_k\}_{k\neq j}$ is incomplete;
where S is the corresponding frame operator.

We now introduce a lemma.

Lemma 3.15. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame of $\mathcal{H}$ with analysis operator T and frame operator S. Let $x \in \mathcal{H}$ and suppose that $x = \sum_{j\in\mathbb{J}} a_j x_j$, where $a_j \in \mathcal{A}$ for each $j \in \mathbb{J}$. Then*

$$\begin{aligned}\textstyle\sum_{j\in\mathbb{J}} a_j a_j^* &= \textstyle\sum_{j\in\mathbb{J}}\langle x, S^{-1}x_j\rangle\langle S^{-1}x_j, x\rangle \\ &\quad + \textstyle\sum_{j\in\mathbb{J}}(a_j - \langle x, S^{-1}x_j\rangle)(a_j^* - \langle S^{-1}x_j, x\rangle).\end{aligned}$$

Proof. For each $j \in \mathbb{J}$ we can write $a_j = (a_j - \langle x, S^{-1}x_j\rangle) + \langle x, S^{-1}x_j\rangle$.

Since $\{x_j : j \in \mathbb{J}\}$ is a frame, we have $x = \sum_{j\in\mathbb{J}}\langle x, S^{-1}x_j\rangle x_j$, and so

$$\sum_{j\in\mathbb{J}}(a_j - \langle x, S^{-1}x_j\rangle)x_j = 0,$$

i.e. $\{a_j - \langle x, S^{-1}x_j\rangle\}_{j\in\mathbb{J}} \in KerT^*$.

Note that $\{\langle x, S^{-1}x_j\rangle\}_{j\in\mathbb{J}} = \{\langle S^{-1}x, x_j\rangle\}_{j\in\mathbb{J}} \in Rang(T)$.

Since $l_2(\mathcal{A}) = KerT^* \oplus \overline{T(\mathcal{H})}$, we see that

$$\{\langle x, S^{-1}x_j\rangle\}_{j\in\mathbb{J}} \perp \{a_j - \langle x, S^{-1}x_j\rangle\}_{j\in\mathbb{J}},$$

which completes the proof.

We now generalize Theorem 3.14 to the situation of frames in Hilbert C^*-modules.

Theorem 3.16. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame for $\mathcal{H}$ and $1_{\mathcal{A}}$ the identity element of $\mathcal{A}$. We have the following statements.*

(1) if $1_{\mathcal{A}} - \langle x_n, S^{-1}x_n\rangle$ is invertible in $\mathcal{A}$, then $\{x_j : j \neq n\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{H}$;

(2) if $1_{\mathcal{A}} - \langle x_n, S^{-1}x_n\rangle$ is not invertible in $\mathcal{A}$, then $\{x_j : j \neq n\}_{j\in\mathbb{J}}$ is not a frame for $\mathcal{H}$.

Proof. By the frame decomposition we have $x_n = \sum_{j\in\mathbb{J}}\langle x_n, S^{-1}x_j\rangle x_j$.

Define, for notational convenience, $a_j = \langle x_n, S^{-1}x_j\rangle$, for each $j \in \mathbb{J}$. Then $x_n = \sum_{j\in\mathbb{J}} a_j x_j$.

(1) Suppose that $1_{\mathcal{A}} - \langle x_n, S^{-1}x_n\rangle = 1_{\mathcal{A}} - a_n$ is invertible.

From $x_n = \sum_{j\in\mathbb{J}} a_j x_j = a_n x_n + \sum_{j\neq n} a_j x_j$, we have $x_n = (1_{\mathcal{A}} - a_n)^{-1}\sum_{j\neq n} a_j x_j$.

Now for any $x \in \mathcal{H}$, we see that

$$\begin{aligned}
&\langle x, x_n\rangle\langle x_n, x\rangle \\
= \;& \langle x, (1_{\mathcal{A}} - a_n)^{-1}\textstyle\sum_{j\neq n} a_j x_j\rangle\langle (1_{\mathcal{A}} - a_n)^{-1}\sum_{j\neq n} a_j x_j, x\rangle \\
= \;& \langle x, \textstyle\sum_{j\neq n} a_j x_j\rangle((1_{\mathcal{A}} - a_n)^{-1})^*(1_{\mathcal{A}} - a_n)^{-1}\langle\sum_{j\neq n} a_j x_j, x\rangle \\
\leq \;& \|(1_{\mathcal{A}} - a_n)^{-1}\|^2\langle x, \textstyle\sum_{j\neq n} a_j x_j\rangle\langle\sum_{j\neq n} a_j x_j, x\rangle) \\
= \;& \|(1_{\mathcal{A}} - a_n)^{-1}\|^2\textstyle\sum_{j\neq n}(\langle x, x_j\rangle a_j^*)\sum_{j\neq n}(a_j\langle x_j, x\rangle) \\
\leq \;& \|(1_{\mathcal{A}} - a_n)^{-1}\|^2\|\textstyle\sum_{j\neq n} a_j a_j^*\|^2\sum_{j\neq n}(\langle x, x_j\rangle\langle x_j, x\rangle).
\end{aligned}$$

Therefore

$$\begin{aligned}
C\langle x, x\rangle \;&\leq\; \textstyle\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \\
&=\; \langle x, x_n\rangle\langle x_n, x\rangle + \textstyle\sum_{j\neq n}\langle x, x_j\rangle\langle x_j, x\rangle \\
&\leq\; (\|(1_{\mathcal{A}} - a_n)^{-1}\|^2\|\textstyle\sum_{j\neq n} a_j a_j^*\|^2 + 1)\sum_{j\neq n}\langle x, x_j\rangle\langle x_j, x\rangle,
\end{aligned}$$

showing that $\{x_j : j \neq n\}_{j\in\mathbb{J}}$ satisfies the lower frame condition.

Obviously, $\{x_j : j \neq n\}_{j\in\mathbb{J}}$ also satisfies the upper frame condition.

(2) Suppose that $1_{\mathcal{A}} - \langle x_n, S^{-1}x_n\rangle = 1_{\mathcal{A}} - a_n$ is not invertible in $\mathcal{A}$.

Assume on the contrary that $\{x_j : j \neq n\}_{j \in \mathbb{J}}$ is a frame, since $S^{-\frac{1}{2}}$ is invertible, it follows that $\{S^{-\frac{1}{2}} x_j : j \neq n\}_{j \in \mathbb{J}}$ is also a frame with frame bound $\tilde{C}$ and $\tilde{D}$. Then

$$\tilde{C}\langle x, x\rangle \leq \sum_{j \neq n} \langle x, S^{-\frac{1}{2}} x_j\rangle \langle S^{-\frac{1}{2}} x_j, x\rangle \leq \tilde{D}\langle x, x\rangle$$

holds for all $x \in \mathcal{H}$.

In particular, for $x = S^{-\frac{1}{2}} x_n$ we have

$$\tilde{C}\langle S^{-\frac{1}{2}} x_n, S^{-\frac{1}{2}} x_n\rangle \leq \sum_{j \neq n} \langle S^{-\frac{1}{2}} x_n, S^{-\frac{1}{2}} x_j\rangle \langle S^{-\frac{1}{2}} x_j, S^{-\frac{1}{2}} x_n\rangle,$$

i.e. $\tilde{C}\langle x_n, S^{-1} x_n\rangle \leq \sum_{j \neq n} \langle x_n, S^{-1} x_j\rangle \langle S^{-1} x_j, x_n\rangle$.

This implies that

$$\tilde{C} a_n \leq \sum_{j \neq n} a_j a_j^*.$$

From Lemma 3.15 we see that $a_n = a_n^2 + \sum_{j \neq n} a_j a_j^*$.

Then we have $\tilde{C} a_n \leq a_n - a_n^2$, and so $\tilde{C} t \leq t - t^2$ holds for any t in $\sigma(a_n)$, the spectrum of a_n.

Since $1_{\mathcal{A}} - a_n$ is not invertible, it follows that $1 \in \sigma(a_n)$. Therefore $\tilde{C} \cdot 1 \leq 1 - 1 \cdot 1 = 0$, a contradiction. This completes the proof.

CHAPTER 4

DUALS AND MODULAR RIESZ BASES

The main purpose of this chapter is to investigate the Riesz bases in Hilbert C^*-modules. It is well-known that in Hilbert spaces every Riesz basis has a unique dual which is also a Riesz basis. But in Hilbert C^*-modules, due to the zero-divisors, not all Riesz bases have unique duals and not every dual is a Riesz basis. We will present several such examples showing that the duals of Riesz bases in Hilbert C^*-modules are much different and more complicated than the Hilbert space cases. We give a complete characterization of all the dual sequences for a Riesz basis, and a necessary and sufficient condition for a dual sequence of a Riesz basis to be a Riesz basis.

4.1 Characterizations of Riesz Bases in Hilbert C^*-modules

In this section we shall give a characterization of Riesz bases in Hilbert C^*-modules which will be used in the latter part of this thesis.

Note that in Hilbert space case a frame is a Riesz basis if and only if its analysis operator is surjective [37]. This is no longer true for Hilbert C^*-module frames.

We first introduce a notation.

Let P_n be the projection on $l^2(\mathcal{A})$ that maps each element to its n-th component, i.e. $P_n x = \{u_j\}_{j\in\mathbb{J}}$, where

$$u_j = \begin{cases} x_n & \text{if } j = n, \\ 0 & \text{if } j \neq n, \end{cases}$$

for each $x = \{x_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$.

We now prove the first main result of this chapter.

Theorem 4.1. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis if and only if $x_n \neq 0$ and $P_n(Rang(T_X)) \subseteq Rang(T_X)$ for all $n \in \mathbb{J}$, where T_X is the analysis operator of $\{x_j\}_{j\in\mathbb{J}}$.*

Proof. Suppose first that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

Note that for any $a = \{a_j\}_{j\in\mathbb{J}}$ in $l^2(\mathcal{A})$, if $\sum_{j\in\mathbb{J}} a_j x_j = 0$, then $a_j x_j = 0$ for all j.

Therefore, if $a \perp Rang(T_X)$, then $a \perp P_n(Rang(T_X))$. It follows that $P_n(Rang(T_X)) \subseteq Rang(T_X)$.

Suppose now that $P_n(Rang(T_X)) \subseteq Rang(T_X)$ for each n. We want to show that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

Suppose that $\sum_{j\in\mathbb{J}} a_j x_j = 0$, where $a_j \in \mathcal{A}$.

Fix an $n \in \mathbb{J}$, then $P_n T_X x \in Rang(T_X)$, so there exists $y_n \in \mathcal{H}$ such that $T_X y_n = P_n T_X x$.

Therefore we get

$$\langle y_n, x_j \rangle = \begin{cases} \langle x, x_n \rangle & \text{if } j = n, \\ 0 & \text{if } j \neq n. \end{cases}$$

Now for any $x \in \mathcal{H}$ we have

$$\begin{aligned} \langle x, a_n x_n \rangle &= \langle x, x_n \rangle a_n^* = \sum_{j\in\mathbb{J}} \langle y_n, x_j \rangle a_j^* \\ &= \sum_{j\in\mathbb{J}} \langle y_n, a_j x_j \rangle = \langle y_n, \sum_{j\in\mathbb{J}} a_j x_j \rangle = 0, \end{aligned}$$

which implies that $a_n x_n = 0$. □

Note that in Hilbert spaces, if $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis and $\sum_{j\in\mathbb{J}} c_j x_j$ converges for a sequence $\{c_j\} \subseteq \mathbb{C}$, then $\{c_j\} \in l^2$. But this is not the case in the setting of Hilbert C^*-modules. We have the following example.

Example 4.2. Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbb{N}}$ and $v = \{v_j\}_{j\in\mathbb{N}}$ in l^∞, we define

$$uv = \{u_j v_j\}_{j\in\mathbb{N}}, \quad u^* = \{\bar{u}_j\}_{j\in\mathbb{N}} \quad \textit{and} \quad \|u\| = \max_{j\in\mathbb{N}} |u_j|.$$

Then $\mathcal{A} = \{l^\infty, \|\cdot\|\}$ is a C^*-algebra.

Let $\mathcal{H} = c_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{H}$ we define

$$\langle u, v\rangle = uv^* = \{u_j \bar{v}_j\}_{j\in\mathbb{N}}.$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

Obviously, $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$.

For each j we let $c_j = \sqrt{j} e_{j+1}$.

Then $c_j e_j = 0$ and so $\sum_{j=1}^\infty c_j e_j = 0$.

But $\sum_{j=1}^\infty c_j c_j^* = \sum_{j=2}^\infty j e_j$ doesn't converge in $\mathcal{A}$. Thus $\{c_j\} \notin l^2(\mathcal{A})$.

Following the definition of Riesz bases in Hilbert C^*-modules, to test a sequence $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis, we need to show that if $\sum_{j\in\mathbb{J}} c_j x_j = 0$ for some sequence $\{c_j\}_{j\in\mathbb{J}} \subseteq \mathcal{A}$, then $c_j x_j = 0$ for each j. We claim that we can restrict the sequence $\{c_j\}_{j\in\mathbb{J}}$ in $l^2(\mathcal{A})$.

Corollary 4.3. *Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a frame of $\mathcal{H}$, then $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis if and only if*

(1) $x_j \neq 0$ for each j;

(2) if $\sum_{j\in\mathbb{J}} c_j x_j = 0$ for some sequence $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$, then $c_j x_j = 0$ for each j.

Proof. See the proof of Theorem 4.1. □

4.2 Duals of Riesz Bases in Hilbert C^*-modules

The aim of this section is to have a detailed investigation on the dual sequences of Riesz bases in Hilbert C^*-modules. Some of the results presented in this section will be needed in proving Theorem 6.2.

We first introduce the following definition.

Definition 4.4. Suppose that $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame and $\{y_j\}_{j\in\mathbb{J}}$ a sequence of $\mathcal{H}$. Then $\{y_j\}_{j\in\mathbb{J}}$ is said to be a *dual sequence* of $\{x_j\}_{j\in\mathbb{J}}$ if

$$x = \sum_{j\in\mathbb{J}} \langle x, y_j\rangle x_j \tag{4.1}$$

holds for all $x \in \mathcal{H}$, where the sum in (4.1) converges in norm. The pair $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ are called a *dual frame pair* when $\{y_j\}_{j\in\mathbb{J}}$ is also a frame.

It should be mentioned that , contrasting to the Hilbert space situation, Riesz bases of Hilbert C^*-modules may posses infinitely many dual frames due to the existence of zero-divisors in the C^*-algebra of coefficients. The following three simple examples show that the dual of Riesz bases of Hilbert C^*-modules are quite different from and more complicated than the Hilbert space cases.

The first example shows that in Hilbert C^*-modules the dual Riesz basis of a Riesz basis is not unique.

Example 4.5. Let $\mathcal{A} = M_{2\times 2}(\mathbb{C})$ denote the C^*-algebra of all 2×2 complex matrices. Let $\mathcal{H} = \mathcal{A}$ and for any $A, B \in \mathcal{H}$ define

$$\langle A, B\rangle = AB^*.$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

Let $E_{i,j}$ be the 2×2 matrix with 1 in the (i, j)-th entry and 0 elsewhere, where $1 \leq i, j \geq 2$.

Then $\{E_{1,1}, E_{2,2}\}$ is a Riesz basis of $\mathcal{H}$ and it is a dual of itself.

One can check that $\{E_{1,1} + E_{2,1}, E_{2,2}\}$ is also a dual Riesz basis of $\{E_{1,1}, E_{2,2}\}$.

It is well-known that if $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis and $\{y_j\}_{j\in\mathbb{J}}$ is a dual sequence of $\{x_j\}_{j\in\mathbb{J}}$ in a Hilbert space H, then $\{y_j\}_{j\in\mathbb{J}}$ is a Riesz basis which is the unique dual of $\{x_j\}_{j\in\mathbb{J}}$. The following example shows that this is not the case in Hilbert C^*-modules.

Example 4.6. Suppose $\mathcal{H}$ and $\mathcal{A}$ are the same as in Example 4.2.

Note that $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$.

Now let $x_j = e_j$ and

$$y_j = \begin{cases} e_1 & \text{if } j = 1, \\ e_j + je_{j-1} & \text{if } j \neq 1. \end{cases}$$

One can verify that

$$x = \sum_{j\in\mathbb{N}} \langle x, y_j\rangle x_j$$

holds for all $x \in H$, but $\{y_j\}_{j\in\mathbb{N}}$ is not a Riesz basis, even not a Bessel sequence.

Note that even the dual sequence of a Riesz basis in Hilbert C^*-modules is a Bessel sequence, it still has the chance not to be a Riesz basis. We have the following example.

Example 4.7. Suppose $\mathcal{H}$ and $\mathcal{A}$ are the same as in Example 4.2.

Now let $x_j = e_j$ and

$$y_j = \begin{cases} e_1 + e_2 & \text{if } j = 1, 2, \\ e_j & \text{if } j \neq 1, 2. \end{cases}$$

Then $\{y_j\}_{j\in\mathbb{N}}$ is a Bessel sequence, and satisfies

$$x = \sum_{j\in\mathbb{N}} \langle x, y_j\rangle x_j$$

for all $x \in \mathcal{H}$.

Therefore, $\{y_j\}_{j\in\mathbb{N}}$ is a frame of $\mathcal{H}$. But obviously $\{y_j\}_{j\in\mathbb{N}}$ is not a Riesz basis.

The following lemma will be needed in several places in the rest of this thesis (in particular, it will be needed in the proof of Theorem 6.2).

Lemma 4.8. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Suppose that $\{y_j\}_{j\in\mathbb{J}}$ and $\{z_j\}_{j\in\mathbb{J}}$ are dual frames of $\{x_j\}_{j\in\mathbb{J}}$ with the property that either $Rang(T_Y) \subseteq Rang(T_Z)$ or $Rang(T_Z) \subseteq Rang(T_Y)$, where T_Y and T_Z are the analysis operators of $\{y_j\}_{j\in\mathbb{J}}$ and $\{z_j\}_{j\in\mathbb{J}}$ respectively. Then $y_j = z_j$ for all $j \in \mathbb{J}$.*

Proof. Suppose that $Rang(T_Z) \subseteq Rang(T_Y)$. Then for each $x \in \mathcal{H}$ there exists $y_x \in \mathcal{H}$ such that

$$T_Y y_x = T_Z x.$$

Applying T_X^* on the both sides we arrive at

$$y_x = T_X^* T_Y y_x = T_X^* T_Z x = x,$$

and so $T_Y x = T_Z x$ for all $x \in \mathcal{H}$.

Equivalently,

$$\sum_{j\in\mathbb{J}} \langle x, y_j\rangle e_j - \sum_{j\in\mathbb{J}} \langle x, z_j\rangle e_j = 0,$$

i.e. $\sum_{j\in\mathbb{J}} \langle x, y_j - z_j\rangle e_j$. Hence $y_j = z_j$ for all j. □

We now give a necessary and sufficient condition about the uniqueness of dual frames in Hilbert C^*-modules. We also prove that if a frame has a unique dual frame then it must be a Riesz basis.

Theorem 4.9. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame of $\mathcal{H}$ with analysis operator T_X, then the following statements are equivalent:*

(1) $\{x_j\}_{j\in\mathbb{J}}$ has a unique dual frame;

(2) $Rang(T_X) = l^2(\mathcal{A})$.

In case the equivalent conditions are satisfied, $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

Proof. (2)⇒(1). Let $\{x_j^*\}_{j\in\mathbb{J}}$ be the canonical dual of $\{x_j\}_{j\in\mathbb{J}}$ with analysis operator T_{X^*}. Then $x_j^* = S_X^{-1} x_j$, where S_X is the frame operator of $\{x_j\}_{j\in\mathbb{J}}$.

Let $\{y_j\}_{j\in\mathbb{J}}$ be any dual frame of $\{x_j\}_{j\in\mathbb{J}}$ with analysis operator T_Y, then

$$Rang(T_Y) \subseteq l^2(\mathcal{A}) = Rang(T_X) = Rang(T_{X^*}).$$

By Lemma 4.8, $y_j = x_j^*$ for all j.

(1)⇒(2). Assume on the contrary that $Rang(T_X) \neq l^2(\mathcal{A})$.

By Theorem 2.57, we have

$$l^2(\mathcal{A}) = Rang(T_X) \oplus KerT_X^*.$$

Let P_X be the orthogonal projection from $l^2(\mathcal{A})$ onto $Rang(T_X)$, then

$$l^2(\mathcal{A}) = P_X l^2(\mathcal{A}) \oplus P_X^{\perp} l^2(\mathcal{A}).$$

Therefore $P_X^{\perp} l^2(\mathcal{A}) = KerT_X^* \neq \{0\}$.

Choose e_{j_0} such that $P_X^{\perp} e_{j_0} \neq 0$ and define an operator $U : P_X^{\perp} l^2(\mathcal{A}) \to \mathcal{H}$ by

$$Uw = \langle w, P_X^{\perp} e_{j_0}\rangle x_{j_0}.$$

Then U is an adjointable linear operator.

Now let $\{x_j^*\}_{j\in\mathbb{J}}$ be the canonical dual of $\{x_j\}_{j\in\mathbb{J}}$ with upper bound D_{X^*} and set $y_j = x_j^* + UP_X^{\perp} e_j$.

We have

$$\begin{aligned}
\sum_{j\in\mathbb{J}} \langle x, y_j\rangle\langle y_j, x\rangle &= \sum_{j\in\mathbb{J}} \langle x, x_j^* + UP_X^{\perp} e_j\rangle\langle x_j^* + UP_X^{\perp} e_j, x\rangle \\
&\leq 2\Big(\sum_{j\in\mathbb{J}} \langle x, x_j^*\rangle\langle x_j^*, x\rangle + \sum_{j\in\mathbb{J}} \langle P_X^{\perp} U^* x, e_j\rangle\langle e_j, P_X^{\perp} U^* x\rangle\Big) \\
&\leq 2\Big(D_{X^*}\langle x, x\rangle + \sum_{j\in\mathbb{J}} \langle P_X^{\perp} U^* x, P_X^{\perp} U^* x\rangle\Big),
\end{aligned}$$

which implies that $\{y_j\}_{j\in\mathbb{J}}$ is a Bessel sequence.

Now for any $x \in \mathcal{H}$,

$$\begin{aligned}
\sum_{j\in\mathbb{J}} \langle x, UP_X^{\perp} e_j\rangle x_j &= T_X^* \sum_{j\in\mathbb{J}} \langle x, UP_X^{\perp} e_j\rangle e_j = T_X^* \sum_{j\in\mathbb{J}} \langle P_X^{\perp} U^* x, e_j\rangle e_j \\
&= T_X^* P_X^{\perp} U^* x = 0.
\end{aligned}$$

This yields that $x = \sum_{j\in\mathbb{J}}\langle x, y_j\rangle x_j$ for all $x \in \mathcal{H}$. By Proposition 3.9, $\{y_j\}_{j\in\mathbb{J}}$ is a dual frame of $\{x_j\}_{j\in\mathbb{J}}$ and is different from $\{x_j^*\}_{j\in\mathbb{J}}$, which contradicts with the uniqueness of the dual frame of $\{x_j\}_{j\in\mathbb{J}}$.

To complete the proof it remains to prove that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis if one of the equivalent conditions holds. Suppose now that $Rang(T_X) = l^2(\mathcal{A})$.

If $\sum_{j\in\mathbb{J}} a_j x_j = 0$ for $a_j \in \mathcal{A}$. Then

$$0 = \sum_{j\in\mathbb{J}} a_j x_j = \sum_{j\in\mathbb{J}} a_j T_X^* e_j = T_X^* \sum_{j\in\mathbb{J}} a_j e_j.$$

Therefore $\sum_{j\in\mathbb{J}} a_j e_j = 0$ as T_X^* is injective. Hence $a_j = 0$ for all j.

Note that $x_j = T_X^* e_j$ for each $j \in \mathbb{J}$. It follows from the injectiveness of T_X^* that $x_j \neq 0$. □

Remark 4.10. By the above theorem, Example 4.5 shows that, thought $\{E_{1,1}, E_{2,2}\}$ is a Riesz basis of $\mathcal{H}$, the corresponding analysis operator is not surjective which is different from the case in Hilbert spaces.

We also have another characterization on the uniqueness of dual frames of Riesz bases in Hilbert C^*-modules.

Proposition 4.11. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$, then $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis with unique dual frame if and only if*

(1) $\overline{span\{x_j : j \in \mathbb{J}\}} = \mathcal{H}$;

(2) there exist $C, D \geq 0$ *such that*

$$\sqrt{C}\|\{c_j\}\| \leq \|\sum_{j\in\mathbb{J}} c_j x_j\| \leq \sqrt{D}\|\{c_j\}\|, \quad \forall \{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A}).$$

Proof. It follows from Theorem 4.9 and Proposition 3.13. □

We now study the dual sequences of Riesz bases in Hilbert C^*-modules. The following theorem is straightforward.

Theorem 4.12. *Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Let $\{y_j\}_{j\in\mathbb{J}}$ be a sequence of $\mathcal{H}$. Then the following statements are equivalent.*

(1) $\{y_j\}_{j\in\mathbb{J}}$ is a dual frame of $\{x_j\}_{j\in\mathbb{J}}$;

(2) $\{y_j\}_{j\in\mathbb{J}}$ is a dual Bessel sequence of $\{x_j\}_{j\in\mathbb{J}}$;

(3) for each $j \in \mathbb{J}$, $y_j = S^{-1}x_j + z_j$, where S is the frame operator of $\{x_j\}_{j\in\mathbb{J}}$, and $\{z_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathcal{H}$ satisfying $\langle x, z_j\rangle x_j = 0$ for all $x \in \mathcal{H}$ and $j \in \mathbb{J}$.

For the case of a dual sequence of a Riesz basis to be a Riesz basis, we have the following characterization.

Theorem 4.13. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a Riesz basis and $\{y_j\}_{j\in\mathbb{J}}$ a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then $\{y_j\}_{j\in\mathbb{J}}$ is a dual Riesz basis of $\{x_j\}_{j\in\mathbb{J}}$ if and only if for each $j \in \mathbb{J}$, $y_j = S^{-1}x_j + z_j$, where S is the frame operator of $\{x_j\}_{j\in\mathbb{J}}$, and $\{z_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathcal{H}$ with the property that for each $j \in \mathbb{J}$ there exists $b_j \in \mathcal{A}$ such that $z_j = b_jS^{-1}x_j$ and $\langle x, x_j\rangle b_jx_j = 0$ holds for all $x \in \mathcal{H}$.*

Proof. "$\Rightarrow$". Suppose that $\{y_j\}_{j\in\mathbb{J}}$ is a dual Riesz basis of $\{x_j\}_{j\in\mathbb{J}}$ and let $z_j = y_j - S^{-1}x_j$.

Then it is easy to see that $\{z_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathcal{H}$.

Now fix an $n \in \mathbb{J}$.

From $y_n = \sum_{j\in\mathbb{J}}\langle y_n, x_j\rangle y_j$ we can infer that $y_n = \langle y_n, x_n\rangle y_n$, i.e.

$$S^{-1}x_n + z_n = \langle S^{-1}x_n + z_n, x_n\rangle(S^{-1}x_n + z_n).$$

Consequently, we have

$$z_n = \langle z_n, x_n\rangle S^{-1}x_n + \langle S^{-1}x_n, x_n\rangle z_n + \langle z_n, x_n\rangle z_n.$$

To show that $\langle S^{-1}x_n, x_n\rangle z_n + \langle z_n, x_n\rangle z_n = 0$, it suffices to show that

$$\langle S^{-1}x_n, x_n\rangle\langle z_n, x\rangle + \langle z_n, x_n\rangle\langle z_n, x\rangle = 0$$

holds for all $x \in \mathcal{H}$.

Note that

$$x = \sum_{j\in\mathbb{J}} \langle x, y_j\rangle x_j = \sum_{j\in\mathbb{J}} \langle x, S^{-1}x_j\rangle x_j + \sum_{j\in\mathbb{J}} \langle x, x_j\rangle x_j = x + \sum_{j\in\mathbb{J}} \langle x, z_j\rangle x_j,$$

which implies that $\sum_{j\in\mathbb{J}} \langle x, z_j\rangle x_j = 0$ and so $\langle x, z_j\rangle x_j = 0$ for all $x \in \mathcal{H}$ and $j \in \mathbb{J}$.

Particularly, we have $\langle x, z_n\rangle x_n = 0$ for all $x \in \mathcal{H}$. This yields that

$$\langle x, z_n\rangle\langle x_n, z_n\rangle = 0 \quad \text{and} \quad \langle x, z_n\rangle\langle x_n, S^{-1}x_n\rangle = 0.$$

Equivalently, $\langle z_n, x_n\rangle\langle z_n, x\rangle = 0$ and $\langle S^{-1}x_n, x_n\rangle\langle z_n, x\rangle = 0$.

Therefore $z_n = b_n S^{-1}x_n$, where $b_n = \langle z_n, x_n\rangle$.

From $\langle x_n, z_n\rangle x_n = 0$, we have

$$\langle y, x_n\rangle\langle x_n, z_n\rangle\langle x_n, x\rangle = 0$$

for all $x, y \in \mathcal{H}$, which is equivalent to $\langle x, x_n\rangle\langle z_n, x_n\rangle\langle x_n, y\rangle = 0$, this implies that

$$\langle x, x_n\rangle b_n x_n = \langle x, x_n\rangle\langle z_n, x_n\rangle x_n = 0.$$

"$\Leftarrow$". Suppose now that for each $j \in \mathbb{J}$ there exists $b_j \in \mathcal{A}$ such that $z_j = b_j S^{-1}x_j$ and $\langle x, x_j\rangle b_j x_j = 0$ holds for all $x \in \mathcal{H}$. Then for all $x, y \in \mathcal{H}$ we have

$$\langle x, x_j\rangle b_j\langle x_j, y\rangle = 0.$$

Equivalently,

$$\langle y, x_j\rangle b_j^*\langle x_j, x\rangle = 0.$$

This implies that $\langle y, x_j\rangle b_j^* x_j = 0$ for all $y \in \mathcal{H}$.

Now for arbitrary $x \in \mathcal{H}$,

$$\begin{aligned}\sum_{j\in\mathbb{J}} \langle x, y_j\rangle x_j &= \sum_{j\in\mathbb{J}} \langle x, S^{-1}x_j\rangle x_j + \sum_{j\in\mathbb{J}} \langle x, z_j\rangle x_j \\ &= x + \sum_{j\in\mathbb{J}} \langle x, b_j S^{-1}x_j\rangle x_j \\ &= x + \sum_{j\in\mathbb{J}} \langle x, S^{-1}x_j\rangle b_j^* x_j \\ &= x + \sum_{j\in\mathbb{J}} \langle S^{-1}x, x_j\rangle b_j^* x_j \\ &= x,\end{aligned}$$

which implies that $\{y_j\}_{j\in\mathbb{J}}$ is a dual sequence of $\{x_j\}_{j\in\mathbb{J}}$.

One can easily see that $\{y_j\}_{j\in\mathbb{J}}$ is a dual frame of $\{x_j\}_{j\in\mathbb{J}}$ by Proposition 3.9.

To complete the proof, we need to show that $\{y_j\}_{j\in\mathbb{J}}$ is a Riesz basis of $\mathcal{H}$.

Suppose that $\sum_{j\in\mathbb{J}} a_j y_j = 0$, then we have

$$0 = \sum_{j\in\mathbb{J}} a_j(S^{-1}x_j + b_j S^{-1}x_j) = \sum_{j\in\mathbb{J}} a_j(1+b_i)S^{-1}x_j.$$

Therefore $a_j(1+b_j)S^{-1}x_j = 0$, i.e. $a_j y_j = 0$ for all j.

We now show that $y_j \neq 0$ for each $j \in \mathbb{J}$.

Assume on the contrary that $y_n = 0$ for some $n \in \mathbb{J}$. Then $z_n = -S^{-1}x_n$. It follows that

$$0 = \langle x, x_n\rangle b_n x_n = \langle x, x_n\rangle S z_n = -\langle x, x_n\rangle x_n$$

holds for all $x \in \mathcal{H}$.

In particular, letting $x = S^{-1}x_n$, we have $0 = -\langle S^{-1}x_n, x_n\rangle x_n = -x_n$, and so $x_n = 0$, a contradiction. This completes the proof. □

Corollary 4.14. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. If $\mathcal{A}$ is commutative, then every Riesz basis of $\mathcal{H}$ has a unique dual Riesz basis.*

Proof. Choose an arbitrary Riesz basis $\{x_j\}_{j\in\mathbb{J}}$ of $\mathcal{H}$. Suppose that $\{S^{-1}x_j + z_j\}_{j\in\mathbb{J}}$ is a dual Riesz basis of $\{x_j\}_{j\in\mathbb{J}}$, where S is the frame operator of $\{x_j\}_{j\in\mathbb{J}}$.

Then by Theorem 4.13, for each $j \in \mathbb{J}$ there exists $b_j \in \mathcal{A}$ such that $z_j = b_j S^{-1}x_j$ and $\langle x, x_j\rangle b_j x_j = 0$ holds for all $x \in \mathcal{H}$.

Since $\mathcal{A}$ is commutative, we have $b_j\langle x, x_j\rangle x_j = 0$ for all $x \in \mathcal{H}$ and $j \in \mathbb{J}$.

Let $x = S^{-1}x_j$. We have

$$0 = b_j\langle S^{-1}x_j, x_j\rangle x_j = b_j\langle x_j, S^{-1}x_j\rangle x_j = b_j x_j,$$

which yields that $z_j = b_j S^{-1}x_j = 0$. □

Note that under that conditions of Corollary 4.14, though a Riesz basis has a unique dual Riesz basis, it may have many dual frames. We have the following example.

Example 4.15. Let $\mathcal{A} = D_{2\times 2}(\mathbb{C})$ denote the C^*-algebra of all 2×2 complex diagonal matrices. Let $\mathcal{H} = \mathcal{A}$ and for any $A, B \in \mathcal{H}$ define

$$\langle A, B \rangle = AB^*.$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

It is obvious that $\mathcal{A}$ is commutative.

Let $E_{i,j}$ be the 2×2 matrix with 1 in the (i, j)-th entry and 0 elsewhere, where $1 \leq i, j \leq 2$.

Then $\{E_{1,1}, E_{2,2}\}$ is a Riesz basis of $\mathcal{H}$, and so it has a unique dual Riesz basis which is itself.

But the dual frame of $\{E_{1,1}, E_{2,2}\}$ is not unique. For example, one can verify that $\{E_{1,1}+\alpha E_{2,2}, \beta E_{1,1}+E_{2,2}\}$ is also a dual frame of $\{E_{1,1}, E_{2,2}\}$ for any $\alpha, \beta \in \mathbb{C}$.

The following example shows that the converse of Corollary 4.14 is not true, namely, if every Riesz basis of a Hilbert $\mathcal{A}$-module $\mathcal{H}$ has a unique dual Riesz basis, $\mathcal{A}$ is not necessarily commutative.

Example 4.16. Let

$$\mathcal{H} = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \forall a \in \mathbb{C} \}$$

and

$$\mathcal{A} = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} : \forall a, b, c, d, e \in \mathbb{C} \}.$$

For any $A, B \in \mathcal{H}$ define

$$\langle A, B \rangle = AB^*.$$

Then $\mathcal{H}$ is a $\mathcal{A}$-module.

Note that $\mathcal{A}$ is not commutative.

Let

$$E_\alpha = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\{E_\alpha\}$ is a Riesz basis of $\mathcal{H}$.

It is easy to see that any Riesz basis of $\mathcal{H}$ has the form of $\{E_\alpha\}$ for some nonzero $\alpha \in \mathbb{C}$. And one can also check that every dual Riesz basis of $\{E_\alpha\}$ for each nonzero α is unique.

We now introduce a notation. For a Hilbert $\mathcal{A}$-module $\mathcal{H}$, let

$$\mathcal{C}_{\mathcal{A}}(\mathcal{H}) = \{a \in \mathcal{A} : abx = bax, \forall b \in \mathcal{A}, x \in \mathcal{H}\}.$$

To complete this section we pose a conjecture as follows.

Conjecture. Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. Then every Riesz basis of $\mathcal{H}$ has a unique dual Riesz basis if and only if $\mathcal{A} = \mathcal{C}_{\mathcal{A}}(\mathcal{H})$.

CHAPTER 5

STRUCTURED MODULAR FRAMES

The purpose of this chapter is to focus on structured frames in Hilbert C^*-modules. More precisely, we work on two closely related issues: frame vector parameterizations and Parseval frame approximations. In the case that the underlying C^*-algebra is a commutative W^*-algebra, we prove that the set of the complete Parseval frame vectors for a unitary group can be parameterized by the set of all the unitary operators in the double commutant of the unitary group. Similar result holds for the set of all the complete frame generators where the unitary operators are replaced by invertible and adjointable operators. Consequently, the set of all the complete Parseval frame vectors is path-connected. We also prove the existence and uniqueness of the best Parseval multi-frame approximations for multi-frame generators of unitary groups on Hilbert C^*-modules when the underlying C^*-algebra is commutative.

5.1 Modular Frame Vector Parameterizations

The aim of this section is to investigate the parameterizations of frame vectors in Hilbert C^*-modules.

Let's first introduce a few more notation. Let $\mathcal{S} \subseteq End_{\mathcal{A}}(\mathcal{H})$, we denote its *commutant* $\{A \in End_{\mathcal{A}}(\mathcal{H}) : AS = SA, S \in \mathcal{S}\}$ by $\mathcal{S}'$. Let $x \in \mathcal{H}$ be a nonzero vector, the *local commutant* $C_x(\mathcal{S})$ of $\mathcal{S}$ at x is defined by

$$C_x(\mathcal{S}) = \{A \in End_{\mathcal{A}}(\mathcal{H}) : ASx = SAx, S \in \mathcal{S}\}.$$

A *unitary system* $\mathcal{U}$ on $\mathcal{H}$ is a set of unitary operators acting on $\mathcal{H}$ which contains the identity operator.

A vector ψ in $\mathcal{H}$ is called a *complete frame vector* (resp. *complete Parseval frame vector, complete Riesz basis vector, Bessel sequence vector*) for a unitary system $\mathcal{U}$ on $\mathcal{H}$ if $\mathcal{U}\psi = \{U\psi : U \in \mathcal{U}\}$ is a frame (resp. Parseval frame, Riesz basis, Bessel sequence) for $\mathcal{H}$. If $\mathcal{U}\psi$ is an orthonormal basis of $\mathcal{H}$, then ψ is called a *complete wandering vector* for $\mathcal{U}$.

For a unitary system $\mathcal{U}$ on $\mathcal{H}$, let $l^2_{\mathcal{U}}(\mathcal{A})$ be the Hilbert $\mathcal{A}$-module defined by

$$l^2_{\mathcal{U}}(\mathcal{A}) = \{\{a_U\}_{U\in\mathcal{U}} \subseteq \mathcal{A} : \sum_{U\in\mathcal{U}} a_U a_U^* \text{ converges in } \|\cdot\|\}.$$

Let $\{e_U\}_{\mathcal{U}}$ denote the standard orthonormal basis of $l^2_{\mathcal{U}}(\mathcal{A})$, where e_U takes value $1_{\mathcal{A}}$ at U and $0_{\mathcal{A}}$ at everywhere else. For the case that $\mathcal{U}$ is a unitary group on $\mathcal{H}$, we have the *left and right regular representation* for each $U \in \mathcal{U}$ which are defined by

$$L_U e_V = e_{UV} \quad \text{and} \quad R_U e_V = e_{VU}.$$

Note that $L_U^{-1} = L_U^* = L_{U^*}$ and $R_U^{-1} = R_U^* = R_{U^*}$.

The following two propositions can be viewed as the analogue of the corresponding results for Hilbert space frames obtained in [19].

Proposition 5.1. *Let $\mathcal{U}$ be a unitary system on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Suppose that $\mathcal{H}$ has orthonormal bases and η is a complete wandering vector for $\mathcal{U}$. For $\xi \in \mathcal{H}$, we have*

(1) ξ is a complete wandering vector for $\mathcal{U}$ if and only if there exists a unitary $T \in C_\eta(\mathcal{U})$ such that $\xi = T\eta$.

(2) ξ is a complete Riesz basis vector for $\mathcal{U}$ if and only if there exists an invertible and adjointable operator $T \in C_\eta(\mathcal{U})$ such that $\xi = T\eta$.

(3) ξ is a complete Parseval frame vector for $\mathcal{U}$ if and only if there exists a co-isometry $T \in C_\eta(\mathcal{U})$ such that $\xi = T\eta$.

(4) ξ is a complete frame vector for $\mathcal{U}$ if and only if there exists an adjointable operator $T \in C_\eta(\mathcal{U})$ with $C\langle x, x\rangle \leq \langle T^*x, T^*x\rangle$ for some $C > 0$ and any $x \in \mathcal{H}$ such that $\xi = T\eta$.

(5) ξ is a complete Bessel sequence vector for $\mathcal{U}$ if and only if there exists an adjointable operator $T \in C_\eta(\mathcal{U})$ such that $\xi = T\eta$.

Proof. We will prove (3). And others go similarly.

Suppose that $\xi = T\eta$ for some unitary operator $T \in C_\eta(\mathcal{U})$. Then for any $x \in \mathcal{H}$, we have

$$\begin{aligned}\langle x, x\rangle &= \langle T^*x, T^*x\rangle = \sum_{U\in\mathcal{U}} \langle T^*x, U\eta\rangle\langle U\eta, T^*x\rangle \\ &= \sum_{U\in\mathcal{U}} \langle x, TU\eta\rangle\langle TU\eta, x\rangle \\ &= \sum_{U\in\mathcal{U}} \langle x, UT\eta\rangle\langle UT\eta, x\rangle \\ &= \sum_{U\in\mathcal{U}} \langle x, U\xi\rangle\langle U\xi, x\rangle,\end{aligned}$$

which implies that ξ is a complete Parseval frame vector for $\mathcal{U}$.

We now assume that ξ is a complete Parseval frame vector for $\mathcal{U}$.

Define two operators T_η and T_ξ from $\mathcal{H}$ to $l^2_{\mathcal{U}}(\mathcal{A})$ respectively by

$$T_\eta x = \sum_{U\in\mathcal{U}} \langle x, U\eta\rangle e_U \quad \text{and} \quad T_\xi x = \sum_{U\in\mathcal{U}} \langle x, U\xi\rangle e_U.$$

It is easy to check that both T_η and T_ξ are well-defined and adjointable.

Let $T = T_\xi^* T_\eta$. Then for any $x \in \mathcal{H}$, we have

$$Tx = \sum_{U\in\mathcal{U}} \langle x, U\eta\rangle U\xi$$

and

$$T^*x = \sum_{U\in\mathcal{U}} \langle x, U\xi\rangle U\eta.$$

We now show that T is a co-isometry.

Indeed, for any $x \in \mathcal{H}$, we see that

$$\begin{aligned}\langle T^*x, T^*x\rangle &= \langle\sum_{U\in\mathcal{U}}\langle x, U\xi\rangle U\eta, \sum_{U\in\mathcal{U}}\langle x, U\xi\rangle U\eta\rangle \\ &= \sum_{U\in\mathcal{U}}\langle x, U\xi\rangle\langle U\xi, x\rangle = \langle x, x\rangle.\end{aligned}$$

To complete the proof, it remains to prove that $\xi = T\eta$ and $T \in C_\eta(\mathcal{U})$.

For any V and U in $\mathcal{U}$,

$$\begin{aligned}\langle V\xi, TU\eta\rangle &= \langle V\xi, \sum_{W\in\mathcal{U}}\langle U\eta, W\eta\rangle W\xi\rangle \\ &= \sum_{W\in\mathcal{U}}\langle V\xi, W\xi\rangle\langle W\eta, U\eta\rangle \\ &= \langle V\xi, U\xi\rangle,\end{aligned}$$

this implies that $TU\eta = U\xi$. Particularly, we have $T\eta = \xi$. And so $TU\eta = U\xi = UT\eta$, as desired. □

Note that if $\mathcal{U}$ is a unitary system which is not a group, and if $\mathcal{U}$ has a complete wandering vector, then $\mathcal{U}$ is not even a semigroup. We have the following result.

Proposition 5.2. *Let $\mathcal{S}$ be a unital semigroup of unitaries on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Suppose that $\mathcal{S}$ has a complete wandering vector. Then $\mathcal{S}$ is a group.*

Proof. Let η be a complete wandering vector for $\mathcal{S}$ and $U \in \mathcal{U}$. For any $x \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{V\in\mathcal{U}}\langle x, V\eta\rangle\langle V\eta, x\rangle &= \langle x, x\rangle = \langle U^*x, U^*x\rangle \\ &= \sum_{V\in\mathcal{S}}\langle U^*x, V\eta\rangle\langle V\eta, U^*x\rangle \\ &= \sum_{V\in\mathcal{S}}\langle x, UV\eta\rangle\langle UV\eta, x\rangle.\end{aligned}$$

Assume on the contrary that $\mathcal{S}$ is not a group, then there exists $U_0 \in \mathcal{S}$ such that $U_0^{-1} \notin \mathcal{S}$. Then $I \notin U_0\mathcal{S}$. Since $U_0\mathcal{S} \subseteq \mathcal{S}$, it follows from (5.1) that

$$\langle x, \eta\rangle\langle \eta, x\rangle = 0, \quad \forall x \in \mathcal{H}.$$

In particular, if $x = \eta$, we have

$$\langle \eta, \eta \rangle \langle \eta, \eta \rangle = 0$$

and hence $\langle \eta, \eta \rangle = 0$, therefore $\eta = 0$, a contradiction. □

To prove our first main result, we need the following:

Lemma 5.3. *Let $\mathcal{G}$ be a unitary group on a Hilbert $\mathcal{A}$-module over a commutative unital C^*-algebra $\mathcal{A}$, then*

$$\mathcal{M} = \mathcal{N}' = \{R_U : U \in \mathcal{G}\}' \quad \textit{and} \quad \mathcal{N} = \mathcal{M}' = \{L_U : U \in \mathcal{G}\}',$$

where $\mathcal{M} = \{L_U : U \in \mathcal{G}\}''$ and $\mathcal{N} = \{R_U : U \in \mathcal{G}\}''$.

Proof. Note that $R_U L_V = L_V R_U$ holds for any $U, V \in \mathcal{G}$. Therefore to prove this lemma it suffices to show that $TS = ST$ for arbitrary $T \in \mathcal{M}'$ and $S \in \mathcal{N}'$.

Suppose that

$$Te_I = \sum_{U \in \mathcal{G}} a_U e_U \quad \text{and} \quad Se_I = \sum_{U \in \mathcal{G}} b_U e_U$$

for some $a_U, b_U \in \mathcal{A}$.

Now for any $V \in \mathcal{G}$, on one hand, we have

$$\begin{aligned} STe_V &= STL_V e_I = SL_V Te_I \\ &= SL_V(\sum_{U \in \mathcal{G}} a_U e_U) = S(\sum_{U \in \mathcal{G}} a_U e_{VU}) \\ &= S(\sum_{U \in \mathcal{G}} a_U R_{VU} e_I) = \sum_{U \in \mathcal{G}} a_U R_{VU} Se_I \\ &= \sum_{U \in \mathcal{G}} a_U R_{VU}(\sum_{W \in \mathcal{G}} b_W e_W) \\ &= \sum_{U, W \in \mathcal{G}} a_U b_W e_{WVU}. \end{aligned}$$

On the other hand

$$\begin{aligned}
TSe_V &= TSR_Ve_I = TR_VSe_I \\
&= TR_V(\sum_{W\in\mathcal{G}} b_We_W) = T(\sum_{W\in\mathcal{G}} b_We_{WV}) \\
&= T(\sum_{W\in\mathcal{G}} b_WL_{WV}e_I) = \sum_{W\in\mathcal{G}} b_WL_{WV}Te_I \\
&= \sum_{W\in\mathcal{G}} b_WL_{WV}(\sum_{U\in\mathcal{G}} a_Ue_U) \\
&= \sum_{U,W\in\mathcal{G}} b_Wa_Ue_{WVU}.
\end{aligned}$$

Since $\mathcal{A}$ is commutative, it follows that $STe_V = TSe_V$, and so $ST = TS$. $\square$

We now define a natural conjugate $\mathcal{A}$-linear isomorphism π from $\mathcal{M}$ onto $\mathcal{M}' = \mathcal{N}$ by

$$\pi(A)Be_I = BA^*e_I, \quad \forall A, B \in \mathcal{M}.$$

In particular, $\pi(A)e_I = A^*e_I$.

Now we are in a position to prove the parameterization of complete Parseval frames for unitary groups.

Theorem 5.4. *Let $\mathcal{G}$ be a unitary group on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a commutative unital W^*-algebra $\mathcal{A}$ such that $l^2_{\mathcal{G}}(\mathcal{A})$ is self-dual. Suppose that $\eta \in \mathcal{H}$ be a complete Parseval frame vector for $\mathcal{G}$. For ξ in $\mathcal{H}$ we have*

(1) ξ is a complete Parseval frame vector for $\mathcal{G}$ if and only if there exists a unitary operator $A \in \mathcal{G}''$ such that $\xi = A\eta$.

(2) ξ is a complete frame vector for $\mathcal{G}$ if and only if there exists an invertible and adjointable operator $A \in \mathcal{G}''$ such that $\xi = A\eta$.

(3) ξ is a complete Bessel sequence vector for $\mathcal{G}$ if and only if there exists an adjointable operator $A \in \mathcal{G}''$ such that $\xi = A\eta$.

Proof. We will prove (1). The proof of (2) and (3) is similar and we leave it to the interested readers.

Let $\mathcal{M} = \{L_U : U \in \mathcal{G}\}''$. As η is a complete Parseval frame vector for $\mathcal{G}$, we have the corresponding analysis operator T_η which is given by

$$T_\eta x = \sum_{U \in \mathcal{G}} \langle x, U\eta \rangle e_U.$$

From the proof of Theorem 6.1, we know that T_η is an isometry with closed range and satisfies

$$T_\eta U = L_U T_\eta \;\; and \;\; T_\eta \eta = Pe_I,$$

where P is the orthogonal projection from $l^2_{\mathcal{G}}(\mathcal{A})$ onto the range of T_η, and we also have $P \in \mathcal{M}'$. Note that $\mathcal{G}$ is unitarily equivalent to the group $\{L_U, U \in \mathcal{G}\}$. Therefore, to prove this theorem is equivalent to prove the theorem for the case that $\tilde{\mathcal{G}} = \{L_U|_{Rang(P)}, U \in \mathcal{G}\}$ and $\tilde{\eta} = Pe_I$.

Sufficiency. Suppose that we have a unitary operator $A \in \tilde{\mathcal{G}}''$ such that $\tilde{\xi} = A\tilde{\eta}$.

We now show that $A\tilde{\eta}$ is a complete Parseval frame vector for $\tilde{\mathcal{G}}$. For any $x \in Rang(P) = Rang(T_\eta)$, we have

$$\begin{aligned}
\sum_{\tilde{U} \in \tilde{\mathcal{G}}} \langle x, \tilde{U}A\tilde{\eta} \rangle \langle \tilde{U}A\tilde{\eta}, x \rangle &= \sum_{U \in \mathcal{G}} \langle x, L_U PA\tilde{\eta} \rangle \langle L_U PA\tilde{\eta}, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle x, L_U PAPe_I \rangle \langle L_U PAPe_I, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle x, L_U PAe_I \rangle \langle L_U PAe_I, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle x, PL_U Ae_I \rangle \langle PL_U Ae_I, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle Px, L_U Ae_I \rangle \langle L_U Ae_I, Px \rangle \\
&= \sum_{U \in \mathcal{G}} \langle x, L_U Ae_I \rangle \langle L_U Ae_I, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle x, \pi(A^*) L_U e_I \rangle \langle \pi(A^*) L_U e_I, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle (\pi(A^*))^* x, e_U \rangle \langle e_U, (\pi(A^*))^* x \rangle \\
&= \langle (\pi(A^*))^* x, (\pi(A^*))^* x \rangle = \langle x, x \rangle,
\end{aligned}$$

which means that $A\tilde{\eta}$ is a complete Parseval frame vector for $\tilde{\mathcal{G}}$.

Necessity. Let $\tilde{\xi} \in Rang(P)$ be a complete Parseval frame vector for $\tilde{\mathcal{G}}$. We want to find a unitary operator $A \in \tilde{\mathcal{G}}''$ such that $\tilde{\xi} = A\tilde{\eta}$.

To this aim, we first define an operator $B : l^2_{\mathcal{G}}(\mathcal{A}) \to l^2_{\mathcal{G}}(\mathcal{A})$ by

$$e_U \longmapsto L_U\xi, \quad U \in \mathcal{G}.$$

One can check that B is an adjointable operator and $B^*e_V = \sum_{W\in\mathcal{G}}\langle L_{W^{-1}}L_V\tilde{\eta}, \tilde{\xi}\rangle e_W$ for any $V \in \mathcal{G}$.

Now for any $U, V \in \mathcal{G}$, we see that

$$\begin{aligned}
& \langle (BB^* - P)e_U, e_V\rangle \\
= & \langle \sum_{W\in\mathcal{G}} \langle L_{W^{-1}}L_U\tilde{\eta}, \tilde{\xi}\rangle e_W, \sum_{S\in\mathcal{G}} \langle L_{S^{-1}}L_V\tilde{\eta}, \tilde{\xi}\rangle e_S\rangle - \langle T_\eta U\eta, e_V\rangle \\
= & \sum_{W\in\mathcal{G}} \langle L_{W^{-1}}L_U\tilde{\eta}, \tilde{\xi}\rangle\langle \tilde{\xi}, L_{W^{-1}}L_V\tilde{\eta}\rangle - \langle \sum_{W\in\mathcal{G}} \langle U\eta, W\eta\rangle e_W, e_V\rangle \\
= & \sum_{W\in\mathcal{G}} \langle L_U\tilde{\eta}, L_W\tilde{\xi}\rangle\langle L_W\tilde{\xi}, L_V\tilde{\eta}\rangle - \langle U\eta, V\eta\rangle \\
= & \langle L_U\tilde{\eta}, L_V\tilde{\eta}\rangle - \langle U\eta, V\eta\rangle \\
= & \langle L_UPe_I, L_VPe_I\rangle - \langle U\eta, V\eta\rangle \\
= & \langle L_UT_\eta\eta, L_VT_\eta\eta\rangle - \langle U\eta, V\eta\rangle \\
= & \langle T_\eta U\eta, T_\eta V\eta\rangle - \langle U\eta, V\eta\rangle = 0,
\end{aligned}$$

this lead to the fact that $P = BB^*$.

From

$$BL_Ue_V = Be_{UV} = L_{UV}\tilde{\xi} = L_UL_V\tilde{\xi} = L_UBe_V,$$

we see that $B \in \mathcal{M}'$. Hence B is a partial isometry in $\mathcal{M}'$.

Let $Q = B^*B$, then P and Q are equivalent projections in $\mathcal{M}'$.

Since $l^2_{\mathcal{G}}(\mathcal{A})$ is self-dual, by [49], $End^*_{\mathcal{A}}(l^2_{\mathcal{G}}(\mathcal{A}))$ is a W^*-algebra.

Let $(End^*_{\mathcal{A}}(l^2_{\mathcal{G}}(\mathcal{A})))_*$ be its predual. One can check that $\mathcal{M}$ and $\mathcal{M}'$ are $\sigma(End^*_{\mathcal{A}}(l^2_{\mathcal{G}}(\mathcal{A})), (End^*_{\mathcal{A}}(l^2_{\mathcal{G}}(\mathcal{A})))_*)$-closed in $End^*_{\mathcal{A}}(l^2_{\mathcal{G}}(\mathcal{A}))$, and so both $\mathcal{M}$ and $\mathcal{M}'$ are W^*-algebras (see [52]).

Claim. $\mathcal{M}$ and $\mathcal{M}'$ are finite W^*-algebras.

We now define $\phi : \mathcal{M} \to \mathcal{A}$ by

$$\phi(A) = \langle Ae_I, e_I \rangle, \quad \forall A \in \mathcal{M}.$$

We want to show that ϕ is a faithful $\mathcal{A}$-value trace for $\mathcal{M}$.

Since $\overline{span\{L_U e_I, U \in \mathcal{G}\}} = l^2_{\mathcal{G}}(\mathcal{A})$, for any $A, B \in \mathcal{M}$, we have

$$Ae_I = \lim_n A_n e_I \quad and \quad Be_I = \lim_n B_n e_I,$$

where

$$A_n e_I = \sum_{i=1}^{k_n} a_i^{(n)} L_{V_i^{(n)}} e_I \quad and \quad B_n e_I = \sum_{j=1}^{l_n} b_j^{(n)} L_{W_j^{(n)}} e_I$$

for some $a_i^{(n)}, b_j^{(n)} \in \mathcal{A}$ and $V_i^{(n)}, W_j^{(n)} \in \mathcal{G}$.

Then

$$\phi(AB) = \langle ABe_I, e_I \rangle = \lim_m \lim_n \langle \sum_{j=1}^{l_m} \sum_{i=1}^{k_n} b_j^{(m)} a_i^{(n)} L_{W_j^{(m)}} L_{V_i^{(n)}} e_I, e_I \rangle.$$

While

$$\phi(BA) = \lim_n \lim_m \langle \sum_{i=1}^{k_n} \sum_{j=1}^{l_m} a_i^{(n)} b_j^{(m)} L_{V_i^{(n)}} L_{W_j^{(m)}} e_I, e_I \rangle.$$

Note that

$$\langle L_{W_j^{(m)}} L_{V_i^{(n)}} e_I, e_I \rangle = \langle L_{V_i^{(n)}} L_{W_j^{(m)}} e_I, e_I \rangle.$$

Therefore $\phi(AB) = \phi(BA)$.

If $A \in \mathcal{M}$ is positive and $\phi(A) = 0$, then

$$\langle A^{\frac{1}{2}} e_I, A^{\frac{1}{2}} e_I \rangle = \langle Ae_I, e_I \rangle = \phi(A) = 0.$$

Thus $A^{\frac{1}{2}} e_I = 0$.

Now for any $U \in \mathcal{G}$, we have

$$A^{\frac{1}{2}} e_U = A^{\frac{1}{2}} R_U e_I = R_U A^{\frac{1}{2}} e_I = 0.$$

Therefore $A^{\frac{1}{2}} = 0$, and so $A = 0$. Similarly we can prove that $\mathcal{M}'$ is also finite.

It follows from Corollary 2.44 that $I - P$ and $I - Q$ are equivalent projections in $\mathcal{M}'$. Therefore there exists a partial isometry $C \in \mathcal{M}'$ such that $CC^* = I - P$ and $C^*C = I - Q$.

Let $T = B + C$. Then T is a unitary operator in $\mathcal{M}'$, and so $A = (\pi^{-1}(T))^*$ is a unitary operator in $\mathcal{M}$.

In order to complete the proof it remains to prove that $A\tilde{\eta} = \tilde{\xi}$.

In fact,

$$\begin{aligned} A\tilde{\eta} &= (\pi^{-1}(T))^* Pe_I = P(\pi^{-1}(T))^* e_I \\ &= P\pi(\pi^{-1}(T))e_I = PTe_I \\ &= P(B + C)e_I = PBe_I + PCe_I \\ &= P\tilde{\xi} = \tilde{\xi}, \end{aligned}$$

which completes the proof. □

We now can easily have the following:

Corollary 5.5. *Let $\mathcal{G}$ be a unitary group on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a commutative unital W^*-algebra $\mathcal{A}$ such that $l^2_{\mathcal{G}}(\mathcal{A})$ is self-dual, then the set of all complete Parseval frame vectors for $\mathcal{G}$ is path-connected.*

5.2 Parseval Frame Approximations

Many interesting frames are generated by some (usually finite number of) "window" functions under the action of a collection of unitary operators. For example, Gabor frames and wavelet frames are of this kind.

Definition 5.6. Let $\mathcal{U}$ be a unitary system on a Hilbert C^*-module $\mathcal{H}$. $\Phi = (\phi_1, \ldots, \phi_N)$, where $\phi_j \in \mathcal{H}$ for all j, is called a *multi-frame generator* of length N for $\mathcal{U}$ if $\{U\phi_j : U \in \mathcal{U}, 1 \leq j \leq N\}$ is a frame.

Definition 5.7. Let $\Phi = (\phi_1, \ldots, \phi_N)$ be a multi-frame generator for a unitary system $\mathcal{U}$ on a Hilbert C^*-module $\mathcal{H}$. Then a Parseval multi-frame generator $\Psi = (\psi_1, \ldots, \psi_N)$ for $\mathcal{U}$ is called a *best Parseval multi-frame approximation* for Φ if the inequality

$$\sum_{k=1}^{N} \langle \phi_k - \psi_k, \phi_k - \psi_k \rangle \leq \sum_{k=1}^{N} \langle \phi_k - \xi_k, \phi_k - \xi_k \rangle$$

is valid for all the Parseval multi-frame generator $\Xi = (\xi_1, \ldots, \xi_N)$ for $\mathcal{U}$.

Let $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$ be a multi-frame generator for a unitary system $\mathcal{U}$ on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. We use T_Φ to denote the analysis operator from $\mathcal{H}$ to $l^2_{\mathcal{U}\times\{1,2,\ldots,N\}}(\mathcal{A})$ defined by

$$T_\Phi x = \sum_{j=1}^{N} \sum_{U \in \mathcal{U}} \langle x, U\phi_j \rangle e_{(U,j)}, \quad \forall x \in \mathcal{H},$$

where $\{e_{(U,j)} : U \in \mathcal{U}, j = 1, 2, \ldots, N\}$ is the standard orthonormal basis for $l^2_{\mathcal{U}\times\{1,2,\ldots,N\}}(\mathcal{A})$.

Note that T_Φ is adjointable and its adjoint operator satisfying

$$T_\Phi^* e_{(U,j)} = U\phi_j, \quad U \in \mathcal{U}, \ j = 1, 2, \ldots, N.$$

Lemma 5.8. *Let $\mathcal{G}$ be a unitary group on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a commutative C^*-algebra $\mathcal{A}$. Suppose that $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$ and $\Psi = \{\psi_1, \psi_2, \ldots, \psi_N\}$ be two multi-frame generators for $\mathcal{G}$, then*

$$\sum_{k=1}^{N} \langle \phi_k, \phi_k \rangle = \sum_{k=1}^{N} \langle \psi_k, \psi_k \rangle.$$

Proof. We compute

$$\begin{aligned}
\sum_{k=1}^{N}\langle\phi_k,\phi_k\rangle &= \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{U\in\mathcal{G}}\langle\phi_k,U\psi_j\rangle\langle U\psi_j,\phi_k\rangle\\
&= \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{U\in\mathcal{G}}\langle U^*\phi_k,\psi_j\rangle\langle\psi_j,U^*\phi_k\rangle\\
&= \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{U\in\mathcal{G}}\langle\psi_j,U^*\phi_k\rangle\langle U^*\phi_k,\psi_j\rangle\\
&= \sum_{j=1}^{N}\langle\psi_j,\psi_j\rangle.
\end{aligned}$$

□

Theorem 5.9. *Let $\mathcal{G}$ be a unitary group on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a commutative unital C^*-algebra $\mathcal{A}$. Suppose that $\Phi = \{\phi_1,\phi_2,\ldots,\phi_N\}$ is a multi-frame generator for $\mathcal{G}$. Then $S^{\frac{1}{2}}\Phi$ is the unique best Parseval multi-frame approximation for Φ, where S is the frame operator for the multi-frame $\{U\phi_j : j = 1,\ldots,N, U\in\mathcal{G}\}$.*

Proof. We first show that $S\in\mathcal{G}'$.

For arbitrary $V\in\mathcal{G}$ and $x\in\mathcal{H}$ we have

$$\begin{aligned}
SVx &= \sum_{k=1}^{N}\sum_{U\in\mathcal{G}}\langle Vx,U\phi_k\rangle U\phi_k\\
&= \sum_{k=1}^{N}\sum_{U\in\mathcal{G}}\langle x,V^*U\phi_k\rangle U\phi_k\\
&= V(\sum_{k=1}^{N}\sum_{U\in\mathcal{G}}\langle x,V^*U\phi_k\rangle V^*U\phi_k)\\
&= V(\sum_{k=1}^{N}\sum_{U\in\mathcal{G}}\langle x,U\phi_k\rangle U\phi_k)\\
&= VSx.
\end{aligned}$$

This shows that $S\in\mathcal{G}'$.

Since $End^*_{\mathcal{A}}(\mathcal{H})$ is a C^*-algebra, by the spectral decomposition for positive elements in C^*-algebra, we can infer that $S^{-\frac{1}{2}}, S^{-\frac{1}{4}} \in \mathcal{G}'$.

Therefore $\{S^{-\frac{1}{2}}\phi_1, S^{-\frac{1}{2}}\phi_2, \ldots, S^{-\frac{1}{2}}\phi_N\}$ is a complete Parseval multi-frame generator for $\mathcal{G}$.

Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_N\}$ be any Parseval multi-frame generator for $\mathcal{G}$. We claim that

$$\sum_{k=1}^{N} \langle T_{S^{-\frac{1}{2}}\Phi} S^{-\frac{1}{4}}\phi_k, T_\Psi S^{-\frac{1}{4}}\phi_k\rangle = \sum_{k=1}^{N} \langle \psi_k, \phi_k\rangle,$$

where $T_{S^{-\frac{1}{2}}\Phi}$ and T_Ψ are the analysis operators with respect to the Parseval multi-frame generators $S^{-\frac{1}{2}}\Phi$ and Ψ respectively.

We compute

$$\begin{aligned}
& \sum_{k=1}^{N} \langle T_{S^{-\frac{1}{2}}\Phi} S^{-\frac{1}{4}}\phi_k, T_\Psi S^{-\frac{1}{4}}\phi_k\rangle \\
= & \sum_{k=1}^{N} \langle \sum_{j=1}^{N} \sum_{U\in\mathcal{G}} \langle S^{-\frac{1}{4}}\phi_k, US^{-\frac{1}{2}}\phi_j\rangle e_{(U,j)}, \sum_{i=1}^{N}\sum_{V\in\mathcal{G}} \langle S^{-\frac{1}{4}}\phi_k, V\psi_i\rangle e_{(V,i)}\rangle \\
= & \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{U\in\mathcal{G}} \langle S^{-\frac{1}{4}}\phi_k, US^{-\frac{1}{2}}\phi_j\rangle \langle U\psi_j, S^{-\frac{1}{4}}\phi_k\rangle \\
= & \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{U\in\mathcal{G}} \langle U\psi_j, S^{-\frac{1}{4}}\phi_k\rangle \langle S^{-\frac{1}{4}}\phi_k, US^{-\frac{1}{2}}\phi_j\rangle \\
= & \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{U\in\mathcal{G}} \langle S^{\frac{1}{4}}\psi_j, U^*S^{-\frac{1}{2}}\phi_k\rangle \langle U^*S^{-\frac{1}{2}}\phi_k, S^{-\frac{1}{4}}\phi_j\rangle \\
= & \sum_{j=1}^{N} \langle S^{\frac{1}{4}}\psi_j, S^{-\frac{1}{4}}\phi_j\rangle = \sum_{j=1}^{N} \langle \psi_j, \phi_j\rangle
\end{aligned}$$

We now want to prove that $S^{\frac{1}{2}}\Phi$ is a best Parseval multi–frame approximation for Φ. We need to show that

$$\sum_{k=1}^{N} \langle \psi_k - \phi_k, \psi_k - \phi_k\rangle \geq \sum_{k=1}^{N} \langle S^{-\frac{1}{2}}\phi_k - \phi_k, S^{-\frac{1}{2}}\phi_k - \phi_k\rangle.$$

By Lemma 5.8, it suffices to prove that

$$\sum_{k=1}^{N}(\langle S^{-\frac{1}{2}}\phi_k,\phi_k\rangle+\langle \phi_k,S^{-\frac{1}{2}}\phi_k\rangle-\langle\psi_k,\phi_k\rangle-\langle\phi_k,\psi_k\rangle)\geq 0.$$

In fact, we have

$$\begin{aligned}
&\sum_{k=1}^{N}(\langle S^{-\frac{1}{2}}\phi_k,\phi_k\rangle+\langle \phi_k,S^{-\frac{1}{2}}\phi_k\rangle-\langle\psi_k,\phi_k\rangle-\langle\phi_k,\psi_k\rangle)\\
=&\sum_{k=1}^{N}(\langle S^{-\frac{1}{4}}\phi_k,S^{-\frac{1}{4}}\phi_k\rangle+\langle S^{-\frac{1}{4}}\phi_k,S^{-\frac{1}{4}}\phi_k\rangle)\\
&-\langle T_{S^{-\frac{1}{2}}\Phi}S^{-\frac{1}{4}}\phi_k,T_\Psi S^{-\frac{1}{4}}\phi_k\rangle-\langle T_\Psi S^{-\frac{1}{4}}\phi_k,T_{S^{-\frac{1}{2}}\Phi}S^{-\frac{1}{4}}\phi_k\rangle)\\
=&\sum_{k=1}^{N}(\langle T_{S^{-\frac{1}{2}}\Phi}S^{-\frac{1}{4}}\phi_k,T_{S^{-\frac{1}{2}}\Phi}S^{-\frac{1}{4}}\phi_k\rangle+\langle T_\Psi S^{-\frac{1}{4}}\phi_k,T_\Psi S^{-\frac{1}{4}}\phi_k\rangle\\
&-\langle T_{S^{-\frac{1}{2}}\Phi}S^{-\frac{1}{4}}\phi_k,T_\Psi S^{-\frac{1}{4}}\phi_k\rangle-\langle T_\Psi S^{-\frac{1}{4}}\phi_k,T_{S^{-\frac{1}{2}}\Phi}S^{-\frac{1}{4}}\phi_k\rangle)\\
=&\sum_{k=1}^{N}\langle(T_{S^{-\frac{1}{2}}\Phi}-T_\Psi)S^{-\frac{1}{4}}\phi_k,(T_{S^{-\frac{1}{2}}\Phi}-T_\Psi)S^{-\frac{1}{4}}\phi_k\rangle\geq 0.
\end{aligned}$$

This implies that $S^{-\frac{1}{2}}\Phi$ is a best Parseval multi-frame approximation for Φ.

For the uniqueness, assume that $\Xi=\{\xi_1,\xi_2,\ldots,\xi_N\}$ be another best Parseval multi-frame approximation for Φ. Then we have

$$\sum_{k=1}^{N}\langle\xi_k-\phi_k,\xi_k-\phi_k\rangle=\sum_{k=1}^{N}\langle S^{-\frac{1}{2}}\phi_k-\phi_k,S^{-\frac{1}{2}}\phi_k-\phi_k\rangle. \tag{5.1}$$

By Lemma 5.8, we also have

$$\sum_{k=1}^{N}\langle\xi_k,\xi_k\rangle=\sum_{k=1}^{N}\langle S^{-\frac{1}{2}}\phi_k,S^{-\frac{1}{2}}\phi_k\rangle. \tag{5.2}$$

Identities (5.1) and (5.2) imply that

$$\begin{aligned}
\sum_{k=1}^{N}(\langle\xi_k,\phi_k\rangle+\langle\phi_k,\xi_k\rangle) &= \sum_{k=1}(\langle S^{-\frac{1}{2}}\phi_k,\phi_k\rangle+\langle\phi_k,S^{-\frac{1}{2}}\phi_k\rangle)\\
&= 2\sum_{k=1}^{N}\langle S^{-\frac{1}{4}}\phi_k,S^{-\frac{1}{4}}\phi_k\rangle.
\end{aligned}$$

We claim that

$$\sum_{k=1}^{N}\langle S^{\frac{1}{4}}\xi_k, S^{\frac{1}{4}}\xi_k\rangle = \sum_{k=1}^{N}\langle S^{-\frac{1}{4}}\phi_k, S^{-\frac{1}{4}}\phi_k\rangle.$$

In fact,

$$\begin{aligned}
& \sum_{k=1}^{N}\langle S^{\frac{1}{4}}\xi_k, S^{\frac{1}{4}}\xi_k\rangle \\
= & \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{U\in\mathcal{G}}\langle S^{\frac{1}{4}}\xi_k, US^{-\frac{1}{2}}\phi_j\rangle\langle US^{-\frac{1}{2}}\phi_j, S^{\frac{1}{4}}\xi_k\rangle \\
= & \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{U\in\mathcal{G}}\langle S^{-\frac{1}{4}}\phi_j, U^*\xi_k\rangle\langle U^*\xi_k, S^{-\frac{1}{4}}\phi_j\rangle \\
= & \sum_{j=1}^{N}\langle S^{-\frac{1}{4}}\phi_j, S^{-\frac{1}{4}}\phi_j\rangle.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \sum_{k=1}^{N}\langle S^{\frac{1}{4}}\xi_k - S^{-\frac{1}{4}}\phi_k, S^{\frac{1}{4}}\xi_k - S^{-\frac{1}{4}}\phi_k\rangle \\
= & \sum_{k=1}^{N}(\langle S^{\frac{1}{4}}\xi_k, S^{\frac{1}{4}}\xi_k\rangle - \langle S^{\frac{1}{4}}\xi_k, S^{-\frac{1}{4}}\phi_k\rangle \\
& -\langle S^{-\frac{1}{4}}\phi_k, S^{\frac{1}{4}}\xi_k\rangle + \langle S^{-\frac{1}{4}}\phi_k, S^{-\frac{1}{4}}\phi_k\rangle) \\
= & \sum_{k=1}^{N}(2\langle S^{-\frac{1}{4}}\phi_k, S^{-\frac{1}{4}}\phi_k\rangle - \langle\xi_k, \phi_k\rangle - \langle\phi_k, \xi_k\rangle) \\
= & \ 0.
\end{aligned}$$

This implies that

$$S^{\frac{1}{4}}\xi_k = S^{-\frac{1}{4}}\phi_k, \quad k = 1, 2, \ldots, N.$$

Therefore

$$\xi_k = S^{-\frac{1}{2}}\phi_k, \quad k = 1, 2, \ldots, N.$$

i.e. $\Xi = S^{-\frac{1}{2}}\Phi$, as expected. □

To complete this chapter we ask

Problem 5.10. *Does Theorem 5.9 hold when the underlying C^*-algebra $\mathcal{A}$ is non-commutative?*

CHAPTER 6

DILATIONS OF FRAMES IN HILBERT C^*-MODULES

In this chapter we investigate the dilation of frames in Hilbert C^*-modules. Our first result shows that a complete Parseval frame vector for a unitary group on Hilbert C^*-module can be dilated to a complete wandering vector. For any dual frame pair in any Hilbert C^*-module, we prove that the pair are orthogonal compressions of a Riesz basis and its canonical dual basis for some larger Hilbert C^*-module. In other words, the dilation theorem for Hilbert space dual frame pairs is still valid for Hilbert C^*-module dual frame pairs. This dilation property remains valid even when restricted to structured frames.

6.1 Dilation of Parseval Frame Vectors

It was proved in ([26]) that each Parseval frame of Hilbert C^*-modules can be dilated to an orthonormal basis. It is natural to ask whether a complete Parseval frame vector for a unitary group on Hilbert C^*-module can be dilated to a complete wandering vector. We answer this question in the following theorem.

Theorem 6.1. *Let $\mathcal{G}$ be a unitary group on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Suppose that η is a complete Parseval frame vector for $\mathcal{G}$. Then there exists a Hilbert $\mathcal{A}$-module $\tilde{\mathcal{H}} \supseteq \mathcal{H}$ and a unitary group $\tilde{\mathcal{G}}$ on $\tilde{\mathcal{H}}$ such that $\tilde{\mathcal{G}}$ has complete wandering vectors in $\tilde{\mathcal{H}}$, $\mathcal{H}$ is an invariant subspace of $\tilde{\mathcal{G}}$ such that $\tilde{\mathcal{G}}|_{\mathcal{H}} = \mathcal{G}$, and the map $G \mapsto G|_{\mathcal{H}}$ is a group isomorphism from $\tilde{\mathcal{G}}$ onto $\mathcal{G}$.*

Proof. Let $\tilde{\mathcal{H}} = l^2_{\mathcal{G}}(\mathcal{A})$.

Now for each $U \in \mathcal{G}$, let L_U be the left regular representation defined by

$$L_U e_V = e_{UV}, \quad \forall V \in \mathcal{G},$$

where e_V is the characteristic function at the single point set $\{V\}$.

Let $\tilde{\mathcal{G}} = \{L_U, U \in \mathcal{G}\}$. It is easy to check that $\tilde{\mathcal{G}}$ is a unitary group isomorphic to $\mathcal{G}$.

We now define an operator $T : \mathcal{H} \to \tilde{\mathcal{H}}$ by

$$T(x) = \sum_{U \in \mathcal{G}} \langle x, U\eta \rangle e_U.$$

One can check that T is an adjointable isometry. Also it is routine to show that the range of T is closed in $\tilde{\mathcal{H}}$. Therefore, by Proposition 2.54 and Theorem 2.57, we see that

$$\tilde{\mathcal{G}} = (T(\mathcal{H}))^{\perp} \oplus T(\mathcal{H}).$$

Hence we have the orthogonal projection P from $l^2_{\mathcal{G}}(\mathcal{A})$ onto $T(\mathcal{H})$, the range of T.

We claim that $P(e_U) = T(U\eta)$ for each $U \in \mathcal{G}$. To see this, let $V \in \mathcal{G}$ be arbitrary, then

$$\begin{aligned}
\langle T(V\eta), P(e_U) \rangle &= \langle PT(V\eta), e_U \rangle = \langle T(V\eta), e_U \rangle \\
&= \langle \sum_{W \in \mathcal{G}} \langle V\eta, W\eta \rangle e_W, e_U \rangle = \langle V\eta, U\eta \rangle \\
&= \langle T(V\eta), T(U\eta) \rangle
\end{aligned}$$

We now show that $L_U T = TU$ on $\mathcal{H}$ for any $U \in \mathcal{G}$.

For each $V \in \mathcal{G}$, we have

$$\begin{aligned}
L_U T(V\eta) &= L_U(\sum_{W \in \mathcal{G}} \langle V\eta, W\eta \rangle e_W) = \sum_{W \in \mathcal{G}} \langle V\eta, W\eta \rangle e_{UW} \\
&= \sum_{W \in \mathcal{G}} \langle UV\eta, UW\eta \rangle e_{UW} = \sum_{W \in \mathcal{G}} \langle UV\eta, W\eta \rangle e_W \\
&= TU(V\eta).
\end{aligned}$$

Thus $L_U T = TU$.

Finally we want to prove that $P \in \tilde{\mathcal{G}}'$.

Indeed, for any $U, V \in \mathcal{G}$,

$$PL_U(e_V) = Pe_{UV} = T(UV\eta) = L_U T(V\eta) = L_U P(e_V),$$

which implies that $PL_U = L_U P$ for any $U \in \mathcal{G}$.

By identifying $\mathcal{H}$ with $T(\mathcal{H})$, we now complete the proof. □

6.2 Dilation of Dual Modular Frame Pairs

The aim of this section is to prove the dilation theorem for dual frame pairs in Hilbert C^*-modules. Our approach is different from that in [13] which involves some results that are uncertain in the Hilbert C^*-module setting.

It is well known that every frame in Hilbert space is a direct summand of Riesz basis. More generally, a dual frame pair in Hilbert space can be dilated to a Riesz basis and its dual Riesz basis (see [13]). This remains true for Hilbert C^*-module frames:

Theorem 6.2. *Let $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ be alternate dual frames for a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Then there exist a Hilbert $\mathcal{A}$-module $\mathcal{K} \supseteq \mathcal{H}$ and a Riesz basis $\{u_j\}_{j\in\mathbb{J}}$ of $\mathcal{K}$ which has a unique dual $\{u_j^*\}_{j\in\mathbb{J}}$ and satisfies*

$$Pu_j = x_j \quad \text{and} \quad Pu_j^* = y_j \quad \forall j \in \mathbb{J},$$

where P is the projection from $\mathcal{K}$ onto $\mathcal{H}$.

Proof. Let T_X and T_Y be the analysis operators of $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$, and P_X and P_Y be the orthogonal projections from $l^2(\mathcal{A})$ onto the range of T_X and T_Y respectively.

For all $x \in \mathcal{H}$ we have

$$\begin{aligned} \langle T_Y x, T_Y S_Y^{-1} y_j \rangle &= \langle x, T_Y^* T_Y S_Y^{-1} y_j \rangle = \langle x, S_Y S_Y^{-1} y_j \rangle \\ &= \langle x, y_j \rangle = \langle T_Y x, e_j \rangle = \langle T_Y x, P_Y e_j \rangle. \end{aligned} \tag{6.1}$$

Therefore $P_Y e_j = T_Y S_Y^{-1} y_j$.

Observe that for all $x \in \mathcal{H}$ we have

$$\begin{aligned} P_Y T_X x &= P_Y \sum_{j \in \mathbb{J}} \langle x, x_j \rangle e_j = \sum_{j \in \mathbb{J}} \langle x, x_j \rangle P_Y e_j \\ &= \sum_{j \in \mathbb{J}} \langle x, x_j \rangle T_Y S_Y^{-1} y_j = T_Y S_Y^{-1} \sum_{j \in \mathbb{J}} \langle x, x_j \rangle y_j \\ &= T_Y S_Y^{-1} x. \end{aligned} \tag{6.2}$$

Note that in the third equality we use (6.1).

Let

$$\mathcal{K} = \mathcal{H} \oplus P_Y^{\perp} l^2(\mathcal{A}) \quad \text{and} \quad u_j = x_j \oplus P_Y^{\perp} e_j.$$

It is easy to see that $\{u_j\}_{j \in \mathbb{J}}$ is a Bessel sequence of $\mathcal{K}$. Then we have the corresponding analysis operator $T_U : \mathcal{K} \to l^2(\mathcal{A})$ given by

$$T_U(x \oplus w) = \sum_{j \in \mathbb{J}} \langle x \oplus w, x_j \oplus P_Y^{\perp} e_j \rangle e_j$$

for any $x \in \mathcal{H}$ and $w \in P_Y^{\perp} l^2(\mathcal{A})$.

Note that for all $x \in \mathcal{H}$ and $w \in P_Y^{\perp} l^2(\mathcal{A})$ we have

$$\begin{aligned} T_U(x \oplus w) &= \sum_{j \in \mathbb{J}} \langle x \oplus w, x_j \oplus P_Y^{\perp} e_j \rangle e_j \\ &= \sum_{j \in \mathbb{J}} (\langle x, x_j \rangle + \langle w, P_Y^{\perp} e_j \rangle) e_j = T_X x + w. \end{aligned} \tag{6.3}$$

We claim that T_U is a bijection.

We first prove that the range of T_U is closed.

To see this, suppose $\phi_n \in Rang(T_U)$ and $\phi_n \to \phi$ as $n \to \infty$. Then there exist $x_n \in \mathcal{H}$ and $w_n \in P_Y^{\perp} l^2(\mathcal{A})$ such that $T_U(x_n \oplus w_n) = \phi_n$.

It follows from identity (6.3) that $T_X x_n + w_n = \phi_n \to \phi$ as $n \to \infty$.

Applying T_Y^* on the both sides we get

$$T_Y^*(T_X x_n + w_n) = T_Y^* T_X x_n = x_n \to T_Y^* \phi,$$

as $n \to \infty$.

Since the range of T_X is closed, it follows that $Rang(T_U)$ is also closed.

To show that T_U is onto, by Theorem 2.57, it is equivalent to show that T_U^* is one-to-one.

Suppose that $T_U^* \sum_{j\in\mathbb{J}} a_j e_j = 0$ for some $\{a_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$. Then we have

$$\begin{aligned} 0 &= T_U^* \sum_{j\in\mathbb{J}} a_j e_j = \sum_{j\in\mathbb{J}} a_j (x_j \oplus P_Y^\perp e_j) \\ &= \sum_{j\in\mathbb{J}} a_j x_j \oplus \sum_{j\in\mathbb{J}} a_j P_Y^\perp e_j = \sum_{j\in\mathbb{J}} a_j x_j \oplus P_Y^\perp \sum_{j\in\mathbb{J}} a_j e_j. \end{aligned} \tag{6.4}$$

Therefore $P_Y^\perp \sum_{j\in\mathbb{J}} a_j e_j = 0$, and so $\sum_{j\in\mathbb{J}} a_j e_j \in Rang(P_Y) = Rang(T_Y)$. Then there exists an element $z \in \mathcal{H}$ such that $T_Y z = \sum_{j\in\mathbb{J}} a_j e_j$.

From $T_Y z = \sum_{j\in\mathbb{J}} \langle z, y_j \rangle e_j$, we have $a_j = \langle z, y_j \rangle$ for all j.

Identity (6.4) also implies $\sum_{j\in\mathbb{J}} a_j x_j = 0$. Therefore

$$0 = \sum_{j\in\mathbb{J}} a_j x_j = \sum_{j\in\mathbb{J}} \langle z, y_j \rangle x_j = z,$$

which yields that $a_j = 0$ for all j.

We now prove that T_U is injective.

Suppose that $T_U(x \oplus w) = 0$ for some $x \in \mathcal{H}$ and $w \in P_Y^\perp l^2(\mathcal{A})$.

By (6.3) we have $T_X x + w = 0$, and so $T_X x = -w$.

Applying P_Y on the both sides we arrive at $P_Y T_X x = P_Y(-w) = 0$.

By (6.2) we can see that $0 = P_Y T_X x = T_Y S_Y^{-1} x$. Hence $x = 0$ as both T_Y and S_Y^{-1} are injective.

By Theorem 2.57, we can infer that T_U^* is also a bijection.

Now let $S_U = T_U^* T_U$.

Then S_U^{-1} is adjointable and hence bounded. Thus S_U is an invertible bounded $\mathcal{A}$-linear operator, then, by Lemma 3.6, $\{u_j\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{K}$.

Let $\{u_j^*\}_{j\in\mathbb{J}}$ be the canonical dual frame of $\{u_j\}_{j\in\mathbb{J}}$ and write $u_j^* = z_j \oplus w_j$.

For any $x \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in\mathbb{J}}\langle x, z_j\rangle\langle z_j, x\rangle &= \sum_{j\in\mathbb{J}}\langle x\oplus 0, z_j\oplus w_j\rangle\langle z_j\oplus w_j, x\oplus 0\rangle \\ &\leq D\langle x\oplus 0, x\oplus 0\rangle = D\langle x, x\rangle,\end{aligned}$$

where D is the upper bound of $\{u_j^*\}_{j\in\mathbb{J}}$. Therefore $\{z_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathcal{H}$. We denote the corresponding analysis operator by T_Z.

We claim that $x = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle z_j$ for all $x \in \mathcal{H}$.

In fact, for every $x \in \mathcal{H}$, we get

$$\begin{aligned}x\oplus 0 &= \sum_{j\in\mathbb{J}}\langle x\oplus 0, x_j\oplus P_Y^{\perp}e_j\rangle z_j\oplus w_j \\ &= \sum_{j\in\mathbb{J}}\langle x, x_j\rangle z_j \oplus \sum_{j\in\mathbb{J}}\langle x, x_j\rangle w_j,\end{aligned}$$

which implies that $x = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle z_j$.

Now for any $x \in \mathcal{H}$,

$$\begin{aligned}x\oplus 0 &= \sum_{j\in\mathbb{J}}\langle x\oplus 0, z_j\oplus w_j\rangle x_j\oplus P_Y^{\perp}e_j \\ &= \sum_{j\in\mathbb{J}}\langle x, z_j\rangle x_j \oplus \sum_{j\in\mathbb{J}}\langle x, z_j\rangle P_Y^{\perp}e_j,\end{aligned}$$

so $P_Y^{\perp}T_Z x = 0$ for all $x \in \mathcal{H}$. This yields that $Rang(T_Z) \subseteq Rang(T_Y)$.

By Lemma 4.8 we can infer that $z_j = y_j$ for all j.

Furthermore the analysis operator of $\{u_j^*\}_{j\in\mathbb{J}}$ is onto. Then, again by Lemma 4.8, we get the uniqueness of the dual of $\{u_j\}_{j\in\mathbb{J}}$.

To complete the proof, it remains to prove that both $\{u_j\}_{j\in\mathbb{J}}$ and $\{u_j^*\}_{j\in\mathbb{J}}$ are Riesz bases of $\mathcal{K}$.

Note that we have already proved that T_U is onto. Then by Theorem 4.9, we see that $\{u_j\}_{j\in\mathbb{J}}$ is a Riesz basis of $\mathcal{K}$, and so $\{u_j^*\}_{j\in\mathbb{J}}$ is also a Riesz basis as $u_j^* = S_U^{-1}u_j$ and S_U is invertible.

□

We end this section by pointing out that the dilation theorem still holds when restricted to structured frames (i.e., frames induced by unitary representations of groups). Recall that two vectors $\phi, \psi \in \mathcal{H}$ are called *dual frame vectors* (resp. *dual Riesz basis vectors*) for a unitary group $\mathcal{U}$ on $\mathcal{H}$ if $\mathcal{U}\phi$ and $\mathcal{U}\psi$ are dual frames (resp. dual Riesz bases) of $\mathcal{H}$.

The following theorem shows that if two frames generated by unitary groups are dual frames, they must be generated by the same unitary group.

Theorem 6.3. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$, and $\xi, \eta \in \mathcal{H}$ be complete frame vectors for unitary groups $\mathcal{G}_1, \mathcal{G}_2$ on $\mathcal{H}$ respectively. Suppose that $\pi : \mathcal{G}_1 \to \mathcal{G}_2$ is a group isomorphism. If $\mathcal{G}_1\xi$ and $\mathcal{G}_2\eta$ are dual frames, then $\pi(U) = U$ for each $U \in \mathcal{G}_1$.*

Proof. For arbitrary $U \in \mathcal{G}_1$ and $x \in \mathcal{H}$, we have

$$\begin{aligned} Ux &= \sum_{V\in\mathcal{G}_1} \langle Ux, V\xi\rangle \pi(V)\eta = \sum_{V\in\mathcal{G}_1} \langle x, U^{-1}V\xi\rangle \pi(V)\eta \\ &= \pi(U) \sum_{V\in\mathcal{G}_1} \langle x, U^{-1}V\xi\rangle \pi(U^{-1}V)\eta = \pi(U)x, \end{aligned}$$

which implies that $\pi(U) = U$ for all $U \in \mathcal{G}_1$. □

To complete this section we consider the dilation property of structured dual frame pairs in Hilbert C^*-modules.

Theorem 6.4. *Suppose that $\mathcal{G}$ is a unitary group on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Let ξ and η be two complete dual frame vectors for $\mathcal{G}$. Then there exists a Hilbert $\mathcal{A}$-module $\mathcal{K} \supseteq \mathcal{H}$*

and a unitary group $\tilde{\mathcal{G}}$, and a complete Riesz vector $\tilde{\xi}$ with a unique dual vector $\tilde{\eta}$ for $\tilde{\mathcal{G}}$ such that

$$P\tilde{U}\tilde{\xi} = U\xi \quad \textit{and} \quad P\tilde{U}\tilde{\eta} = U\eta,$$

where P is the projection from $\mathcal{K}$ onto $\mathcal{H}$.

Proof. Let T_ξ, T_η be the analysis operator of $\mathcal{G}\xi, \mathcal{G}\eta$, and P_ξ, P_η be the orthogonal projections from $l^2_{\mathcal{G}}(\mathcal{A})$ onto the range of T_ξ, T_η respectively.

Let $\mathcal{K} = \mathcal{H} \oplus P_Y^\perp l^2_{\mathcal{G}}(\mathcal{A})$ and $\tilde{U} = U \oplus L_U$ for each $U \in \mathcal{G}$.

One can easily verify that $\tilde{G} = \{\tilde{U} : U \in \mathcal{G}\}$ is a group of unitary operators on $\mathcal{K}$.

Let $\tilde{\xi} = \xi \oplus P_\eta^\perp e_I$, then $\tilde{U}\tilde{\xi} = U\xi \oplus L_U P_\eta^\perp e_I = U\xi \oplus P_\eta^\perp e_U$.

Then, by Theorem 6.2, $\tilde{\xi}$ is a complete frame vector for $\tilde{\mathcal{G}}$.

Let S be the frame operator of $\tilde{\mathcal{G}}\tilde{\xi}$, and $\tilde{\eta} = S^{-1}\tilde{\xi}$, as desired. □

6.3 Projective Frames

In this section we will discuss the characterization of projective frames. We just list some basic observations on this topic. We will continue this work in future.

Definition 6.5. Suppose that $\{x_n\}_{n=1}^\infty$ is a sequence of Banach space X. Then $\{x_n\}_{n=1}^\infty$ is called a *projective frame* of X if it is the projective image (i.e. apply a bounded projection) of a (bounded unconditional) basis for a larger Banach space.

In [13], it was proved that a sequence $\{x_n\}_{n=1}^\infty$ of a Banach space X is a projective frame if and only if there exists a sequence $\{y_n\}_{n=1}^\infty \subseteq X^*$ such that

$$x = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n \tag{6.5}$$

holds true for all $x \in X$.

We want to find a intrinsic characterization of projective frames in Banach spaces, which does not assume the additional hypothesis of the associated dual. In other words, we are interested in finding a characterization of a sequence $\{x_n\}_{n=1}^{\infty}$ in Banach space X that admits a generalized dual $\{y_n\}_{n=1}^{\infty}$ in the sense of equation (6.5)

We first look at the Hilbert space case.

Theorem 6.6. *In finite dimensional Hilbert spaces every generating sequence is a projective frame.*

Proof. Let H be a Hilbert space and $\{x_n\}_{n=1}^{\infty}$ a generating sequence of H.

We can find an integer $N > 0$ such that $\{x_1, x_2, \dots, x_N\}$ generates H. Since the dimension of H is finite, it follows that $\{x_1, x_2, \dots, x_N\}$ is a frame of H.

Let $\{x_1^*, x_2^*, \dots, x_N^*\}$ be any dual frame of $\{x_1, x_2, \dots, x_N\}$.

Now let

$$y_n = \begin{cases} x_n^* & \text{if } 1 \leq n \leq N; \\ 0 & \text{if } n > N, \end{cases}$$

One can easily check that $x = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n$ holds true for all $x \in H$. □

The following example shows that Theorem 6.6 is no longer true in general for infinite dimensional Hilbert space.

Example 6.7. Suppose that H is a infinite-dimensional Hilbert space with an orthonormal basis $\{e_1, e_2, \dots\}$.

For each n, let $x_n = \sum_{i=1}^{n} e_i$.

Then $\{x_n\}_{n=1}^{\infty}$ is a generating sequence of H, but it is not a projective sequence.

Indeed, assume on the contrary that $\{x_n\}_{n=1}^{\infty}$ is a projective frame.Then there exist a sequence $\{y_n\}_{n=1}^{\infty} \subseteq H$ such that for each $x \in H$ we have

$$x = \sum_{n-1}^{\infty} \langle x, y_n \rangle x_n.$$

We first consider the case that $x = e_1$. We have

$$e_1 = \sum_{n=1}^{\infty} \langle e_1, y_n \rangle (e_1 + e_2 + \cdots + e_n).$$

It follows that

$$\begin{array}{rcl} 1 & = & \sum_{n=1}^{\infty} \langle e_1, y_n \rangle \\ 0 & = & \sum_{n=2}^{\infty} \langle e_1, y_n \rangle \\ 0 & = & \sum_{n=3}^{\infty} \langle e_1, y_n \rangle \\ \vdots & \vdots & \vdots \end{array}$$

which yields that $\langle e_1, y_1 \rangle = 1$ and $\langle e_1, y_n \rangle = 0$ for $n \geq 2$.

Therefore $y_1 = e_1$.

For the case of $x = e_2$, we have

$$e_2 = \sum_{n=1}^{\infty} \langle e_2, y_n \rangle (e_1 + e_2 + \cdots + e_n),$$

which implies that

$$\begin{array}{rcl} 0 & = & \sum_{n=1}^{\infty} \langle e_2, y_n \rangle \\ 1 & = & \sum_{n=2}^{\infty} \langle e_2, y_n \rangle \\ 0 & = & \sum_{n=3}^{\infty} \langle e_2, y_n \rangle \\ \vdots & \vdots & \vdots \end{array}$$

It follows that $\langle e_2, y_1 \rangle = -1, \langle e_2, y_2 \rangle = 1$ and $\langle e_2, y_n \rangle = 0$ for all $n \geq 3$.

But we already know that $y_1 = e_1$ and so $\langle e_2, y_1 \rangle = \langle e_2, e_1 \rangle = 0$, a contradiction.

In future we will focus on finding an intrinsic characterization of projective frames in infinite dimensional Hilbert spaces as well as in Banach spaces, even more generally, in Hilbert C^*-modules.

CHAPTER 7

PALEY-WIENER TYPE PERTURBATION OF FRAMES AND RIESZ BASES IN HILBERT C^*-MODULES

In this chapter we shall see that in a finitely or countably generated Hilbert C^*-module $\mathcal{H}$ if a sequence $\{y_j\}_{j\in\mathbb{J}}$ in $\mathcal{H}$ is "sufficiently near" to a given frame in $\mathcal{H}$, then $\{y_j\}_{j\in\mathbb{J}}$ is also a frame of $\mathcal{H}$. Hence it will follows that various properties of frames $\{x_j\}_{j\in\mathbb{J}}$ in Hilbert C^*-module $\mathcal{H}$ are "stable" in the sense that they are conserved by every sequence $\{y_j\}_{j\in\mathbb{J}}$ "sufficiently near" to the frame $\{x_j\}_{j\in\mathbb{J}}$. Our first result in this chapter extends the Casazza-Christensen's perturbation theorem of frames to Hilbert C^*-modules. But for the case of Riesz bases in Hilbert C^*-modules, the stability of Riesz bases is quite different from that of frames. We will give a complete characterization on all the Riesz bases for Hilbert C^*-modules such that the perturbation (under Casazza-Christensen's perturbation condition) of a Riesz basis still remains a Riesz basis.

7.1 Perturbation of Modular Frames

In this section we concentrate on the perturbation of frames in Hilbert C^*-modules. We first give a "necessary and sufficient" perturbation theorem of frames in any Hilbert C^*-module.

Theorem 7.1. *Suppose that $\mathcal{H}$ is a Hilbert C^*-module. Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame for $\mathcal{H}$ with frame bounds C_X and D_X and $\{y_j\}_{j\in\mathbb{J}}$ be a sequence of $\mathcal{H}$. Then the following statements are equivalent:*

(1) $\{y_j\}_{j\in\mathbb{J}}$ is a frame of $\mathcal{H}$;

(2) There is a constant $M > 0$ so that for all $x \in \mathcal{H}$ we have

$$\|\sum_{j\in\mathbb{J}}\langle x, x_j - y_j\rangle\langle x_j - y_j, x\rangle\| \leq M\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\|$$

and

$$\|\sum_{j\in\mathbb{J}}\langle x, x_j - y_j\rangle\langle x_j - y_j, x\rangle\| \leq M\|\sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle y_j, x\rangle\|.$$

Moreover, if $\{y_j\}_{j\in\mathbb{J}}$ is a Bessel sequence, then (1) and (2) are equivalent to

(3) There exists a constant $M > 0$ so that

$$\|\sum_{j\in\mathbb{J}}\langle x, x_j - y_j\rangle\langle x_j - y_j, x\rangle\| \leq M\|\sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle y_j, x\rangle\|$$

holds for all $x \in \mathcal{H}$.

Proof. (1)⇒(2). Let C_Y and D_Y be the frame bounds of $\{y_j\}_{j\in\mathbb{J}}$.

For every $x \in \mathcal{H}$ we have

$$\begin{aligned}
& \|\textstyle\sum_{j\in\mathbb{J}}\langle x, x_j - y_j\rangle\langle x_j - y_j, x\rangle\| \\
= & \|\textstyle\sum_{j\in\mathbb{J}}(\langle x, x_j\rangle - \langle x, y_j\rangle)(\langle x_j, x\rangle - \langle y_j, x\rangle)\| \\
= & \|\textstyle\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle + \sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle y_j, x\rangle \\
& -\textstyle\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle y_j, x\rangle - \sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle x_j, x\rangle\| \\
\leq & \|2(\textstyle\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle + \sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle y_j, x\rangle)\| \\
\leq & 2(\|\textstyle\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| + D_Y\|\langle x, x\rangle\|) \\
\leq & 2(\|\textstyle\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\| + \frac{D_Y}{C_X}\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\|) \\
= & 2(1 + \tfrac{D_Y}{C_X})\|\textstyle\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\|.
\end{aligned}$$

Similarly, we have

$$\|\sum_{j\in\mathbb{J}}\langle x, x_j - y_j\rangle\langle x_j - y_j, x\rangle\| \leq 2(1 + \frac{D_X}{C_Y})\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\|.$$

Therefore we can choose $M = max\{2(1 + \frac{D_Y}{C_X}), 2(1 + \frac{D_X}{C_Y})\}$.

(2)⇒(1). Given M in (2) and any $x \in \mathcal{H}$ we have

$$\begin{aligned}
& C_X\|\langle x,x\rangle\| \le \|\textstyle\sum_{j\in\mathbb{J}}\langle x,x_j\rangle\langle x_j,x\rangle\| \\
= \ & \|\textstyle\sum_{j\in\mathbb{J}}(\langle x,x_j-y_j\rangle + \langle x,y_j\rangle)(\langle x_j-y_j,x\rangle + \langle y_j,x\rangle)\| \\
\le \ & 2(\textstyle\sum_{j\in\mathbb{J}}\langle x,x_j-y_j\rangle\langle x_j-y_j,x\rangle + \sum_{j\in\mathbb{J}}\langle x,y_j\rangle\langle y_j,x\rangle) \\
\le \ & 2(M+1)\|\textstyle\sum_{j\in\mathbb{J}}\langle x,y_j\rangle\langle y_j,x\rangle\| \\
= \ & 2(M+1)\|\textstyle\sum_{j\in\mathbb{J}}(\langle x,y_j-x_j\rangle + \langle x,x_j\rangle)(\langle y_j-x_j,x\rangle + \langle x,x_j\rangle)\| \\
\le \ & 4(M+1)\|\textstyle\sum_{j\in\mathbb{J}}\langle x,y_j-x_j\rangle\langle y_j-x_j,x\rangle + \sum_{j\in\mathbb{J}}\langle x,x_j\rangle\langle x_j,x\rangle\| \\
= \ & 4(M+1)\|\textstyle\sum_{j\in\mathbb{J}}\langle x,x_j-y_j\rangle\langle x_j-y_j,x\rangle + \sum_{j\in\mathbb{J}}\langle x,x_j\rangle\langle x_j,x\rangle\| \\
\le \ & 4(M+1)^2\|\textstyle\sum_{j\in\mathbb{J}}\langle x,x_j\rangle\langle x_j,x\rangle\| \\
\le \ & 4(M+1)^2 D_X\|\langle x,x\rangle\|.
\end{aligned}$$

Therefore

$$\frac{C_X}{2(M+1)}\|\langle x,x\rangle\| \le \|\sum_{j\in\mathbb{J}}\langle x,y_j\rangle\langle y_j,x\rangle\| \le 2(M+1)D_X\|\langle x,x\rangle\|.$$

The *moreover* part follows from the proof of (2)⇒(1). □

Before we prove the first main result of this section, we need the following result which is due to Casazza and Christensen ([11]). It is a generalization of the famous Neumann Theorem which states that an operator U on a Banach space is invertible if $\|I - U\| < 1$.

Lemma 7.2. *([11]) Let X be a Banach space, and $U : X \to X$ a linear operator. Assume that there exist constants $\lambda_1, \lambda_2 \in (0,1)$ such that*

$$\|Ux - x\| \le \lambda_1\|x\| + \lambda_2\|Ux\|, \quad \forall x \in X.$$

Then U is bounded and invertible with

$$\|U\| \le \frac{1+\lambda_1}{1-\lambda_2} \quad \text{and} \quad \|U^{-1}\| \le \frac{1+\lambda_2}{1-\lambda_1}.$$

We are now in a position to prove the following theorem.

Theorem 7.3. *Suppose that $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame for $\mathcal{H}$ with frame bounds C and D. Suppose that $\{y_j\}_{j\in\mathbb{J}}$ is a sequence of $\mathcal{H}$ and there exist $\lambda_1, \lambda_2, \mu \geq 0$ such that $\max\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\} < 1$. Then $\{y_j\}_{j\in\mathbb{J}}$ is also a frame for $\mathcal{H}$ with bounds*

$$(\frac{(1-\lambda_1)\sqrt{C}-\mu}{1+\lambda_2})^2 \quad and \quad (\frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2})^2,$$

if one of the following conditions is fulfilled for any finite sequence $\{c_j\}_{j=1}^n \subseteq \mathcal{A}$ and all $x \in \mathcal{H}$:

$$\begin{aligned} &\|\sum_{j\in\mathbb{J}}\langle x, x_j - y_j\rangle\langle x_j - y_j, x\rangle\|^{\frac{1}{2}} \\ \leq\ & \lambda_1\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\|^{\frac{1}{2}} + \lambda_2\|\sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle y_j, x\rangle\|^{\frac{1}{2}} + \mu\|x\|; \end{aligned} \tag{7.1}$$

or

$$\|\sum_{j=1}^n c_j(x_j - y_j)\| \leq \lambda_1\|\sum_{j=1}^n c_j x_j\| + \lambda_2\|\sum_{j=1}^n c_j y_j\| + \mu\|\sum_{j=1}^n c_j c_j^*\|^{\frac{1}{2}}. \tag{7.2}$$

Proof. Let T_X and S_X denote the analysis operator and frame operator of $\{x_j\}_{j\in\mathbb{J}}$ respectively.

Assume first that condition (7.1) holds true for all $x \in \mathcal{H}$.

We now define an operator $T_Y : \mathcal{H} \to l^2(\mathcal{A})$ by

$$T_Y x = \sum_{j\in\mathbb{J}}\langle x, y_j\rangle e_j.$$

Then condition (7.1) turns to be

$$\|T_X x - T_Y x\| \leq \lambda_1\|T_X x\| + \lambda_2\|T_Y x\| + \mu\|x\|.$$

On one hand we have

$$(1-\lambda_2)\|T_Y x\| \leq (1+\lambda_1)\|T_X x\| + \mu\|x\|,$$

which implies that

$$\|T_Y x\| \leq \frac{1}{1-\lambda_2}[(1+\lambda_1)\|T_X x\| + \mu\|x\|] \leq \frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2}\|x\|.$$

Therefore $\{y_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with Bessel bound $(\frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2})^2$.

On the other hand we have

$$(1-\lambda_1)\|T_X x\| - \mu\|x\| \leq (1+\lambda_2)\|T_Y x\|.$$

Therefore

$$\|T_Y x\| \geq \frac{1}{1+\lambda_2}[(1-\lambda_1)\|T_X x\| - \mu\|x\|] \geq \frac{(1-\lambda_1)\sqrt{C}-\mu}{1+\lambda_2}\|x\|,$$

which implies that $\{y_j\}_{j\in\mathbb{J}}$ is a frame.

Suppose now that condition (7.2) holds. Then for each $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$ we have

$$\|\sum_{j=1}^{n} c_j y_j\| \leq \frac{1}{1-\lambda_2}[(1+\lambda_1)\|\sum_{j=1}^{n} c_j x_j\| + \mu\|\sum_{j=1}^{n} c_j c_j^*\|^{\frac{1}{2}}],$$

which yields that

$$\|\sum_{j=1}^{n} c_j y_j\| \leq \frac{1}{1-\lambda_2}[(1+\lambda_1)\|\sum_{j=1}^{\infty} c_j x_j\| + \mu\|\sum_{j=1}^{\infty} c_j c_j^*\|^{\frac{1}{2}}].$$

Furthermore, we obtain

$$\|\sum_{j=1}^{\infty} c_j y_j\| \leq \frac{1}{1-\lambda_2}[(1+\lambda_1)\|\sum_{j=1}^{\infty} c_j x_j\| + \mu\|\sum_{j=1}^{\infty} c_j c_j^*\|^{\frac{1}{2}}].$$

Therefore we can define a bounded operator $U : \mathcal{H} \to l^2(\mathcal{A})$ by

$$U\{c_j\} = \sum_{j\in\mathbb{J}} c_j y_j,$$

which satisfying

$$\|U\{c_j\}\| \leq \frac{1}{1-\lambda_2}[(1+\lambda_1)\|T_X^*\{c_j\}\| + \mu\|\{c_j\}\|] \leq \frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2}\|\{c_j\}\|.$$

By Proposition 3.11, $\{y_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with bound $(\frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2})^2$.

Note that for each $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$ we also have

$$\|\sum_{j\in\mathbb{J}} c_j(x_j - y_j)\| \le \lambda_1\|\sum_{j\in\mathbb{J}} c_j x_j\| + \lambda_2\|\sum_{j\in\mathbb{J}} c_j y_j\| + \mu\|\sum_{j\in\mathbb{J}} c_j c_j^*\|^{\frac{1}{2}}.$$

Then for each $x \in \mathcal{H}$, letting $\{c_j\} = T_X S_X^{-1}x$, we get

$$\begin{aligned}\|x - UT_X S_X^{-1}x\| &\le \lambda_1\|x\| + \lambda_2\|UT_X S_X^{-1}x\| + \mu\|T_X S_X^{-1}x\| \\ &\le \lambda_1\|x\| + \frac{\mu}{\sqrt{C}}\|x\| + \lambda_2\|UT_X S_X^{-1}x\|.\end{aligned}$$

By Lemma 7.2, $UT_X S_X^{-1}$ is invertible with

$$\|UT_X S_X^{-1}\| \le \frac{1 + \lambda_1 + \frac{\mu}{\sqrt{C}}}{1 - \lambda_2}$$

and

$$\|(UT_X S_X^{-1})^{-1}\| \le \frac{1 + \lambda_2}{1 - (\lambda_1 + \frac{\mu}{\sqrt{C}})}.$$

Now for arbitrary $x \in \mathcal{H}$, we have

$$x = UT_X S_X^{-1}(UT_X S_X^{-1})^{-1}x = \sum_{j\in\mathbb{J}} \langle (UT_X S_X^{-1})^{-1}x, S_X^{-1}x_j\rangle y_j.$$

It follows that

$$\begin{aligned}&\|x\|^4 \\ =\ & \|\langle x, x\rangle\|^2 \\ =\ & \|\sum_{j\in\mathbb{J}} \langle (UT_X S_X^{-1})^{-1}x, S_X^{-1}x_j\rangle\langle y_j, x\rangle\|^2 \\ \le\ & \|\sum_{j\in\mathbb{J}} \langle (UT_X S_X^{-1})^{-1}x, S_X^{-1}x_j\rangle\langle S_X^{-1}x_j, (UT_X S_X^{-1})^{-1}x\rangle\| \cdot \|\sum_{j\in\mathbb{J}} \langle x, y_j\rangle\langle y_j, x\rangle\| \\ \le\ & \frac{1}{C}\|\langle (UT_X S_X^{-1})^{-1}x, (UT_X S_X^{-1})^{-1}x\rangle\| \cdot \|\sum_{j\in\mathbb{J}} \langle x, y_j\rangle\langle y_j, x\rangle\| \\ \le\ & \frac{1}{C}\left(\frac{1 + \lambda_2}{1 - (\lambda_1 + \frac{\mu}{\sqrt{C}})}\right)^2\|x\|^2 \cdot \|\sum_{j\in\mathbb{J}} \langle x, y_j\rangle\langle y_j, x\rangle\|.\end{aligned}$$

Note that in the second inequality we apply the fact that $\{S_X^{-1}x_j\}_{j\in\mathbb{J}}$ is a frame with frame bounds $\frac{1}{D}$ and $\frac{1}{C}$.

It follows that

$$(\frac{(1-\lambda_1)\sqrt{C}-\mu}{1+\lambda_2})^2\|x\|^2 \le \|\sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle y_j, x\rangle\|.$$

□

7.2 Perturbation of Modular Riesz Bases

For the extension of the second part of Theorem 1.1, we first point out that if $\mu = 0$ in the condition (7.2) of Theorem 7.3, then $\{y_j\}_{j\in\mathbb{J}}$ is a Riesz basis provided that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

Theorem 7.4. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ and $\{x_j\}_{j\in\mathbb{J}}$ a Riesz basis for $\mathcal{H}$. Suppose that $\{y_j\}_{j\in\mathbb{J}}$ is a sequence of $\mathcal{H}$ and there exist $\lambda_1, \lambda_2 \in [0, 1)$. If*

$$\|\sum_{j\in\mathbb{J}} c_j(x_j - y_j)\| \le \lambda_1\|\sum_{j\in\mathbb{J}} c_jx_j\| + \lambda_2\|\sum_{j\in\mathbb{J}} c_jy_j\| \tag{7.3}$$

holds for all finite sequence $\{c_j\}_{j=1}^n \subseteq \mathcal{A}$, then $\{y_j\}_{j\in\mathbb{J}}$ is also a Riesz basis.

Proof. We first claim that $y_j \neq 0$ for each j.

Assume to the contrary that there exists j_0 such that $y_{j_0} = 0$. Choose $\{c_j\} = e_{j_0}$, then we have

$$\|x_{j_0}\| \le \lambda_1\|x_{j_0}\|,$$

which implies that $x_{j_0} = 0$, a contradiction.

By Theorem 7.3, we see that $\{y_j\}_{j\in\mathbb{J}}$ is also a frame of $\mathcal{H}$. We denote the analysis operators of $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ by T_X and T_Y respectively.

We now claim that $Rang(T_X) = Rang(T_Y)$.

If $\{c_j\} \in KerT_X^*$, then we have

$$\|T_Y^*\{c_j\}\| \le \lambda_2\|T_Y^*\{c_j\}\|,$$

which leads to $\{c_j\} \in KerT_Y^*$.

In the same manner we can show that $KerT_Y^* \subseteq KerT_X^*$, and so $KerT_X^* = KerT_Y^*$.

It follows from Proposition 3.12 that both $Rang(T_X^*)$ and $Rang(T_Y^*)$ are closed, and hence both $Rang(T_X)$ and $Rang(T_Y)$ are closed. Now applying Theorem 15.3.8 in [55] we see that $Rang(T_X) = Rang(T_Y)$.

Then by Theorem 4.1, we can infer that $\{y_j\}_{j\in\mathbb{J}}$ is also a Riesz basis of $\mathcal{H}$. □

As we have seen from Lemma 4.1 that the structure of Hilbert C^*-module Riesz bases is much more complicated than the Hilbert space Riesz bases. Therefore there is no surprise that the perturbation of Riesz bases in Hilbert C^*-modules could be quite different from that in Hilbert spaces. The following example shows that the second part of Theorem 1.1 is no longer true in general for Hilbert C^*-module Riesz bases.

Example 7.5. Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbb{N}}$ and $v = \{v_j\}_{j\in\mathbb{N}}$ in l^∞, we define

$$uv = \{u_j v_j\}_{j\in\mathbb{N}}, \quad u^* = \{\bar{u}_j\}_{j\in\mathbb{N}} \quad \textit{and} \quad \|u\| = \max_{j\in\mathbb{N}} |u_j|.$$

Then $\mathcal{A} = \{l^\infty, \|\cdot\|\}$ is a C^*-algebra.

Let $\mathcal{H} = c_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{H}$ we define

$$\langle u, v\rangle = uv^* = \{u_j \bar{v}_j\}_{j\in\mathbb{N}}.$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

For each j, let $x_j = e_j$. Obviously, $\{x_j\}_{j\in\mathbb{N}}$ is a Parseval Riesz basis of $\mathcal{H}$.

Now let

$$y_j = \begin{cases} e_1 + e_2 & \text{if } j = 1, 2, \\ e_j & \text{if } j \neq 1, 2, \end{cases}$$

and $\lambda_1 = \frac{1}{8}, \lambda_2 = \frac{15}{16}$ and $\mu = \frac{3}{4}$.

Then one can check that condition (7.2) in Theorem 7.3 is satisfied. But $\{y_j\}_{j\in\mathbb{J}}$ is not a Riesz basis.

Our second main result is to give a necessary and sufficient condition under which every perturbation $\{y_j\}_{j\in\mathbb{J}}$ of a Riesz basis $\{x_j\}_{j\in\mathbb{J}}$ is also a Riesz basis in Hilbert C^*-modules.

Theorem 7.6. *Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis of $\mathcal{H}$ with frame bounds C and D, where $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. Assume that there exist $\lambda_1, \lambda_2 \geq 0$ and $\mu > 0$ such that*

$$\max\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\} < 1.$$

Then the following are equivalent:

(i) Every sequence $\{y_j\}_{j\in\mathbb{J}}$ in $\mathcal{H}$ satisfying the following perturbation condition is again a Riesz basis:

$$\|\sum_{j=1}^{n} c_j(x_j - y_j)\| \leq \lambda_1 \|\sum_{j=1}^{n} c_j x_j\| + \lambda_2 \|\sum_{j=1}^{n} c_j y_j\| + \mu \|\sum_{j=1}^{n} c_j c_j^*\|^{\frac{1}{2}} \qquad (7.4)$$

for any $c_1, c_2, \ldots, c_n \in \mathcal{A}$.

(ii) $KerT_X^* = l^2(\mathcal{B})$, where T_X is the analysis operator of $\{x_j\}_{j\in\mathbb{J}}$ and $\mathcal{B} = \{a \in \mathcal{A} : a\mathcal{H} = \{0\}\}$.

In case the equivalent conditions are satisfied, we also have $KerT_Y^* = KerT_X^*$ and $Rang(T_Y) = Rang(T_X)$, where T_Y is the analysis operator of $\{y_j\}_{j\in\mathbb{J}}$.

Proof. From Theorem 7.3 and its proof we can infer that $\{y_j\}_{j\in\mathbb{J}}$ is a frame and satisfies

$$\|\sum_{j\in\mathbb{J}} c_j(x_j - y_j)\| \leq \lambda_1 \|\sum_{j\in\mathbb{J}} c_j x_j\| + \lambda_2 \|\sum_{j\in\mathbb{J}} c_j y_j\| + \mu \|\sum_{j\in\mathbb{J}} c_j c_j^*\|^{\frac{1}{2}}$$

for all $\{c_j\} \in l^2(\mathcal{A})$.

"*(i)* $\Rightarrow$ *(ii)*". Suppose first that any sequence $\{y_j\}_{j\in\mathbb{J}}$ satisfying condition (7.4) is a Riesz basis. We now show that $KerT_X^* = l^2(\mathcal{B})$.

Obviously, $l^2(\mathcal{B}) \subseteq KerT_X^*$.

Now pick an arbitrary $\{a_j\}_{j\in\mathbb{J}} \in KerT_X^*$. We need to prove that $a_j\mathcal{H} = \{0\}$ for each j.

Assume on the contrary that there exists $j_0 \in \mathbb{J}$ such that $a_{j_0}\mathcal{H} \neq \{0\}$. We have two cases:

Case 1 There exists $j_1 \in \mathbb{J}$ such that $a_{j_0}x_{j_1} \neq 0$.

Choose $M > 0$ such that $\frac{\|x_{j_1}\|}{M} \leq \mu$.

Consider sequence $\{z_j\}_{j\in\mathbb{J}}$ given by

$$z_j = \begin{cases} x_{j_0} - \frac{1}{M}x_{j_1}, & \text{if } j = j_0; \\ x_j, & \text{otherwise.} \end{cases}$$

One can check that $\{z_j\}_{j\in\mathbb{J}}$ satisfies condition (7.4).

Now let $\{c_j\}$ be a sequence such that

$$c_j = \begin{cases} Ma_{j_0}, & \text{if } j = j_0; \\ a_{j_0}, & \text{if } j = j_1; \\ a_j, & \text{otherwise.} \end{cases}$$

Observe that

$$\sum_{j\in\mathbb{J}} c_j z_j = \sum_{j\in\mathbb{J}} a_j x_j = 0.$$

But

$$c_{j_0} z_{j_0} = -a_{j_0} x_{j_1} \neq 0.$$

Thus $\{z_j\}_{j\in\mathbb{J}}$ is not a Riesz basis, a contradiction.

Case 2 $a_{j_0}x_j = 0$ for all $j \in \mathbb{J}$.

We pick $z \in \mathcal{H}$ such that $a_{j_0}z \neq 0$, and $N > 0$ such that $\frac{\sqrt{2}}{N}\|z\| \leq \mu$.

Consider a sequence $\{z_j\}_{j\in\mathbb{J}}$ defined by

$$z_j = \begin{cases} x_1 + \frac{1}{N}z, & \text{if } j = 1; \\ x_2 - \frac{1}{N}z, & \text{if } j = 2; \\ x_j, & \text{otherwise.} \end{cases}$$

Note that $\{z_j\}_{j\in\mathbb{J}}$ also satisfies condition (7.4).

Letting $c_j = a_{j_0}$ for all j, we have

$$\sum_{j\in\mathbb{J}} c_j z_j = \sum_{j\in\mathbb{J}} a_{j_0} x_j = 0.$$

But

$$c_1 z_1 = -c_2 z_2 = \frac{a_{j_0}}{N} z \neq 0,$$

which contradicts the fact that $\{z_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

"$(ii) \Rightarrow (i)$". Suppose now that $KerT_X^* = l^2(\mathcal{B})$ and $\{y_j\}_{j\in\mathbb{J}}$ is an arbitrary sequence satisfying condition (7.4).

By Corollary 4.3, we consider any sequence $\{a_j\} \in l^2(\mathcal{A})$ such that $\sum_{j\in\mathbb{J}} a_j y_j = 0$.

We claim that $\{a_j\} \in l^2(\mathcal{B})$.

Assume on the contrary that $\{a_j\} \notin l^2(\mathcal{B})$. By Theorem 2.57 we have

$$l^2(\mathcal{A}) = KerT_X^* \oplus (KerT_X^*)^\perp = l^2(\mathcal{B}) \oplus (l^2(\mathcal{B}))^\perp.$$

Thus $\{a_j\}$ has a unique decomposition

$$\{a_j\} = \{a_j^{(1)}\} \oplus \{a_j^{(2)}\},$$

where $\{a_j^{(1)}\} \in l^2(\mathcal{B})$ and $\{a_j^{(2)}\}$ is a nonzero sequence in $(l^2(\mathcal{B}))^\perp$.

It follows that

$$
\begin{aligned}
&\|\sum_{j\in\mathbb{J}} a_j y_j\| \\
= \;& \|\sum_{j\in\mathbb{J}} (a_j^{(1)} + a_j^{(2)}) y_j\| = \|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\| \\
= \;& \|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j - \sum_{j\in\mathbb{J}} a_j^{(2)} (x_j - y_j)\| \\
\geq \;& \|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j\| - \|\sum_{j\in\mathbb{J}} a_j^{(2)} (x_j - y_j)\| \\
\geq \;& \|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j\| - \lambda_1 \|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j\| - \lambda_2 \|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\| - \mu\|\{a_j^{(2)}\}\| \\
= \;& (1-\lambda_1)\|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j\| - \lambda_2 \|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\| - \mu\|\{a_j^{(2)}\}\| \\
\geq \;& [(1-\lambda_1)\sqrt{C}]\|\{a_j^{(2)}\}\| - \lambda_2 \|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\| - \mu\|\{a_j^{(2)}\}\| \\
= \;& [(1-\lambda_1)\sqrt{C} - \mu]\|\{a_j^{(2)}\}\| - \lambda_2 \|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\|.
\end{aligned}
$$

Note that in the last inequality we apply Proposition 3.13.

It follows that

$$0 = \|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\| \geq \frac{(1-\lambda_1)\sqrt{C} - \mu}{1+\lambda_2}\|\{a_j^{(2)}\}\|,$$

and hence $a_j^{(2)} = 0$ for each j, a contradiction.

Thus we can infer that $KerT_Y^* = l^2(\mathcal{B})$.

To show that $\{y_j\}_{j\in\mathbb{J}}$ is a Riesz basis, it remains to show that $y_j \neq 0$ for each j.

Assume on the contrary that $y_{j_0} = 0$ for some $j_o \in \mathbb{J}$. For any $a \in \mathcal{A}$, let

$$c_j = \begin{cases} a, & \text{if } j = j_0; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{j\in\mathbb{J}} c_j y_j = 0$, i.e. $\{c_j\}_{j\in\mathbb{J}} \in KerT_Y^*$.

Since $KerT_X^* = KerT_Y^*$, we see that $ax_{j_0} = 0$ for any $a \in \mathcal{A}$. Therefore $x_{j_0} = 0$ which leads to a contradiction with the assumption that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis. This completes the proof. □

Remark 7.7. Case 2 in the above proof states that there exists an element $a \in \mathcal{A}$ such that $ax_j = 0$ for all j but $a\mathcal{H} \neq \{0\}$, where $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis of a Hilbert $\mathcal{A}$-module $\mathcal{H}$. Though this never occurs in Hilbert spaces, it could happen in Hilbert C^*-modules. For example, let's consider the C^*-algebra $\mathcal{A} = M_{2\times 2}(\mathbb{C})$ of all 2×2 complex matrices.

Let $\mathcal{H} = \mathcal{A}$ and for any $x, y \in \mathcal{H}$ define

$$\langle x, y\rangle = xy^*.$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

Choose

$$x_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

One can check that $\{x_1, x_2\}$ is a Riesz basis of $\mathcal{H}$.

Pick

$$a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then we have

$$ax_1 = ax_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

But, it is obvious that

$$a\mathcal{H} \neq \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}.$$

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