# ITERATIVELY REWEIGHTED LEAST SQUARES MINIMIZATION WITH PRIOR INFORMATION: A NEW APPROACH

by

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## ABSTRACT

Iteratively reweighted least squares (IRLS) algorithms provide an alternative to the more standard $l_1$-minimization approach in compressive sensing. Daubechies et al. introduced a particularly stable version of an IRLS algorithm and rigorously proved its convergence in 2010. They did not, however, consider the case in which prior information on the support of the sparse domain of the solution is available. In 2009, Miosso et al. proposed an IRLS algorithm that makes use of this information to further reduce the number of measurements required to recover the solution with specified accuracy. Although Miosso et al. obtained a number of simulation results strongly confirming the utility of their approach, they did not rigorously establish the convergence properties of their algorithm. In this paper, we introduce prior information on the support of the sparse domain of the solution into the algorithm of Daubechies et al. We then provide a rigorous proof of the convergence of the resulting algorithm.

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# CHAPTER 1:  INTRODUCTION TO COMPRESSIVE SENSING

Compressive sensing is a novel paradigm that has introduced many improvements over the more traditional methods in fields such as data compression, channel coding, inverse problems, and data acquisition.  For example, in signal recovery, Shannon's theorem has defined the traditional approach.  According to this theorem, the sampling rate must be at least twice the maximum frequency rate present in the signal.  Compressive sensing asserts that certain signals can be recovered even if the original number of measurements is much smaller.  The defining property of such signals is sparsity.  In the field of signal recovery, a signal is sparse if it can be concisely expressed in a particular basis.

To make the presentation of the concepts more concrete, we will continue to restrict our exposition to the fields of signal recovery and data acquisition (relying on [4] ) until we present the mathematical formulation of our problem.  One of the most remarkable results in the field of compressive sensing is the following:  a sensor can efficiently capture the information in a sparse signal without having to acquire the entire data.  The signal can then be recovered from a small number of measurements.  In traditional methods, massive data acquisition is followed by compression, during which only the essential information is stored.  The typical example cited is that of a digital camera which has millions of sensors but stores the picture in only a few hundred kilobytes.  Compressive sensing approach allows one to obtain the essential information without preliminary massive data acquisition.  The main requirement pertains to the sparsity of the signal:  The sparser the signal, the fewer the number of measurements needed to recover it.

Another important idea in the field of compressive sensing is incoherence. The mathematics of compressive sensing usually requires two matrices: one matrix is used for sensing and another is used for representation. Each of these matrices consists of orthogonal bases arranged in matrix form. The smaller the maximal correlation between the vectors from the 'sensing' basis with the vectors of the 'representation' basis, the larger the incoherence between these bases. The larger the incoherence, the smaller the number of samples needed for recovery.

Matrix $\Psi = [\psi_1 \psi_2 .. \psi_n]$, the columns of which are an orthogonal basis, is used to represent a signal $f$. Each $\psi_i \in \mathbb{R}^n$ and $\Psi$ is an $N \times N$ matrix. Hence we can write $f = \Psi^* x$, where $x_i = \langle f, \psi_i \rangle$. With the sensing matrix $\Phi'$, the coherence between the sensing basis $\Phi'$ and the representation basis $\Psi$ is $\mu(\Phi', \Psi) = \sqrt{n} \cdot \max_{1 \le j, j \le n} \left| \langle \varphi_k', \psi_j \rangle \right|$. A commonly adapted strategy is to choose $\Phi'$ to be a random matrix. Such matrices are largely incoherent with any fixed orthogonal basis $\Psi$. Finally, we are ready to treat the problem mathematically.

Let $\Phi = \Phi' \Psi^*$, where $f = \Psi^* x$ and $y = \Phi' f$ so that

$$\Phi x = y \tag{1.1}$$

with $\Phi$ an $m \times N$ matrix such that $m < N$. Let $N := N(\Phi)$ be the null space of $\Phi$ and $x_0$ a solution of (1.1). Then the set $F(y) := \Phi^{-1}(y)$ of all solutions of (1.1) is given by $F(y) := x_0 + N$. The underdetermined system of equations (1.1) has infinitely many solutions, with the *k*-sparse solutions of this system having only $k$ nonzero components. The support of vector $x$ consists of the nonzero components of $x$. Thus a sparse solution has a support of small cardinality.

Combinatorial methods can be used to find the $k$-sparse solutions of (1.1) directly, but these methods are numerically prohibitive. Instead, $l_1$-minimization has been used in many disciplines to obtain the solutions of underdetermined systems. If $\Phi$ and $y$ in (1.1) satisfy certain conditions, and there is a $k$-sparse solution, then the unique solution to the $l_1$-minimization problem

$$x := \arg\min_{z \in F(y)} \|z\|_{l_1^N} \tag{1.2}$$

is also the solution of (1.1).

The following theorem [3] illustrates the use of several concepts introduced above.

*Fix* $f \in \mathbb{R}^N$ *and suppose that the coefficient sequence* $x$ *of* $f$ *in the basis* $\Psi$ *is S-sparse. Select m measurements in the* $\Phi'$ *domain uniformly at random. Then if*

$$m \geq C \cdot \mu^2(\Phi', \Psi) \cdot S \cdot \log(n)$$

*for some positive constant C, the solution to (1.2) is exact with overwhelming probability. More precisely, the probability of success exceeds* $1 - \delta$ *if* $m \geq C \cdot \mu^2(\Phi', \Psi) \cdot S \cdot \log(n/\delta)$ .

Observe that the smaller the sparsity S and coherence $\mu$, the fewer samples are needed for recovery. It is likely that the number of random measurements will be far less than demanded by the signal size. Moreover, it is not necessary to know if $f$ is S-sparse, neither must one know the sparsity pattern. If $f$ is indeed S-sparse, it will be obtained as a solution to the convex optimization problem (1.2).

Next we introduce a widely-used property of matrix $\Phi$. $\Phi$ satisfies the restricted isometry property (RIP) of order $L$ with constant $\delta \in (0,1)$ if for each vector $z$ with sparsity L, condition

$$(1-\delta)\|z\|_{l_2^N} \le \|\Phi z\|_{l_2^m} \le (1+\delta)\|z\|_{l_2^N} \tag{1.3}$$

holds. This property requires that all subsets of S columns of $\Phi$ are nearly orthogonal. Many theoretical results on $l_1$-minimization make use of (1.3) in order to draw conclusions about the nature of the solution recovered. The following result due to Candes ([4]) illustrates this point.

*Assume that* $\delta_{2S} < \sqrt{2}-1$. *Then the solution* $x^*$ *to (1.2) obeys*

$$\begin{aligned} &\|x^*-x\|_{l_2} \le C_0 \cdot \|x-x_S\|_{l_1} / \sqrt{S} \ \ \text{and} \\ &\|x^*-x\|_{l_1} \le C_0 \cdot \|x-x_S\|_{l_1} \end{aligned} \tag{1.4}$$

*for some constant* $C_0$, *where* $x_S$ *is the vector* $x$ *with all but the largest S components set to zero.*

This is a deterministic result relying only on the RIP property of a matrix. If an S-sparse solution exists, then it will be obtained as a result of $l_1$-minimization. Moreover, even if an S-sparse solution does not exist, but the matrix satisfies the RIP property, then S largest components of the solution will be identified. Thus the reconstruction will include the most significant pieces of information even if no particular care was taken to measure those pieces beforehand. Once again, this is perhaps the main purpose of compressive sensing.

A legitimate question is, of course: how can we find matrices that satisfy the RIP? While, considering the combinatorial nature of the problem, determining if a given matrix satisfies the RIP is not plausible for large matrices, certain groups of matrices are known to satisfy the RIP with high probabilities. Candes and Wakin [4] provide the following among examples of such matrices: i.) matrix $\Phi$ formed by uniformly sampling N column vectors at random on the unit sphere of $\mathbb{R}^m$; ii.) $\Phi$ formed by i.i.d. elements sampled from $N(0, \frac{1}{M})$; iii.) $\Phi$ formed by i.i.d. elements taken from symmetric Bernoulli distribution. Provided that

$$m \geq C \cdot S \cdot \log(n / S), \tag{1.5}$$

where C is an instance-specific constant, matrices in i.)-iii.) satisfy the RIP. Moreover, if $\Psi$ is an arbitrary orthobasis and $\Phi$ is a matrix mentioned in i.)-iii.), then $\Phi\Psi$ satisfies the RIP given (1.5), with C being an instance-specific constant.

# CHAPTER 2: ITERATIVELY REWEIGHTED LEAST SQUARES MINIMIZATION

Among the alternative methods that can be both more efficient and simpler than (1.2) is *iteratively reweighted least squares (IRLS) minimization*. The basic result is that if (1.2) has a solution $x*$ with no vanishing coordinates, then the unique solution of

$$x^w := \arg\min_{z \in F(y)} \|z\|_{l_2^N(w)}, \ \textit{where} \ w := (w_1, ..., w_N) \ \textit{and} \ w_j := \left|x_j^*\right|^{-1}, \tag{2.1}$$

coincides with $x*$. The condition on non-vanishing coordinates is rather restrictive, and ad hoc solutions are necessary to handle the weight vectors not conforming to this condition. When these solutions are implemented, the algorithm might not converge [8]. With weights defined in a particular manner, Daubechies et al. [6] proposed an algorithm that does not require this condition. Daubechies et al. [6] also proved the convergence of their algorithm and examined its rate of convergence.

Another variation of an IRLS algorithm was put forth by Miosso et al. [9]. They considered a case in which prior information on the support of $x$ is available. That is, there is information on the positions of the nonzero components. More precisely, let $\Delta$ be the subset of positions in $\{1, 2, \ldots, N\}$ which are known to belong to the support of x, that is,

$$x_k \neq 0 \ \ \forall \, k \in \Delta \, . \tag{2.2}$$

The algorithm of Miosso et al. [9] relies on the observation that if the positions in $\Delta$ are known, then the sparse solution can be obtained by minimizing the number of nonzero components in $\Delta^c$. The algorithm has certain desirable characteristics. First, the number of required measurements is reduced by the cardinality of $\Delta$, denoted by $|\Delta|$. Moreover, the number of iterations and the computation time required for convergence are reduced. Finally, the algorithm is robust with respect to errors in prior information. If incorrect prior information on the support is used in the algorithm, the correct solution can still be obtained if the number of measurements is increased. One of the purposes of this paper is to propose an algorithm that would have the advantages of both the algorithm of Daubechies et al. [6] and that of Miosso et al. [9].

The algorithm of Daubechies et al. [6] follows.

*Algorithm 1. Let*

$$J(z,w,\varepsilon) := \frac{1}{2}\left[\sum_{j=1}^{N} z_j^{\,2} w_j + \sum_{j=1}^{N} (\varepsilon^2 w_j + w_j^{\,-1})\right], \quad z \in \mathbb{R}^N \tag{2.3}$$

*Initialize* $w^0 := (1,...,1)$ *and set* $\varepsilon_0 := 1$. *Recursively define for n = 0, 1, . . . ,*

$$x^{n+1} := \underset{z\in F(y)}{\arg\min}\, J(z, w^n, \varepsilon_n) = \underset{z\in F(y)}{\arg\min} \|z\|_{l_2(w^n)} \tag{2.4}$$

*and*

$$\varepsilon_{n+1} := \min\left(\varepsilon_n, \frac{r(x^{n+1})_{K+1}}{N}\right), \tag{2.5}$$

*where K is a fixed integer described later. Moreover, define*

$$w^{n+1} := \underset{w>0}{\arg\min}\, J(x^{n+1}, w, \varepsilon_{n+1}) \,. \tag{2.6}$$

*Stop the algorithm if* $\varepsilon_n = 0$, *in which case* $x^j := x^n$ *for* $j > n$. *If* $\varepsilon_n \neq 0$, *the algorithm will generate an infinite sequence* $(x^n)_{n\in\mathbb{N}}$ *of distinct vectors.*

Each step of the algorithm requires the solution of a least squares problem. In matrix form,

$$x^{n+1} = D_n \Phi^T (\Phi D_n \Phi^T)^{-1} y . \tag{2.7}$$

Matrix $\Phi$ contains

$$w_j^{n+1} = \left[ (x_j^{n+1})^2 + \varepsilon_{n+1}^2 \right]^{-1/2}, \; j = 1, \ldots, N, \tag{2.8}$$

on the diagonal.

Let the vector obtained from $\eta$ by setting all coordinates $\eta_i$ for $i \notin S \subset \{1, 2, \ldots, N\}$ equal to zero be denoted by $\eta_S$. $\Phi$ has the *Null Space Property* (NSP) of order L for $\gamma > 0$ if

$$\|\eta_T\|_{l_1} \le \gamma \|\eta_{T^c}\|_{l_1} \tag{2.9}$$

for all sets $T$ of cardinality not exceeding $L$ and all $\eta \in N$. It can be shown (see [6]) that if a matrix satisfies the restricted isometry property of order $L := J + J'$ for given $\delta \in (0,1)$, where $J, J' \ge 1$ are integers, then $\Phi$ has the NSP of order $J$ for

$$\gamma := \frac{1+\delta}{1-\delta}\sqrt{\frac{J}{J'}} .$$

The main result of Daubechies et al [6] is the theorem that follows.

*Theorem 1. Let K (the same index as used in the update rule (2.5)) be chosen so that* $\Phi$ *satisfies* *the null space property (2.9) of order K, with $\gamma < 1$. Then, for each $y \in \mathbb{R}^m$, the output of Algorithm 1 converges to a vector $\bar{x} \in F(y)$, with $r(\bar{x})_{K+1} = N \cdot \lim_{n\to\infty} \varepsilon_n$ and the following hold:*

*(i) If $\varepsilon := \lim_{n\to\infty} \varepsilon_n = 0$, then $\bar{x}$ is K-sparse; in this case there is therefore a unique $l_1$-minimizer $x^*$, and $\bar{x} = x^*$. Moreover, we have, for $k \le K$, and any $z \in F(y)$,*

$$\|z-\bar{x}\|_{l_1} \le c\sigma_k(z)_{l_1} \quad \text{with } c := \frac{2(1+\gamma)}{1-\gamma}. \tag{2.10}$$

*(ii) If $\varepsilon := \lim_{n\to\infty} \varepsilon_n > 0$, then $\bar{x} = x^{\varepsilon}$.*

*(iii) In this last case, if $\gamma$ satisfies the stricter bound $\gamma < 1 - \frac{2}{K+2}$, then we have, for all $z \in F(y)$ and any $k < K - \frac{2\gamma}{1-\gamma}$, that*

$$\|z-\bar{x}\|_{l_1} \le \tilde{c}\,\sigma_k(z)_{l_1}, \quad \text{with } \tilde{c} := \frac{2(1+\gamma)}{1-\gamma}\left[\frac{K-k+3/2}{K-k-2\gamma/(1-\gamma)}\right]. \tag{2.11}$$

*(iv) If $F(y)$ contains a vector $z$ of sparsity $k < K - \frac{2\gamma}{1-\gamma}$, then $\varepsilon = 0$ and $\bar{x} = x^* = z$.*

In order to prove their result, Daubechies et al. [6] used a weighted $l_2(w)$-norm. If $w_j > 0$ for all $j \in \{1,\dots,N\}$, $l_2(w)$ is a Hilbert space with inner product

$$\langle u, w\rangle_w := \sum_{j=1}^{N} w_j u_j w_j. \tag{2.12}$$

Moreover, they defined

$$x^w := \arg\min_{z\in F(y)} \|z\|_{l_2^N(w)}. \tag{2.13}$$

The minimizer in (2.13) is unique by strict convexity of the weighted norm. By a well-known characterization of the best approximation on a Hilbert space, the minimizer satisfies

$$\left\langle x^w, \eta \right\rangle_w = 0 \ \ \forall\, \eta \in N. \tag{2.14}$$

Moreover, any element that satisfies (2.14) is equal to $x^w$.

Certain other constructions are also useful. First is the $l_1$-error $\sigma_j(z)_{l_1}$. Let $\Sigma_k$ be the set of all $x \in \mathbb{R}^N$ with support that has cardinality at most *k*. For any $z \in \mathbb{R}^N$ and any $j = 1,\ldots,N,$ let

$$\sigma_j(z)_{l_1} := \inf_{w\in\Sigma_j} \|z-w\|_{l_1^N}. \tag{2.15}$$

Note that $\sigma_j(z)_{l_1} = \sum_{v>j} r(z)_v$.

Second is the following functional used in the proof of convergence:

$$f_\varepsilon(z) := \sum_{j=1}^{N} ({z_j}^2 + \varepsilon^2)^{1/2}. \tag{2.16}$$

The unique minimizer of this strictly convex functional is

$$x^\varepsilon := \arg\min_{z\in F(y)} f_\varepsilon(z). \tag{2.17}$$

# CHAPTER 3: CONVERGENCE OF IRLS

In this chapter, we present the main steps in the proof of convergence of IRLS, with more elaborated proofs in many cases. For ease of reference, we will adopt the lemma and theorem numbers of Daubechies et al. [6]. The proof of Theorem 1 relies on a number of lemmas, most of which establish certain inequalities. The following lemma implies that if $\Phi$ has full rank, then unique $l_1$-minimizers are $k$-sparse for some $k \le m$.

Lemma 2.1 *An element* $x \in F(y)$ *has minimal* $l_1$*-norm among all elements* $z \in F(y)$ *if and only if*

$$\left| \sum_{x_i \neq 0} sign(x_i)\eta_i \right| \le \sum_{x_i = 0} |\eta_i| \ \ \forall\, \eta \in N . \tag{2.18}$$

*Moreover,x is unique if and only if we have strict inequality in* (2.18) *for all* $\eta \in N$ *which are not identically zero.*

Proof : Assume that $x \in F(y)$ is a minimum $l_1$-norm element. Since $g(t) = x + t\eta$ is continuous, for any $\eta \in N$ and any $t \in \mathbb{R}$ ,

$$\sum_{i=1}^{N} |x_i + t\eta_i| \ge \sum_{i=1}^{N} |x_i| . \tag{2.19}$$

Now break up the summation into a part for which $x_i = 0$ and a part for which $x_i \neq 0$, then choose $t$ of an appropriate sign. More precisely, for a fixed $\eta \in N$ and for a sufficiently small $t$, $x_i + t\eta_i$ , and $x_i$ will have the same sign $s_i := sign(x_i)$ whenever $x_i \neq 0$. Hence (2.19) can be written as

$$t\sum_{x_i\neq 0} s_i\eta_i + \sum_{x_i=0}|t\eta_i| \geq 0\,.$$

The above inequality implies (2.18) if $t$ is chosen to have an appropriate sign. If $x$ is unique, (2.19), and thus (2.18) hold with strict inequalities for all $\eta \in N\setminus\{0\}$.

For the other direction, the idea is to add and subtract $s_i\eta_i$, use (2.18), then use the definition of absolute value ( $|s_i(x_i+\eta_i)| = |s_i||(x_i+\eta_i)| = |(x_i+\eta_i)|$). More precisely, for each $\eta \in N$,

$$\begin{aligned}\sum_{i=1}^{N}|x_i| &= \sum_{x_i\neq 0} s_i x_i = \sum_{x_i\neq 0} s_i(x_i+\eta_i) - \sum_{x_i\neq 0} s_i\eta_i \\ &\leq \sum_{x_i\neq 0} s_i(x_i+\eta_i) + \sum_{x_i=0}|\eta_i| \leq \sum_{i=1}^{N}|x_i+\eta_i|\end{aligned}\,. \tag{2.20}$$

Thus $x_i$ has minimal norm among the elements in $F(y)$. If we assume that strict inequality holds in (2.18), then we have a strict inequality in (2.20). In this case, $x$ must be unique. □

The following lemma shows that the behavior of the rearrangements and approximation errors is 'controlled'. The lemma plays a crucial role in the proof of the main convergence result.

Lemma 4.1 *The map* $z \mapsto r(z)$ *is Lipschitz continuous on* $(\mathbb{R}^N, \|\cdot\|_{l_\infty})$ *with Lipschitz constant 1; i.e., for any* $z, z' \in \mathbb{R}^N$,

$$\|r(z)-r(z')\|_{l_\infty} \le \|z-z'\|_{l_\infty}. \tag{2.21}$$

*Moreover, for any j, we have*

$$\left|\sigma_j(z)_{l_1} - \sigma_j(z')_{l_1}\right| \le \|z-z'\|_{l_1}, \tag{2.22}$$

*and for any J > j, we have*

$$(J-j)r(z)_J \le \|z-z'\|_{l_1} + \sigma_j(z')_{l_1}. \tag{2.23}$$

Proof: To prove that $\|r(z)-r(z')\|_{l_\infty} \le \|z-z'\|_{l_\infty}$, use the definition of $r(\cdot)$ and the triangle inequality on the $l_\infty$-norm. The details follow. Pick $z$ and $z'$, and any $j \in \{1,\ldots,N\}$. Let the set of indices corresponding to $j-1$ largest elements in $z'$ be denoted by $\Lambda$. Then

$$\begin{aligned} r(z)_j &\le \max_{i\in\Lambda^c}|z_i| \le \max_{i\in\Lambda^c}|z_i'| + \max_{i\in\Lambda^c}|z-z_i'| \le \max_{i\in\Lambda^c}|z_i'| + \|z-z'\|_{l_\infty} \\ &= r(z')_j + \|z-z'\|_{l_\infty} \end{aligned}. \tag{2.24}$$

Reverse the roles of $z$ and $z'$ to complete the proof of (2.21).

To prove that $\left|\sigma_j(z)_{l_1} - \sigma_j(z')_{l_1}\right| \le \|z-z'\|_{l_1}$, use the definition of $\sigma_j(z)_{l_1}$ and the triangle inequality as follows

$$\begin{aligned} \sigma_j(z)_{l_1} &= \inf_{u\in\Sigma_j}\|z-u\|_{l_1} \le \|z-u\|_{l_1} \le \|z-z'\|_{l_1} + \|z'-u\|_{l_1} \\ &= \|z-z'\|_{l_1} + \sigma_j(z')_{l_1} \end{aligned}. \tag{2.25}$$

Reverse the roles of $z$ and $z'$ to complete the proof of (2.22). □

The next lemma establishes an approximate reverse triangle inequality. If two points $z, z' \in F(y)$ have close $l_1$-norms and one of them is close to a sparse vector, then the two points are close to each other. In the proof of the main convergence result, it is used once explicitly and twice implicitly (through Lemma 4.3).

Lemma 4.2 *Assume that NSP holds for some* $L$ *and* $\gamma < 1$. *Then for any* $z, z' \in F(y)$, *we have*

$$\|z' - z\|_{l_1} \le \frac{1+\gamma}{1-\gamma}(\|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}) . \tag{2.26}$$

Outline of the Proof: $z^T - w^T = [z_1 ..... z_L \; z_{L+1} ..... z_N] - [w_1 ............ w_N]$, where $w$ can be chosen arbitrarily as long as it has at most L nonzero components. Choose $w$ that is equal to $z$ at the $L$ largest entries of $z$. Thus, $\|z_{T^c}\|_{l_1} = \inf_{w \in \Sigma_L} \|z - w\|_{l_1^N} = \sigma_L(z)_{l_1}$. This fact and a clever rearrangement of terms, aided by the addition of zero, allow us to use Lemma 4.1 to obtain the desired inequality.

Proof: Let $T$ be the set of indices corresponding to the $L$ largest entries in $z$. Then

$$\begin{aligned}
&\|(z'-z)_{T^c}\|_{l_1} \le \|z'_{T^c}\|_{l_1} + \|z_{T^c}\|_{l_1} = \|z'\|_{l_1} - \|z'_T\|_{l_1} + \|z_{T^c}\|_{l_1} \\
&= \|z'\|_{l_1} - \|z'_T\|_{l_1} + \sigma_L(z)_{l_1} = \|z\|_{l_1} + \|z'\|_{l_1} - \|z\|_{l_1} - \|z'_T\|_{l_1} + \sigma_L(z)_{l_1} \\
&= \|z_T\|_{l_1} - \|z'_T\|_{l_1} + \|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1} \\
&\le \|(z'-z)_T\|_{l_1} + \|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}
\end{aligned} \tag{2.27}$$

Since $z, z' \in F(y) := x_0 + N$ , $(z - z') \in N$, hence by the null space property and (2.27),

$$\begin{aligned}\|(z'-z)_T\|_{l_1} &\le \gamma\|(z'-z)_T\|_{l_1}\\ &\le \gamma\left(\|(z'-z)_T\|_{l_1} + \|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}\right).\end{aligned} \tag{2.28}$$

Rearrange (2.28) to obtain

$$\|(z'-z)_T\|_{l_1} \le \frac{\gamma}{1-\gamma}\left(\|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}\right). \tag{2.29}$$

In order to make the following rearrangement more transparent, let $S := \|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}$.

Then from (2.27) and (2.29),

$$\begin{aligned}\|z'-z\|_{l_1} &= \|(z'-z)_{T^c}\|_{l_1} + \|(z'-z)_T\|_{l_1} =\\ &\le \|(z'-z)_T\|_{l_1} + \|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1} + \frac{\gamma}{1-\gamma}\left(\|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}\right)\\ &\le \frac{\gamma}{1-\gamma}\left(\|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}\right) + \|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1} + \frac{\gamma}{1-\gamma}\left(\|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}\right),\\ &= \left(\frac{\gamma}{1-\gamma} + 1 + \frac{\gamma}{1-\gamma}\right)\left(\|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}\right) = \frac{1+\gamma}{1-\gamma}\left(\|z'\|_{l_1} - \|z\|_{l_1} + 2\sigma_L(z)_{l_1}\right)\end{aligned} \tag{2.30}$$

which completes the proof. □

If $\Phi$ satisfies the null space property and the solution set contains an L-sparse vector, this vector is the unique $l_1$-minimizer as the following result states.

Lemma 4.3 *Assume that NSP holds for some* $L$ *and* $\gamma < 1$. *Suppose that* $F(y)$ *contains an L-sparse vector. Then this vector is the unique* $l_1$*-minimizer in* $F(y)$*; denoting it by x*, we have, moreover, for all* $v \in F(y)$,

$$\|v - x^*\|_{l_1} \le 2\frac{1+\gamma}{1-\gamma}\sigma_L(v)_{l_1}. \tag{2.31}$$

Proof: The idea is to choose an arbitrary solution $v$ and apply Lemma 4.1 in view of the fact that $x_s$ is L-sparse. In this case, $\sigma_L(x_s)_{l_1} = 0$, and since $v$ is arbitrary, (2.26) implies that $x_s$ is an $l_1$-minimizer. Choose another minimizer and argue uniqueness from (2.26). The details follow.

Let $x_s$ be the L-sparse vector in $F(y)$. Note that $\sigma_L(x_s)_{l_1} := \inf_{v\in\Sigma_L}\|x_s - v\|_{l_1^N}$, with $\Sigma_L$ containing all $x \in \mathbb{R}^N$ with support of cardinality at most L. Clearly, $\Sigma_L$ also contains the L-sparse vector x, and hence $\sigma_L(x_s)_{l_1} = 0$. In view of this, apply (2.26) with $z' = v$ and $z = x_s$ to obtain

$$\|v - x_s\|_{l_1} \le \frac{1+\gamma}{1-\gamma}\left[\|v\|_{l_1} - \|x_s\|_{l_1}\right].$$

Since v is arbitrary, the $0 \le \|v\|_{l_1} - \|x_s\|_{l_1}$ for all $v \in F(y)$, implying that $x_s$ is an $l_1$-minimizer .

To show uniqueness, suppose that there is another $l_1$-minimizer in $x_s \in F(y)$. Clearly, $\|x'\|_{l_1} = \|x_s\|_{l_1}$ and thus

$$\|x' - x_s\|_{l_1} \le \frac{1+\gamma}{1-\gamma}\left[\|x'\|_{l_1} - \|x_s\|_{l_1}\right] = 0.$$

Since $\|x' - x_s\|_{l_1} = 0$, we have $x' = x_s$.

For the ‘moreover’ part, let $z' = x*$ and $z = v$ in (2.26). Since $x*$ is the unique $l_1$-minimizer, $\|x^*\|_{l_1} \le \|v\|_{l_1}$ and (2.31) follows. □

Daubechies et al. [6] established certain useful results concerning the functional $J$. Substitute (2.8) into (2.3) to obtain

$$J(x^{n+1}, w^{n+1}, \varepsilon_{n+1}) = \sum_{j=1}^{N} \left[ (x_j^{n+1})^2 + \varepsilon_{n+1}^2 \right]^{1/2}. \tag{2.32}$$

Moreover, J obeys the following monotonicity property for $n \ge 0$

$$J(x^{n+1}, w^{n+1}, \varepsilon_{n+1}) \le J(x^{n+1}, w^{n}, \varepsilon_{n+1}) \le J(x^{n+1}, w^{n}, \varepsilon_{n}) \le J(x^{n}, w^{n}, \varepsilon_{n}). \tag{2.33}$$

The first inequality follows from the minimization property (2.6), the second from inequality from $\varepsilon_{n+1} \le \varepsilon_n$, and the third inequality from the minimization property (2.4).

Lemma 4.4 *For each* $n \ge 1$, *we have*

$$\|x^n\|_{l_1} \le J(x^1, w^0, \varepsilon_0) =: A \tag{2.34}$$

*and*

$$w_j^n \ge A^{-1}, \quad j = 1,\ldots,N. \tag{2.35}$$

Proof: By monotonicity of J,

$$\|x^n\|_{l_1} = \sum_{j=1}^{N} |x_j^n| \le \sum_{j=1}^{N} \left( (x_j^n)^2 + \varepsilon^2 \right)^{1/2} = J(x^n, w^n, \varepsilon_n) = J(x^1, w^0, \varepsilon_0),$$

$$(w_j^n)^{-1} = \left((x_j^n)^2 + \varepsilon^2\right)^{1/2} \le J(x^n, w^n, \varepsilon_n) \le J(x^1, w^0, \varepsilon_0) = A,$$

implying (2.35). □

The following lemma states that the iterations of the algorithm eventually 'stay close'.

Lemma 5.1 *Given any* $y \in \mathbb{R}^m$*, the* $x^n$ *satisfy*

$$\sum_{n=1}^{\infty} \left\| x^{n+1} - x^n \right\|_{l_2}^2 \le 2A^2 \qquad (2.36)$$

*where A is the constant of Lemma 4.4. In particular, we have*

$$\lim_{n\to\infty}(x^n - x^{n+1}) = 0. \qquad (2.37)$$

Outline of the Proof: Use monotonicity of J and the fact that $x^{n+1} - x^n \in \mathcal{N}$. Sum over $n \ge 1$ to arrive at the desired result.

Proof: For each $n = 1,2,\ldots,$

$$\begin{aligned}
&2\left[J(x^n, w^n, \varepsilon_n) - J(x^{n+1}, w^{n+1}, \varepsilon_{n+1})\right] \\
&\ge 2\left[J(x^n, w^n, \varepsilon_n) - J(x^{n+1}, w^n, \varepsilon_{n+1})\right] \\
&= \sum_{j=1}^{N} \frac{((x_j^n)^2 - (x_j^{n+1})^2)}{((x_j^n)^2 + \varepsilon_n^2)^{1/2}} = \left\langle x^n, x^n \right\rangle_{w^n} - \left\langle x^{n+1}, x^{n+1} \right\rangle_{w^n} \\
&= \left\langle x^n, x^n \right\rangle_{w^n} + \left\langle x^{n+1}, x^n \right\rangle_{w^n} - \left\langle x^{n+1}, x^n \right\rangle_{w^n} - \left\langle x^{n+1}, x^{n+1} \right\rangle_{w^n} \\
&= \left\langle x^n - x^{n+1}, x^n - x^{n+1} \right\rangle_{w^n} \\
&= \sum_{j=1}^{N} (x_j^n - x_j^{n+1})^2 w_j^n \ge A^{-1} \left\| x_j^n - x_j^{n+1} \right\|_{l_2}^2
\end{aligned}$$

where the fifth equality relies on the fact that $\left\langle x^{n+1}, x^n - x^{n+1}\right\rangle_{w^n} = 0$ (note that $x^{n+1} - x^n \in N$ and use (2.14)). Observe that $J(x^1, w^1, \varepsilon_1) \le A$ and sum over $n \ge 1$ to arrive at (2.36). □

The following lemma provides a characterization of $x^{\varepsilon} \triangleq \arg\min_{z \in F(y)} f_{\varepsilon}(z)$. This characterization is crucial in establishing the convergence of the algorithm if $\varepsilon > 0$.

Lemma 5.2 *Let* $\varepsilon > 0$ *and* $z \in F(y)$ *. Then* $z = x^{\varepsilon}$ *if and only if*

$$\langle z, \eta \rangle_{\tilde{w}(z,\varepsilon)} = 0 \;\; \forall\, \eta \in N, \tag{2.38}$$

*where* $\tilde{w}(z,\varepsilon)_i = (z_i^2 + \varepsilon^2)^{-1/2}, \, i = 1, \ldots, N.$

Proof: Construct a function $G_{\varepsilon}(t) \triangleq f_{\varepsilon}(z + t\eta) - f_{\varepsilon}(z)$. It is analytic, $G_{\varepsilon}(0) = 0$, and if $z = x^{\varepsilon}$, $G_{\varepsilon}(t) \ge 0 \;\; \forall\, t \in \mathbb{R}$. That is, $G_{\varepsilon}(\cdot)$ is nonnegative and equal to zero at $t = 0$, and in view of the fact that it is analytic, it must either have a saddle point at $t = 0$ or be equal to zero around $t = 0$. Thus $G_{\varepsilon}^{'}(0) = 0$. Differentiate $G_{\varepsilon}(\cdot)$ using chain rule and use the definition of weighted inner product (2.6) to arrive at

$$G_{\varepsilon}^{'}(0) = \sum_{i=1}^{N} \frac{\eta_i z_i}{\left[ z_i^2 + \varepsilon^2 \right]^{1/2}} = \langle z, \eta \rangle_{\tilde{w}(z,\varepsilon)}, \tag{2.39}$$

implying (2.38).

Now assume that $z \in F(y)$ and $\langle z, \eta \rangle_{\hat{w}(z,\varepsilon)} = 0 \;\; \forall \eta \in N$. Consider $g(u) \square \left[ u^2 + \varepsilon^2 \right]^{1/2}$.

Since $g''(u) = u^2 (u^2 + \varepsilon^2)^{-3/2} \geq 0$, $g(\cdot)$ is convex. Hence for an arbitrary $u_0$, the line segment passing through $u_0$ and tangent to $g(\cdot)$ is below $g(\cdot)$. This is expressed by

$$\left[ u^2 + \varepsilon^2 \right]^{1/2} \geq \left[ u_0^2 + \varepsilon^2 \right]^{1/2} + \left[ u_0^2 + \varepsilon^2 \right]^{-1/2} u_0 (u - u_0) . \tag{2.40}$$

The $N$ -dimensional version of this inequality produces

$$\begin{aligned} f_\varepsilon(v) &\geq f_\varepsilon(z) + \sum_{i=1}^{N} (z_i^2 + \varepsilon^2)^{-1/2} z_i (v_i - z_i) \\ &= f_\varepsilon(z) + \langle z, v - z \rangle_{\hat{w}(z,\varepsilon)} = f_\varepsilon(z) \end{aligned} , \tag{2.41}$$

where the last inequality follows from (2.14) since $(v - z) \in N$. Since $v$ is arbitrary, $z = x^\varepsilon$. □

We are now ready to provide the proof of the main result of Daubechies et al. [6] .

Proof of Theorem 1:

(i) Consider the case when $\varepsilon = 0$.

If $\varepsilon = 0$, then either $\varepsilon_{n_0} = 0$ for some $n_0$ or $\varepsilon_{n_0} > 0$ but $\varepsilon_n \to 0$.

If $\varepsilon_{n_0} = 0$ for some $n_0$, the algorithm stops and $\bar{x} = x^{n_0}$. Since $\varepsilon_{n_0} = 0$, (1.7) implies that $r(x^{n_0})_{K+1} = 0$ and hence $\bar{x} = x^{n_0}$ is K-sparse. By Lemma 4.3, this solution is the unique $l_1$-minimizer and $\bar{x} = x^*$.

If $\varepsilon_{n_0} > 0$ and $\varepsilon_n \to 0$, there is an increasing sequence of indices $(n_i)$ such that $\varepsilon_{n_i} < \varepsilon_{n_i - 1}$ for all $i$ (otherwise there is a contradiction). Since $\varepsilon_{n_i} < \varepsilon_{n_i - 1}$, definition (2.5) implies that $r(x^{n_i})/N$ is being chosen when $\varepsilon_{n_i}$ is updated. Thus $r(x^{n_i}) < N\varepsilon_{n_i - 1}$ $\forall i$. Since $\left(x^n\right)_{n\in\mathbb{N}}$ is a bounded sequence (by Lemma 4.4), it contains a convergent subsequence (Bolzano-Weierstrass) with indices $(p_j)_{j\in\mathbb{N}}$ out of $(n_i)$. Let $\tilde{x} \in F(y)$ be the limit of $\left(x^{p_j}\right)_{j\in\mathbb{N}}$. By Lemma 4.1, since $\left(x^{p_j}\right)_{j\in\mathbb{N}} \to \tilde{x}$, $r(x^{p_j})_{K+1} \to r(\tilde{x})_{K+1}$. Inequality

$$r(\tilde{x})_{K+1} = \lim_{j\to\infty} r(x^{p_j})_{K+1} \le \lim_{j\to\infty} N\varepsilon_{p_{j-1}} = 0 \tag{2.42}$$

immediately follows in view of the previous observation that $r(x^{n_i}) < N\varepsilon_{n_i - 1}$. Thus $\tilde{x}$ is K-sparse. By Lemma 4.3, $\tilde{x} = x^*$, the unique $l_1$-minimizer.

We have shown that $\left(x^{p_j}\right)_{j\in\mathbb{N}} \to \tilde{x}$, and it only remains to show that $x^n \to \tilde{x}$. Since $\left(x^{p_j}\right)_{j\in\mathbb{N}} \to \tilde{x}$ and $\varepsilon_{p_j} \to 0$, (2.32) implies that $J(x^{p_j}, w^{p_j}, \varepsilon_{p_j}) \to \|x^*\|_{l_1}$. By monotonicity property (2.33), $J(x^n, w^n, \varepsilon_n) \to \|x^*\|_{l_1}$. From (2.32),

$$J(x^n, w^n, \varepsilon_n) = \sum_{j=1}^{N}((x_j^n)^2 + \varepsilon^2)^{1/2} \le \sum_{j=1}^{N}((x_j^n)^2 + 2\left|x_j^n\right|\varepsilon + \varepsilon^2)^{1/2} = \sum_{j=1}^{N}\left|x_j^n\right| + \varepsilon = \left\|x^n\right\|_{l_1} + N\varepsilon\ . \tag{2.43}$$

This observation, together with the previous deduction that $J(x^n, w^n, \varepsilon_n) \to \|x^*\|_{l_1}$ imply (2.43), which in turn means that $\left\|x^n\right\| \to \|x^*\|$. Invoke Lemma 5.2 to show that $x^n \to x^*$.

(2.10) follows from (2.31) of Lemma 4.3 and the observation that $\sigma_n(z) \geq \sigma_{n'}(z)$ for $n \leq n'$.

(ii) Consider the case when $\varepsilon > 0$. Let $(x^{n_i}) \to \tilde{x} \in F(y)$ be any convergent subsequence of $(x_n)$ (which exists by Bolzano-Weierstrass theorem). First we show that $\tilde{x} = x^{\varepsilon}$. Since

$$w_j^n = \left[ (x_j^n)^2 + \varepsilon_j^2 \right]^{-1/2} \leq \varepsilon^{-1},$$

$$\lim_{i \to \infty} w_j^{n_i} = \left[ (\tilde{x}_j)^2 + \varepsilon^2 \right]^{-1/2} = \tilde{w}(\tilde{x}, \varepsilon)_j =: \tilde{w}_j \in (0, \infty), \;\; j = 1, \dots, N$$

with the notation of Lemma 5.2. Moreover, by (2.37), $\lim_{i \to \infty} x^{n_i + 1} = \tilde{x}$. Since $x^{n_i} \in F(y)$ for every $i$ and every $\eta \in N$, $\left\langle x^{n_i+1}, \eta \right\rangle_{w^{n_i}} = 0$ by (2.14), and hence

$$\left\langle \tilde{x}, \eta \right\rangle_{\tilde{w}} = \lim_{i \to \infty} \left\langle x^{n_i+1}, \eta \right\rangle_{w^{n_i}} = 0 \;\; \forall\, \eta \in N . \tag{2.44}$$

Lemma 5.2 and (2.44) imply that $\tilde{x} = x^{\varepsilon}$. Thus $(x^n)_{n \in \mathbb{N}} \to \bar{x} = x^{\varepsilon}$, a unique limit.

(iii) (Error Estimate)

$$\left\| x^{\varepsilon} \right\|_{l_1} \leq f_{\varepsilon}(x^{\varepsilon}) \leq f_{\varepsilon}(z) \leq \left\| z \right\|_{l_1} + N\varepsilon , \tag{2.45}$$

where the first inequality follows from (2.16), the second from (2.17), and the third from a calculation identical to the one done in (2.43). Thus $\left\| x^{\varepsilon} \right\|_{l_1} - \left\| z \right\|_{l_1} \leq N\varepsilon$ and Lemma 4.2 implies

$$\left\| x^{\varepsilon} - z \right\|_{l_1} \leq \frac{1+\gamma}{1-\gamma} \left[ N\varepsilon + 2\sigma_k(z)_{l_1} \right], k \leq K, \tag{2.46}$$

in view of the fact that $\sigma_n(z) \geq \sigma_{n'}(z)$ for $n \leq n'$ .

Since r(.) is Lipschitz continuous (by Lemma 4.1), (2.46) and (1.7) imply that

$$N\varepsilon = \lim_{n\to\infty} N\varepsilon_n \leq \lim_{n\to\infty} r(x^n)_{K+1} = r(x^\varepsilon)_{K+1} . \tag{2.47}$$

Together with (4.4) of Lemma 4.1, (2.47) implies that

$$(K+1-k)N\varepsilon \leq \frac{1+\gamma}{1-\gamma}\left[N\varepsilon + 2\sigma_k(z)_{l_1}\right] + \sigma_k(z)_{l_1} . \tag{2.48}$$

Collect $N\varepsilon$ on the left-hand side and use assumptions to get

$$N\varepsilon(((K-k)+1-\frac{1+\gamma}{1-\gamma}) = N\varepsilon((K-k)-\frac{2\gamma}{1-\gamma}) .$$

Moreover, note that

$$\frac{1+\gamma}{1-\gamma}\left[2\sigma_k(z)_{l_1}\right] + \sigma_k(z)_{l_1} = \frac{3+\gamma}{1-\gamma}\sigma_k(z)_{l_1} \text{ and } 3+\frac{4\gamma}{1-\gamma} = \frac{3+\gamma}{1-\gamma} .$$

The above results imply that (5.21) yields

$$N\varepsilon \leq \frac{3+4\gamma/1-\gamma}{(K-k)-2\gamma/1-\gamma}\sigma_k(z)_{l_1} . \tag{2.49}$$

Straightforward substitution of (5.22) in (5.19) yields (5.13).

(iv) Suppose that $\varepsilon > 0$. If the solution set contains a $k$-sparse vector $z$ (so that $\sigma_k(z)_{l_1} = 0$) with $k < K - \frac{2\gamma}{1-\gamma}$, then $N\varepsilon \le 0$, which is a contradiction. Hence the presence of $k$-sparse solution implies that $\varepsilon = 0$. □

# CHAPTER 4: NEW IRLS ALGORITHM WITH PRIOR INFORMATION

In this chapter, we present the new results of this thesis. We integrate prior information on the support of the sparse domain into the algorithm of Daubechies et al. [6]. We assume that this information is perfectly accurate. For the purposes of the algorithm and the proofs that follow, let $x_\Delta$ be the vector derived from $x$ by setting all the components with $j \notin \Delta$ equal to zero. Moreover, let $M = \#\text{supp}(x_\Delta)$ and $L = \#\text{supp}(x_{\Delta^c})$, implying that vector $x$ is $K = M + L$ sparse.

*Algorithm 2. Let*

$$J(z, w, \varepsilon; \tau) := \frac{1}{2}\left[\sum_{j=1}^{N} (z_j^{\,2} + \varepsilon^2) w_j + \tau_j^2 w_j^{-1}\right], \quad z \in \mathbb{R}^N,$$
$$\textit{where } \tau_j = \begin{cases} 1 & \textit{if } j \notin \Delta \\ \textit{fixed number between } 10^{-6} \textit{ and } 10^{-2} & \textit{if } j \in \Delta \end{cases}. \tag{3.1}$$

*Initialize*

$$w_j^{(0)} = \begin{cases} 1 & \textit{if } j \notin \Delta \\ \textit{fixed number between } 10^{-6} \textit{ and } 10^{-2} & \textit{if } j \in \Delta \end{cases}$$

*and set* $\varepsilon_0 := 1$. *For n = 0, 1, . . . , recursively define*

$$x^{n+1} := \underset{z \in F(y)}{\arg\min}\, J(z, w^n, \varepsilon_n; \tau) = \underset{z \in F(y)}{\arg\min} \|z\|_{l_2(w^n)} \tag{3.2}$$

*and*

$$\varepsilon_{n+1} := \min\left(\varepsilon_n, \frac{r((x^{n+1})_{\Delta^c})_{L+1}}{N}\right), \tag{3.3}$$

*where K is a fixed integer. Moreover, define*

$$w^{n+1} := \arg\min_{w>0} J(x^{n+1}, w, \varepsilon_{n+1}; \tau)\,. \tag{3.4}$$

*Stop the algorithm if* $\varepsilon_n = 0$, *in which case* $x^j := x^n$ *for j > n. If* $\varepsilon_n \neq 0$, *the algorithm will generate an infinite sequence* $(x^n)_{n\in\mathbb{N}}$ *of distinct vectors.*

Each step of the algorithm requires the solution of a least squares problem. In matrix form,

$$x^{n+1} = D_n \Phi^T (\Phi D_n \Phi^T)^{-1} y\,. \tag{3.5}$$

Matrix $\Phi$ contains

$$w_j^{n+1} = \tau_j \left[ (x_j^{n+1})^2 + \varepsilon_{n+1}^2 \right]^{-1/2}, \;\; j = 1, \dots, N \tag{3.6}$$

on the diagonal.

Lemma 2.1, Lemma 4.1, Lemma 4.2, and Lemma 4.3 do not rely on the form of the functional $J$ and hence hold without change. Equation (2.32) becomes

$$J(x^{n+1}, w^{n+1}, \varepsilon_{n+1}) = \sum_{j=1}^{N} \tau_j \left[ (x_j^{n+1})^2 + \varepsilon_{n+1}^2 \right]^{1/2}. \tag{3.7}$$

Equation (2.33) still holds by identical reasoning. This equation implies that $x^n$ is bounded from below by 0 and from above by $J(x^1, w^0, \varepsilon_0)$. Moreover, it implies that $\|x_n\|_{l_1} \leq J(x^1, w^0, \varepsilon_0) \leq \sum_{j=1}^{N} (x_j^1)^2 + 1$, and hence $x^n$ is also bounded from above.

Lemma 4.4 *For each* $n \geq 1$, *we have*

$$\left\|(x^n)_{\Delta^c}\right\|_{l_1} \leq J(x^1, w^0, \varepsilon_0) =: A , \qquad (3.8)$$

$$\left\|(x^n)_{\Delta}\right\|_{l_1} \leq (\tau_{j\in\Delta})^{-1} A , \qquad (3.9)$$

*and*

$$w_j^n \geq A^{-1}, \quad j \notin \Delta , \qquad (3.10)$$

$$((w_j^n))_{\Delta}^{-1} \leq (\tau_{j\in\Delta})^{-1} A . \qquad (3.11)$$

Proof: By monotonicity of J,

If $j \notin \Delta$,

$$\left\|(x^n)_{\Delta^c}\right\|_{l_1} = \sum_{j\in\Delta^c}\left|x_j^n\right| \leq \sum_{j\in\Delta^c}\left((x_j^n)^2 + \varepsilon^2\right)^{1/2} \leq J(x^n, w^n, \varepsilon_n) \leq J(x^1, w^0, \varepsilon_0) ,$$

$$((w_j^n))_{\Delta^c}^{-1} = \left(((x_j^n))_{\Delta^c}^2 + \varepsilon^2\right)^{1/2} \leq J(x^n, w^n, \varepsilon_n) \leq J(x^1, w^0, \varepsilon_0) = A,$$

implying (2.35).

If $j \in \Delta$,

$$\tau_{j\in\Delta}\left\|(x^n)_{\Delta}\right\|_{l_1} = \tau_{j\in\Delta}\sum_{j\in\Delta}\left|x_j^n\right| \leq \sum_{j\in\Delta}\tau_j\left((x_j^n)^2 + \varepsilon^2\right)^{1/2} \leq J(x^n, w^n, \varepsilon_n) \leq J(x^1, w^0, \varepsilon_0) ,$$

$$((w_j^n))_{\Delta}^{-1} = (\tau_{j\in\Delta})^{-1}\left(((x_j^n))_{\Delta}^2 + \varepsilon^2\right)^{1/2} \leq (\tau_{j\in\Delta})^{-1} J(x^n, w^n, \varepsilon_n) \leq (\tau_{j\in\Delta})^{-1} J(x^1, w^0, \varepsilon_0) = (\tau_{j\in\Delta})^{-1} A .$$

Lemma 5.1 *Given any* $y \in \mathbb{R}^m$, *the* $x^n$ *satisfy*

$$\sum_{n=1}^{\infty} \left\| (x^{n+1})_{\Delta^c} - (x^n)_{\Delta^c} \right\|_{l_2}^2 \le 2A^2, \qquad (3.12)$$

$$\sum_{n=1}^{\infty} \left\| (x^{n+1})_{\Delta} - (x^n)_{\Delta} \right\|_{l_2}^2 \le \frac{2A^2}{\tau}, \qquad (3.13)$$

*where A is the constant of Lemma 4.4. In particular,*

$$\lim_{n \to \infty} (x^n - x^{n+1}) = 0. \qquad (3.14)$$

Main Idea of the Proof: Use monotonicity of J and the fact that $x^{n+1} - x^n \in \mathrm{N}$. Sum over $n \ge 1$ to arrive at the desired result.

Proof: For each n=1,2,…,

$$\begin{aligned}
&2\left[ J(x^n, w^n, \varepsilon_n) - J(x^{n+1}, w^{n+1}, \varepsilon_{n+1}) \right] \\
&\ge 2\left[ J(x^n, w^n, \varepsilon_n) - J(x^{n+1}, w^n, \varepsilon_n) \right] \\
&= \sum_{j=1}^{N} ((x_j^n)^2 - (x_j^{n+1})^2) w_j^n + \sum_{j=1}^{N} (\varepsilon_n^2 w_j - \varepsilon_n^2 w_j) + (w^{-1} - w^{-1}) \\
&= \sum_{j=1}^{N} \tau_j \frac{((x_j^n)^2 - (x_j^{n+1})^2)}{((x_j^n)^2 + \varepsilon_n^2)^{1/2}} = \left\langle x^n, x^n \right\rangle_{w^n} - \left\langle x^{n+1}, x^{n+1} \right\rangle_{w^n} , \qquad (3.15) \\
&= \left\langle x^n, x^n \right\rangle_{w^n} + \left\langle x^{n+1}, x^n \right\rangle_{w^n} - \left\langle x^{n+1}, x^n \right\rangle_{w^n} - \left\langle x^{n+1}, x^{n+1} \right\rangle_{w^n} \\
&= \left\langle x^n + x^{n+1}, x^n - x^{n+1} \right\rangle_{w^n} = \left\langle x^n - x^{n+1}, x^n - x^{n+1} \right\rangle_{w^n} \\
&= \sum_{j=1}^{N} (x_j^n - x_j^{n+1})^2 w_j^n \ge \sum_{j \in \Delta^c} (x_j^n - x_j^{n+1})^2 w_j^n \ge A^{-1} \left\| (x_j^n)_{j \in \Delta^c} - (x_j^{n+1})_{j \in \Delta^c} \right\|_{l_2}^2
\end{aligned}$$

where the fifth equality relies on the fact that $\left\langle x^{n+1}, x^n - x^{n+1} \right\rangle_{w^n} = 0$ (note that $x^{n+1} - x^n \in N$ and use (2.14)). Observe that $J((x^1)_{j\in\Delta^c}, w^1, \varepsilon_1) \le A$ and sum over $n \ge 1$ to arrive at (2.36).

For $j \in \Phi$, follow the same steps as in (3.15) but change the last line as follows

$$\begin{aligned} 2A &\ge 2\left[ J(x^n, w^n, \varepsilon_n) - J(x^{n+1}, w^{n+1}, \varepsilon_{n+1}) \right] \\ &\ge \sum_{j=1}^{N} (x_j^n - x_j^{n+1})^2 w_j^n \ge \sum_{j\in\Delta} (x_j^n - x_j^{n+1})^2 w_j^n \ge \frac{\tau}{A} \left\| (x_j^n)_{j\in\Delta} - (x_j^{n+1})_{j\in\Delta} \right\|_{l_2}^2 . \end{aligned}$$

□

Re-define $f_\varepsilon(z)$ in (2.16) as

$$f_\varepsilon(z) := \sum_{j=1}^{N} \tau_j (z_j{}^2 + \varepsilon^2)^{1/2} . \tag{3.16}$$

In lemma 5.2, equation (2.39) becomes

$$G_\varepsilon^{'}(0) = \sum_{i=1}^{N} \frac{\tau_i \eta_i z_i}{\left[ z_i^2 + \varepsilon^2 \right]^{1/2}} = \left\langle z, \eta \right\rangle_{w(z,\varepsilon)} , \tag{3.17}$$

hence the “only if” part holds.

For the “if” part, equation (2.40) becomes

$$\tau \left[ u^2 + \varepsilon^2 \right]^{1/2} \ge \tau \left[ u_0^2 + \varepsilon^2 \right]^{1/2} + \tau \left[ u_0^2 + \varepsilon^2 \right]^{-1/2} u_0 (u - u_0)$$

and equation (2.41) becomes

$$f_{\varepsilon}(v) \geq f_{\varepsilon}(z) + \sum_{i=1}^{N} \tau_i (z_i^2 + \varepsilon^2)^{-1/2} z_i (v_i - z_i)$$
$$= f_{\varepsilon}(z) + \langle z, v - z \rangle_{w(z,\varepsilon)} = f_{\varepsilon}(z) \quad .$$

This completes the proof. ∎

Now, we state and prove the main result of the thesis.

*Theorem2. Let K (the same index as used in the update rule (2.5)) be chosen so that* $\Phi$ *satisfies the null space property (2.9) of order K, with* $\gamma < 1$*. Then, for each* $y \in \mathbb{R}^m$*, the output of Algorithm 2 converges to a vector* $\bar{x} \in F(y)$*, with* $r(\bar{x})_{K+1} = N \cdot \lim_{n\to\infty} \varepsilon_n$ *and the following hold:*

*(i) If* $\varepsilon := \lim_{n\to\infty} \varepsilon_n = 0$*, then* $\bar{x}$ *is K-sparse; in this case there is therefore a unique* $l_1$*-minimizer* $x^*$ $\bar{x} = x^*$*. Moreover, for* $k \leq K$ *and any* $z \in F(y)$*,*

$$\left\| z - \bar{x} \right\|_{l_1} \leq c\sigma_k(z)_{l_1} \quad \text{with } c := \frac{2(1+\gamma)}{1-\gamma}. \tag{3.18}$$

*(ii) If* $\varepsilon := \lim_{n\to\infty} \varepsilon_n > 0$*, then* $\bar{x} = x^{\varepsilon}$*.*

*(iii) In this last case, if* $\gamma$ *satisfies the stricter bound* $\gamma < 1 - \frac{2}{K+2}$*, then for all* $z \in F(y)$ *and any* $k < K - \frac{2\gamma}{1-\gamma}$*,*

$$\left\| z^{\#} - \bar{x} \right\|_{l_1} \leq \tilde{c}\,\sigma_k(z^{\#})_{l_1}, \quad \text{with } \tilde{c} := \frac{2(1+\gamma)}{1-\gamma}\left[\frac{K-k+3/2}{K-k-2\gamma/(1-\gamma)}\right]. \tag{3.19}$$

*(iv) If* $F(y)$ *contains a vector* $z$ *of sparsity* $k < K - \frac{2\gamma}{1-\gamma}$*, then* $\varepsilon = 0$ *and* $\bar{x} = x^* = z$*.*

Proof:

(i) Case $\varepsilon = 0$.

If $\varepsilon = 0$, then either $\varepsilon_{n_0} = 0$ for some $n_0$ or $\varepsilon_{n_0} > 0$ but $\varepsilon_n \to 0$.

If $\varepsilon_{n_0} = 0$ for some $n_0$, the proof stays the same.

If $\varepsilon_{n_0} > 0$ and $\varepsilon_n \to 0$, there is an increasing sequence of indices $(n_i)$ such that $\varepsilon_{n_i} < \varepsilon_{n_i - 1}$ for all $i$ (otherwise there is a contradiction). Since $\varepsilon_{n_i} < \varepsilon_{n_i - 1}$, definition (1.7) implies that $r((x^{n_i})_{\Delta^c})/N$ is being chosen when $\varepsilon_{n_i}$ is updated. Thus $r(x^{n_i}) < N\varepsilon_{n_i - 1}$ $\forall i$. Since $\left(x^n\right)_{n \in \mathbb{N}}$ is a bounded sequence (by Lemma 4.4), it contains a convergent subsequence (by Bolzano-Weierstrass theorem) with indices $(p_j)_{j \in \mathbb{N}}$ out of $(n_i)$. Let $(\tilde{x}) \in F(y)$ be the limit of $\left(x^{p_j}\right)_{j \in \mathbb{N}}$. By Lemma 4.1, since $\left(x^{p_j}\right)_{j \in \mathbb{N}} \to (\tilde{x})$, $r(x^{p_j})_{K+1} \to r(\tilde{x})_{K+1}$, using the definition of $r(\cdot)$ that applies to the entire vector. Inequality

$$r(\tilde{x})_{K+1} = \lim_{j \to \infty} r(x^{p_j})_{K+1} \le \lim_{j \to \infty} N\varepsilon_{p_{j-1}} = 0 \tag{3.20}$$

immediately follows in view of the previous observation that $r(x^{n_i}) < N\varepsilon_{n_i - 1}$. Thus vector $\tilde{x}$ is K-sparse. By Lemma 4.3, $\tilde{x} = x^*$, the unique $l_1$-minimizer.

Consider the case in which $j \notin \Delta$. We have shown that $\left(x^{p_j}\right)_{j\in\Box,\Delta^c} \to (\tilde{x})_{\Delta^c}$, and it only remains to show that $(x^n)_{\Delta^c} \to (\tilde{x})_{\Delta^c}$. Since $\left(x^{p_j}\right)_{j\in\Box,\Delta^c} \to (\tilde{x})_{\Delta^c}$ and $\varepsilon_{p_j} \to 0$, (2.33) implies that $J((x^{p_j})_{\Delta^c}, w^{p_j}, \varepsilon_{p_j}) \to \left\|(x^*)_{\Delta^c}\right\|_{l_1}$.

By monotonicity property (4.14), $J((x^n)_{\Delta^c}, w^n, \varepsilon_n) \to \left\|(x^*)_{\Delta^c}\right\|_{l_1}$. From (3.7) ,

$$\begin{aligned} J((x^n)_{\Delta^c}, w^n, \varepsilon_n) &= \sum_{j\in\Delta^c}((x_j^n)^2 + \varepsilon_n^2)^{1/2} \le \sum_{j\in\Delta^c}((x_j^n)^2 + 2\left|x_j^n\right|\varepsilon_n + \varepsilon_n^2)^{1/2} \\ &= \sum_{j\in\Delta^c}(\left|x_j^n\right| + \varepsilon_n) \le \left\|(x^n)_{\Delta^c}\right\|_{l_1} + N\varepsilon_n . \end{aligned} \tag{3.21}$$

This observation implies that

$$J((x^n)_\Delta, w^n, \varepsilon_n) - N\varepsilon_n \le \left\|(x^n)_\Delta\right\|_{l_1} \le J((x^n)_\Delta, w^n, \varepsilon_n), \tag{3.22}$$

which, together with the previous deduction that $J((x^n)_{\Delta^c}, w^n, \varepsilon_n) \to \left\|(x^*)_{\Delta^c}\right\|_{l_1}$ implies that $\left\|(x^n)_{\Delta^c}\right\|_{l_1} \to \left\|(x^*)_{\Delta^c}\right\|_{l_1}$. Invoke Lemma 4.2 with $z' = (x^n)_{\Delta^c}$ and $z = (x^*)_{\Delta^c}$:

$$\limsup_{n\to\infty}\left\|(x^n)_{\Delta^c} - (x^*)_{\Delta^c}\right\|_{l_1} \le \frac{1+\gamma}{1-\gamma}(\lim_{n\to\infty}\left\|(x^n)_{\Delta^c}\right\|_{l_1} - \left\|(x^*)_{\Delta^c}\right\|_{l_1}) = 0, \tag{3.23}$$

which implies that $(x^n)_{\Delta^c} \to (x^*)_{\Delta^c}$.

If $j \in \Delta$, we have shown that $\left(x^{p_j}\right)_{j\in\mathbb{N},\Delta} \to (\tilde{x})_\Delta$, and it only remains to show that $(x^n)_\Delta \to (\tilde{x})_\Delta$. Since $\left(x^{p_j}\right)_{j\in\mathbb{N},\Delta} \to (\tilde{x})_\Delta$ and $\varepsilon_{p_j} \to 0$, (2.33) implies that $J((x^{p_j})_\Delta, w^{p_j}, \varepsilon_{p_j}) \to \tau_{j\in\Delta} \left\|(x^*)_\Delta\right\|_{l_1}$. Moreover, (3.21) becomes

$$
\begin{aligned}
J((x^n)_\Delta, w^n, \varepsilon_n) &= (\tau_{j\in\Delta}) \sum_{j\in\Delta} ((x_j^n)^2 + \varepsilon_n^2)^{1/2} \le (\tau_{j\in\Delta}) \sum_{j\in\Delta} ((x_j^n)^2 + 2\left|x_j^n\right|\varepsilon_n + \varepsilon_n^2)^{1/2} \\
&= (\tau_{j\in\Delta}) \sum_{j\in\Delta} (\left|x_j^n\right| + \varepsilon_n) \le (\tau_{j\in\Delta}) \left\|(x^n)_\Delta\right\|_{l_1} + N\varepsilon_n \tau_{j\in\Delta}
\end{aligned}, \qquad (3.24)
$$

and (3.22) becomes

$$
(\tau_{j\in\Delta}) \sum_{j\in\Delta} ((x_j^n)^2 + \varepsilon_n^2)^{1/2} - N\varepsilon_n \tau_{j\in\Delta} \le (\tau_{j\in\Delta}) \left\|(x^n)_\Delta\right\|_{l_1} \le (\tau_{j\in\Delta}) \sum_{j\in\Delta} ((x_j^n)^2 + \varepsilon_n^2)^{1/2}, \qquad (3.25)
$$

which simplifies to

$$
(\tau_{j\in\Delta})^{-1} J((x^n)_\Delta, w^n, \varepsilon_n) - N\varepsilon_n \le \left\|(x^n)_\Delta\right\|_{l_1} \le (\tau_{j\in\Delta})^{-1} J((x^n)_\Delta, w^n, \varepsilon_n). \qquad (3.26)
$$

This observation, together with a previous deduction ($J(x^n, w^n, \varepsilon_n) \to \|x^*\|_{l_1}$), imply that $\|x^n\| \to \|x^*\|$. Invoke Lemma 4.2 to show that $x^n \to x^*$.

(3.18) follows from (2.31) of Lemma 4.3 and the observation that $\sigma_n(z) \ge \sigma_{n'}(z)$ for $n \le n'$.

(ii) Case $\varepsilon > 0$. Note that

$$
w_j^n = \tau_j \left[ (x_j^n)^2 + \varepsilon_j^2 \right]^{-1/2} \le \tau_j \varepsilon^{-1},
$$

$$\lim_{i\to\infty} w_j^{n_i} = \tau_j \left[ (\tilde{x}_j)^2 + \varepsilon^2 \right]^{-1/2} = \tilde{w}(\tilde{x},\varepsilon)_j =: \tilde{w}_j \in (0,\infty), \;\; j = 1,\ldots, N,$$

in the notation of Lemma 5.2. The rest of the proof is identical to that of Daubechies et al. (2010).

(iii) Error estimate

Let $y^{\#} := (\tau_1 y_1, \tau_2 y_2, ..., \tau_N y_N)$ for any vector y. Then for any $z \in F(y)$,

$$\left\|(x^{\#})^{\varepsilon}\right\|_{l_1} \le f_{\varepsilon}(x^{\varepsilon}) \le f_{\varepsilon}(z) \le \left\|z^{\#}\right\|_{l_1} + N\varepsilon \,, \tag{3.27}$$

where the first inequality follows from (3.16), the second from (2.17), and the third from a calculation identical to the one done in (3.24). Thus $\left\|(x^{\#})^{\varepsilon}\right\|_{l_1} - \left\|z^{\#}\right\|_{l_1} \le N\varepsilon$ and since $\sigma_n(z) \ge \sigma_{n'}(z)$ for $n \le n'$, Lemma 4.2 with $\sigma_j(z^{\#})_{l_1} := \inf_{w^{\#}\in\Sigma_j} \left\|z^{\#} - w^{\#}\right\|_{l_1^N}$ implies that

$$\left\|(x^{\#})^{\varepsilon} - z^{\#}\right\|_{l_1} \le \frac{1+\gamma}{1-\gamma}\left[ N\varepsilon + 2\sigma_k(z^{\#})_{l_1} \right], k \le K. \tag{3.28}$$

Moreover, Since r(.) is Lipschitz-continuous (Lemma 4.1), (3.28) and (3.3) imply that

$$N\varepsilon = \lim_{n\to\infty} N\varepsilon_n \le \lim_{n\to\infty} r((x^{\#})^{n})_{K+1} = r((x^{\#})^{\varepsilon})_{K+1}. \tag{3.29}$$

Together with (4.4) of Lemma 4.1, (3.29) implies that

$$(K+1-k)N\varepsilon \le \frac{1+\gamma}{1-\gamma}\left[ N\varepsilon + 2\sigma_k(z^{\#})_{l_1} \right] + \sigma_k(z^{\#})_{l_1}. \tag{3.30}$$

Collect $N\varepsilon$ on the left side and use assumptions to get

$$N\varepsilon(((K-k)+1-\frac{1+\gamma}{1-\gamma})=N\varepsilon((K-k)-\frac{2\gamma}{1-\gamma}). \qquad (3.31)$$

Moreover, note that

$$\frac{1+\gamma}{1-\gamma}\left[2\sigma_k(z^{\#})_{l_1}\right]+\sigma_k(z^{\#})_{l_1}=\frac{3+\gamma}{1-\gamma}\sigma_k(z^{\#})_{l_1} \text{ and } 3+\frac{4\gamma}{1-\gamma}=\frac{3+\gamma}{1-\gamma}.$$

The above results substituted into (3.30) yield

$$N\varepsilon\le\frac{3+4\gamma/1-\gamma}{(K-k)-2\gamma/1-\gamma}\sigma_k(z^{\#})_{l_1}. \qquad (3.32)$$

Straightforward substitution of (3.32) in (3.28) yields (3.19).

(iv) Suppose that $\varepsilon>0$. If the solution set contains a *k*-sparse vector $z$ (so that $\sigma_k(z)_{l_1}=0$) with $k<K-\frac{2\gamma}{1-\gamma}$, then $z$ is equal to the unique *k*-sparse $l_1$-minimizer by Lemma 4.3. Note that since $\tau_j\in(0,1]$, $\sigma_k(z^{\#})_{l_1}\le\sigma_k(z)_{l_1}$, and $N\varepsilon\le 0$ from (3.32), which is a contradiction to $\varepsilon>0$. Hence the presence of a k-sparse solution implies that $\varepsilon=0$.

∎

# CHAPTER 5: CONCLUSION

We have developed a new IRLS algorithm based on the ideas of Daubechies et al. [6] and Miosso et al. [9]. The work of Daubechies et al. [6] aided us in proving the convergence of our IRLS algorithm, which makes use of prior information on the support of the sparse domain of the solution. This is precisely the type of prior information considered by Miosso et al. [9]. We have thus proposed an algorithm that has the advantages of each algorithm employed by these authors, namely, it is an algorithm that includes prior information on the support of the sparse domain of the solution and it is an algorithm with proven convergence properties. We have not yet supported our work by numerical experiments. This is perhaps the main weakness of our paper, and we hope that it will be remedied in the future.

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