VALUATION OF OVER-THE-COUNTER (OTC) DERIVATIVES WITH COLLATERALIZATION

by

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ABSTRACT

Collateralization in over-the-counter (OTC) derivatives markets has grown rapidly over the past decade, and even faster in the past few years, due to the impact of the recent financial crisis and the particularly important attention to the counterparty credit risk in derivatives contracts. The addition of collateralization to such contracts significantly reduces the counterparty credit risk and allows to offset liabilities in case of default.

We study the problem of valuation of OTC derivatives with payoff in a single currency and with single underlying asset for the cases of zero, partial, and perfect collateralization. We assume the derivative is traded between two default-free counterparties and analyze the impact of collateralization on the fair present value of the derivative. We establish a uniform generalized derivative pricing framework for the three cases of collateralization and show how different approaches to pricing turn out to be consistent. We then generalize the results to include multi-asset and cross-currency arguments, where the underlying and the derivative are in some domestic currency, but the collateral is posted in a foreign currency. We show that the results for the single currency, multi-asset case are consistent with those obtained for the single currency, single asset case.

To my family

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CHAPTER 1

INTRODUCTION

1.1 Financial Markets and Products

Financial markets exist because they enable an efficient allocation of resources across time and across different states of nature. Take, for example, the case of a young individual who just entered the job market with a high salary. If there are financial markets available, the earned income could be invested in financial instruments, such as stocks and bonds, to finance the cost of home ownership, more education, or retirement. The salary may be high in the present but this may not always be the case. This implicit uncertainty makes necessary to prepare for unfortunate states of nature by trying to move resources from the present time to unknown times. If there were no financial markets available, all this individual could do is consume since there would not be mechanisms in place to transfer money from one state of nature, where income is readily available, to another state of nature, where the individual may have more limited income or no income at all. Now, consider the situation of a farmer who produces oranges in Florida. With financial markets, the farmer could use a derivative contract to hedge orange prices. A forward contract, a futures contract, or a weather derivative would be very appropriate to prepare for potential losses caused by unexpected and damaging seasonal effects. If there are no financial markets available, the farmer is just subject to whatever happens at any particular time, with no mechanisms in place to prevent or offset losses. Financial markets, then, have three main important roles that makes them essential to our society. First, they aggregate information from multiple sources, organize it, and make it available to all participants and interacting agents in the market. Markets also aggregate liquidity, preventing fragmentation in order to make supply and demand work together. Finally, markets promote efficiency and fairness to all participants, particularly when there is transparency in prices, eliminating insider information practices that prevent markets from being well-functioning. In all cases, financial functions and implicit risks due to uncertainty define risk-sharing as one of the most important functions of financial markets.

Financial products are created (engineered) in financial markets to satisfy particular needs. For example, our aforementioned farmer may use derivatives to hedge risk. On the other hand, the same products could also be used for speculation (which would be the case of collateralized debt obligations). Products also allow to raise capital (venture capitalists) to fund risky projects or organizations and expect return back from them. They can also be used to fund liabilities, such as buying a house or paying for education (for example through the use of annuities). Financial products are typically traded in markets so that their price gets discovered by looking up and processing information that is readily available to all agents in a fair market. In our work, we assume that the market has the structure of a *perfectly competitive market* in which there are large numbers of buyers and sellers, sellers can easily enter into or exit from the market, and buyers and sellers are well-informed. We also assume that the market is *arbitrage-free*, a situation in which all relevant assets are priced in such an appropriate way that it is not possible for any individual gains to outpace market gains without taking on additional risk. This is commonly known as the *no-arbitrage condition*. Almost every product in the financial market is priced using the no-arbitrage condition with respect to some underlying primary asset like a stock, a bond, or a commodity of some kind. The situation is also known as the *no-free lunch argument*: every nonzero, nonnegative payoff comes with a cost.

1.2 Financial Economics and Financial Engineering

Financial economics is concerned with setting interest rates and pricing bonds, equities, and other primary financial assets[1] by using fundamental equilibrium arguments. Financial engineering relies on financial economics by usually assuming that interest rates and prices of equities are given and uses that information to price derivatives based on *no-arbitrage arguments* with respect to some underlying primary asset like a stock or a bond. Financial engineering is comprised by three major areas: security pricing, portfolio selection, and risk management. *Security pricing* deals with pricing derivatives securities, such as forwards, swaps, futures, options, collateralized debt obligations (CDOs), and collateralized mortgage obligations (CMOs). All these products are developed by financial engineers and it becomes necessary to assign a price to them. The work presented in this thesis falls precisely under this category. In *portfolio selection*, the goal is to choose a portfolio composition and a trading strategy to maximize the expected utility with respect to consumption and final wealth. The other major area, particularly after the recent financial crisis, is *risk management*, which aims to understand the risks inherent in a portfolio and determine the probability of large losses.

1.3 Derivatives Markets and Security Design

The use of derivatives in financial markets has become increasingly important over the last four decades, even though these products have practically been around in some form for hundreds of years. Many different types of derivatives are traded regularly by financial institutions and fund managers in the over-the-counter (OTC) markets, as well as on

[1]Primary assets refer to those from which other assets are constructed

many exchanges throughout the world. Derivatives are frequently added to bond issues, used for executive and employee compensation plans, and embedded in many important capital investment opportunities. This is why it is crucial to understand how derivatives work, how they can be used, and how they can be priced fairly.

In general terms, a *derivative* can be defined as a financial instrument whose value depends on the price of other, more basic, underlying variables, such as the prices of assets (commodities) that can be traded, or even the amount of rain falling at a certain region in some particular season. In a general context, a financial instrument can simply be thought of as a contract or agreement between two parties. A stock option, for example, is a type of derivative whose value is determined by the price of a stock. In this case, we refer to the stock as the *underlying asset* of the derivative. For the case of the amount of rain or any other uncertain condition, a derivative can be thought of as a bet on the price of a commodity dependent on a future outcome of the underlying (weather, in this case), which can be used to provide some kind of insurance and hedge the parties involved against potentially unfavorable outcomes. This contract is called a weather derivative. Note that investors could also use this kind of contract simply to speculate on the price of the commodity, but in this case the contract would not be insurance. Hence, the risk-reducing nature of a derivative does not depend on the derivative contract itself but on how the contract is used and who is using it.

There are three distinct perspectives on derivatives that define how we think about them and how we use them for a specific purpose [13]. The *end-user perspective* considers corporations, investment managers, and investors as end-users. These users enter into derivative contracts with specific goals in mind, such as to manage risk, speculate (as a way of investing), reduce financial transaction costs, or circumvent regulatory restrictions, and are more concerned about how a derivative can help them to achieve the goals. The *market-maker perspective* considers traders or intermediaries as market-makers who are interested in buying or selling derivatives to customers for a profit. In order to make money, market-makers charge a *spread*: they buy at a low price from customers who wish to sell and sell at a higher price from customers who wish to buy. Market-makers typically hedge the risk of supply and demand and thus they are mainly concerned about the mathematical details of derivatives pricing (valuation) and hedging. The work presented in this thesis falls under the market-maker perspective. Finally, the *economic observer perspective* gathers information and analyzes the general use of derivatives, the activities of the market-makers, the organization of markets, and the different elements that hold everything together.

This work deals precisely with one of the major concepts in financial engineering and derivatives: *it is possible to construct a given financial product from other products and create a given derivative payoff in multiple ways*. This is why the general problem of valuation of derivatives is so important and also central in understanding how market-making works. The market-maker sells a derivative contract to an end-user. With the proper pricing, this creates an offsetting position that pays the market-maker if it becomes necessary to pay the customer. This suggests that it is possible to customize the contract to make it more appropriate for particular situations. The idea of customization is based on the idea that a given contract can be replicated, which is why we will use the concept of a *replicating portfolio* as an important tool for derivative pricing in this thesis work.

Finally, it is worth emphasizing the two different kinds of derivative trading markets: exchange-traded markets and over-the-counter markets (OTC markets). A *derivatives exchange* is a market where individuals trade standardized contracts that have been defined by the exchange. Some traditional derivatives exchanges are The Chicago Board of Trade (CBOT), The Chicago Mercantile Exchange (CME), and the Chicago Board Options Exchange (CBOE). Many other exchanges all over the world now trade futures and options, with foreign currencies, stocks, and stock indices as underlying assets. The *OTC market* is an important alternative to exchanges that has become increasingly popular and larger than the exchange-traded market. Essentially, trades are carried out over the phone and are generally between two financial institutions or between a financial institution and one of its clients (such as a fund manager or corporate treasurer) as part of a telephone- and computer- linked network of dealers. A major and characteristic advantage of the OTC market is that the terms of a contract may differ from those specified by an exchange. Market participants have the freedom to negotiate any deal that benefits the parties involved. A resulting disadvantage is that there may be some credit risk in a trade (i.e., a small risk that the contract will not be honored). This key element became increasingly important, particularly after the recent financial crisis, bringing about the idea and need to incorporate the figure of collateralization to OTC derivatives, which completes the framework of this thesis work. We often refer to OTC derivatives with collateralization simply as *collateralized OTC derivatives*.

1.4 Collateralization

In lending agreements, collateral is a deposit of specific property (an asset) as recourse to the lender to secure repayment in case the borrower defaults on the initial loan. Collateralization provides lenders a sufficient reassurance against default risk. If a borrower defaults on a loan, then the borrower must forfeit the collateral asset and the lender takes possession of the asset. For example, in typical mortgage loan transactions, the real estate being acquired by means of a loan serves as collateral. Businesses can also use collateralization for debt offerings through the use of bonds. In this case, such bonds would clearly specify the asset being used as collateral for the repayment of the bond offering in case of default. In banking, lending with collateralization often refers to secured lending, asset-based lending, or lending secured by an asset. This type of lending usually presents unilateral obligations secured by property as collateral. Over the past few years, the use of more complex collateralization agreements to secure trade transactions has increased rapidly, particularly in response to the recent financial crisis. In this type of agreement, also known as capital market collateralization, the obligations are often bilateral and secured by more liquid assets such as cash or securities with corresponding interest rates. Cash is the most popular choice for collateral due to the ease of valuation, transfer, and hold.

To understand how bilateral agreements work, suppose two parties enter into a swap, which is one type of OTC derivative that transforms one kind of cash flow into another. As interest rates change over time, one party will have a *mark-to-market* (MTM) profit on the deal, while the other party will have a loss. If the party losing money were to be in default, the party with MTM profit would have to replace the deal at current market prices and the profit would be lost. Hence, a positive MTM value on the swap is a credit exposure on the other party. For this reason, banks often state credit risk on a swap as the MTM value plus some additional value to offset the potential future credit exposure. This makes collateralization a very important risk management tool to mitigate counterparty credit risk in derivatives contracts. Here, *mark to market* refers to a measure of the fair value of accounts that can change over time, such as assets or liabilities, under the notion of market fairness to all participants, as mentioned previously in Section 1.1.

Collateral management is the term used to describe the process of reducing counterparty credit exposures in derivatives contracts. It is normally used with OTC derivatives, such as swaps and options. When two parties agree to enter into an agreement with collateralization, they negotiate and execute a collateral support document that contains the specific terms and conditions for the collateralization. The trades subject to collateral are regularly marked-to-market and their net valuation is part of the agreement. The party with negative MTM on the trade portfolio must post collateral to the party with positive MTM, which, in turn, must pay the counterparty the margin at the collateral rate. As prices move and new deals are added, the valuation of the trade portfolio will change. Depending on what is agreed, the valuation is repeated at frequent intervals, typically daily, weekly or monthly. However, the collateral settled in a daily basis is the most common practice. This makes, in many cases, the collateral rate to be the overnight index rate of the collateral currency in accordance to the specific terms of the agreement. The collateral position is then adjusted to reflect the new valuation of the portfolio. The process is repeated and the posted collateral changes with the value of the trades. The process continues unless one of the parties defaults. In agreements with collateralization, the trades can be terminated in case of default and the collateral can be used as repayment of the contract. If the collateral is sufficient, the MTM profit is protected and the credit risk is mitigated. Note that this process requires careful analysis of the trades involved in order to determine their accurate value. Collateral management can be thought of as a process that exchanges credit risk for operational risk.

1.5 Security Pricing with Collateralization

Collateralization in OTC derivatives markets has grown rapidly over the past decade, and even faster in the past few years, due to the impact of the recent financial crisis and the particularly important attention to the counterparty credit risk. According to the International Swaps and Derivatives Association (ISDA) Margin Survey [19], about

70% of the trade volumes for OTC derivatives were collateralized at the end of 2009, as opposed to barely 30% in 2003. Coverage has also gone up to 78% and 84% for all the OTC and fixed income derivatives, respectively [3], and now more than 80% of the collateral posted is cash (about 50% in USD). Agreements in trading among dealers to collateralize mutual exposures (and hence reduce credit risk) are based on the *Credit Support Annex* (CSA) to the ISDA master agreement, which gives a detailed specification of all the terms of the transactions. The CSA is essentially a legal document regulating credit support for derivative transactions. Collateralized trades are often referred to as *CSA trades*.

Collateralization significantly reduces the counterparty credit risk (i.e., the party with negative present value of the derivative). As collateral is used to offset liabilities in case of default, it could be thought of as an essentially risk-free investment, so the interested rate on the posted collateral is usually set to be a proxy of a risk-free rate. Purchased assets are often posted as collateral against the funds used to buy them, such as in the repurchase agreement market (simply known as the *repo market* [2]) for shares used in delta hedging. In this case, the goal is to reduce (hedge) the risk associated with price movements in the underlying asset by offsetting long and short positions [1]. Since this thesis work focuses mainly on the area of security pricing with collateralization, we do not consider credit risk factors in the valuation of derivatives. We only consider the difference in pricing between non-collateralized and collateralized derivatives, derived from the cost associated with the collateralization. Moreover, since daily portfolio reconciliation has rapidly become the market standard, we also assume that the collateral account is adjusted continuously, which turns out to be a good approximation. It is finally worth noting that due to the nature of the derivatives we consider in the OTC market, the specific

[2] 'repo'is the name given to a form of short-term borrowing for dealers in government securities. The dealer sells the securities to investors on an overnight basis and buys them back the following day. This practice is a repo for the the party selling the security, which is also agreeing to repurchase it at a later time.

terms of collateralization may vary from case to case. We only seek to cover a number of general scenarios for varying dynamics (processes) of the underlying and collateral but the results can be extended to different cases. Other extensions also allow for partial or perfect collateralization, and the formulas presented in this work consider both features in the general setting under the scenarios covered.

1.6 Mechanisms of Unsecured and Secured Trades with External Funding

To better understand the general effects of collateralization and its increasing importance in the OTC derivatives market, let us now briefly illustrate the mechanisms of unsecured and secured trades (contracts) with external funding, following the description presented in [3]. The first situation is depicted in Figure 1.1.

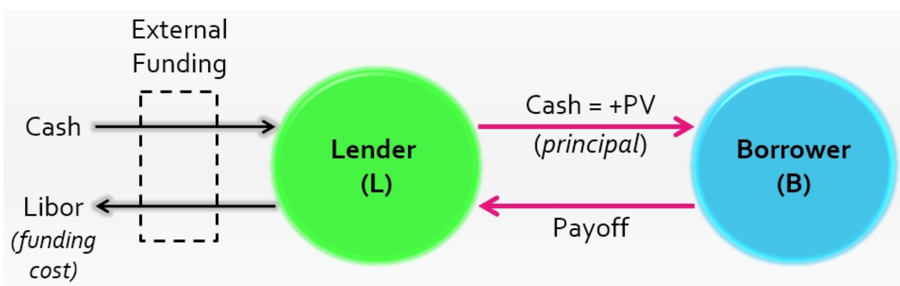

Figure 1.1: Unsecured contract with external funding

Consider two parties, L (for *lender*) and B (for *borrower*), that enter into a derivative contract. L has a positive *present value* (PV) in the contract with B (which is assumed to have high credit quality). By a positive PV we mean a receipt of cash by L at a future time from the contract. From the perspective of L, the situation is equivalent to providing a loan to the counterparty B with the principal value equal to its PV. Since L has to wait for the payment from B until the maturity of the contract, L has to finance its loan to B and hence the pricing of the contract should reflect the funding cost to L. If L has and maintains LIBOR[3] credit quality, the funding cost is given by the LIBOR of its funding currency since it makes the PV of 'funding' zero. This is the main reason why LIBOR is widely used as a proxy of the discounting rate in the OTC derivative pricing.

The above situation changes considerably when collateral is added to the contract, As mentioned previously, we now assume that the trade has been made with a CSA, requiring cash collateral with zero minimum transfer amount. The situation with collateralization is depicted in Figure 1.2.

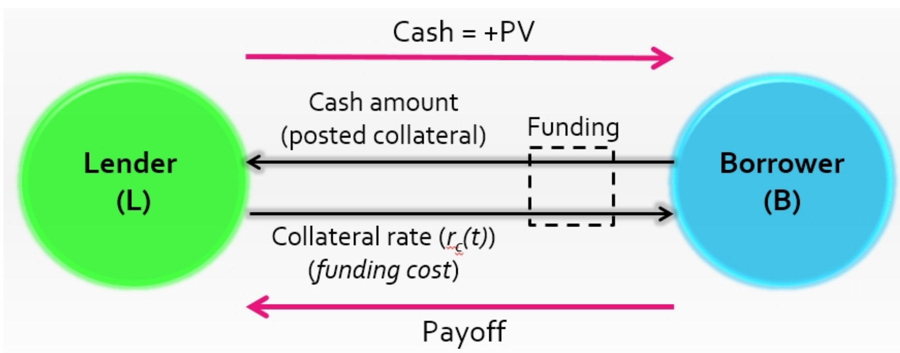

Figure 1.2: Collateralized (secured) contract with external funding

In this case, L does not require external funding since B is posting an amount of cash (collateral) equal to the PV of the contract. However, L has to pay the counterparty B the margin at the *collateral rate* $r_c(t)$ (return rate of the collateral) applied to the posted collateral amount (or *outstanding collateral*). This makes the funding cost of the contract equal to the collateral rate. According to [19], the most popular collateral in the current financial market is the cash of the developed countries and the typical choice of collateral

[3]London Inter-Bank Offered Rate

rate is the *overnight rate* (ONR) of the corresponding currency.

As mentioned in [9], the impact of collateralization to the valuation of OTC derivatives is particular higher when the borrowing rate of the derivative desk (here denoted in general by L) is significantly higher than the collateral rate designated in the corresponding CSA. The conventional LIBOR-OIS[4] spread is usually regarded as an indicator of such a gap.

The models for discounting projected cash flows of the derivative with the collateral rate (often referred to as *collateral rate discounting*) implies several main assumptions [1, 2, 9]:

1. *Full collateralization*: the posted collateral amount equals the PV (or MTM) of the derivative;
2. *Symmetric (or bilateral) collateralization*: each counterparty posts collateral when the MTM of the derivative is negative from its own perspective and receives the same collateral rate;
3. *Continuous adjustment*: the collateral is adjusted immediately the MTM changes;
4. *Domestic collateralization*: the collateral and the derivative payoff are in the same currency;
5. *Cash-equivalent collateral*: the posted collateral must be essentially risk-free and have the highest quality.
6. *No counterparty credit risk*: the counterparties are both assumed to be default-free

[4]Overnight Index Swap

In this thesis work, as in [9], a collateralized derivative that satisfies all the above assumptions is referred to as a *perfectly collateralized derivative*, whereas a *fully collateralized derivative* refers to perfect collateralization with collateral currency different from the payoff currency (assumption 4 is relaxed). Similarly, a *partially collateralized derivative* refers to perfect collateralization with both assumptions 1 and 4 relaxed. In the development of the model for single currency, we will often relax the terminology a little bit more and will simply consider partial collateralization as that occurring when only assumption 1 is relaxed.

1.7 Brief Discussion on Some Related Works

The theoretical foundation of valuation of OTC derivatives for single currency and single asset, under the aforementioned assumptions, has been laid out in the works of Piterbarg [1], Fujii [2], and Castagna [6]. Each of these authors introduce different approaches in a general setting to develop valuation models under specific considerations for the underlying and collateral processes.

Piterbarg [1] uses replicating portfolio and self-financing arguments to derive the general derivative pricing formula, which after some manipulation allows to obtain specific expressions for the cases of zero and perfect collateralization. Piterbarg does not state any particular process for the collateral account but his analysis includes a comprehensive replicating portfolio with underlying stock and a cash amount split among different detailed accounts. Piterbarg's work only focuses on single currency and does not include partial collateralization arguments.

Fujii [2] introduces a stochastic process for the collateral account with an appropriate self-financing strategy and a reinvestment argument that makes the process dependent on the dynamics of the derivative. In order to solve an implicit dependence problem of the collateral and the value of the derivative in his argument, Fujii considers a simple trading strategy for the collateral and the number of positions of the derivative that allows him to obtain the formula for a perfectly collateralized derivative in single currency. Fujii also suggests a different valuation approach for single currency and presents an equivalent formula for the cross-currency scenario, but omits important details in his analysis.

Castagna [6] introduces the concept of *Liquidity Value Adjustment* (LVA) and uses it to make a distinction between a collateralized derivative and one without collateral. Castagna follows the steps of Piterbarg by using the concept of a self-financing replicating portfolio with a modified version of the underlying asset and collateral processes. His work is mainly focused on the development of a general derivative valuation model with an extension to partial collateralization. Even though Castagna achieves the desired results, his work seems to have some inconsistencies in the derivation that are not clearly explained, particularly regarding an apparent conflict in his proposed collateral process and the partial collateralization constraint.

1.8 Our Contribution

We first study the problem of valuation of OTC derivatives with single underlying asset and derivative payoff in single currency for the cases of zero, partial, and perfect collateralization. We assume the derivative is traded between two default-free counterparties and analyze the impact of collateralization on the fair present value of the derivative. We follow the different ideas presented in [1], [2], and [6], present a complete mathematical derivation of the results, and seek to establish a uniform derivative pricing framework for the three cases of collateralization. To achieve the goal, we present different approaches to pricing under specified conditions and show that the results are uniformly consistent despite the differences in the strategies.

In particular, we show how the general results presented in [1] can be extended to allow for partial collateralization and that these results are consistent with those presented in [2] and [6]. As described in [9], we first show how a portfolio including an underlying asset for the derivative and cash positions with various funding sources and corresponding short rates is constructed to replicate the value of the derivative. For the mathematical derivation, we use an approach based on a generalized Black-Scholes-Merton framework with collateral under CSA to obtain a fundamental PDE based on a self-financing condition. Applying the Feynman-Kac formula to the PDE yields the desired pricing solution for perfect collateralization. Finally, we use the partial collateralization argument to extend this result and obtain a generalized pricing formula that covers all states of collateralization. We also introduce at this point the concept of LVA and show how the value of a collateralized derivative is, in fact, equal to the value of the derivative without collateralization plus some adjustment value, known as LVA.

In the second part of the single currency, single asset analysis, and based on [2], we derive a derivative pricing formula for perfect collateralization by using Q-martingale arguments. We define some stochastic process and show that such a process is a Q-martingale. Then, we show that the price process of a collateralized derivative can be expressed as a function of a certain martingale process, which after some manipulation leads to the perfectly collateralized derivative pricing formula.

The second major part of our work focuses on establishing a generalized pricing framework to include cross-currency and multi-asset arguments. Up to this point, both the derivative and the posted collateral were in the same currency. Now, in the the cross-currency, single asset analysis, the underlying asset and the derivative are in some domestic currency, but the collateral is posted in a foreign currency. We refer to this as full collateralization which is, in fact, the case of perfect collateralization with collateral posted in different currency. Finally, we include some general analysis for the case of single currency for a derivative that has multiple underlying assets, and show that the results are consistent with those obtained for the single asset.

CHAPTER 2

FUNDAMENTAL MATHEMATICAL CONCEPTS

In order to develop a unified valuation framework, it is essential to present the fundamental mathematical concepts, definitions, and theorems that will be used throughout this thesis work. Not only this adds consistency to the work, given the diversity in notation and presentation of the concepts used as reference, but also allows to better understand and frame the derived results under a common set of support tools with specified assumptions, considerations, and even limitations. The general purpose is for this thesis work to be as self-contained as possible.

2.1 General Probability Theory

Definition 2.1.1 (**σ-algebra**). *Let Ω be a nonempty set. In particular, let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be the set of all possible outcomes (basic events) of a random experiment. We call Ω the* sample space *of the experiment. Let $\mathcal{F} \subset 2^{\Omega}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a* σ-algebra *provided that:*

1. *$\Omega \in \mathcal{F}$,*
2. *if A is a set such that $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and*
3. *if $A_1, A_2, \ldots$ is a sequence of sets such that $A_n \in \mathcal{F}$ for all $n \geq 1$, then $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{F}$*

This concept is essential for our purposes because if we have a σ-algebra of sets, then all the operations we might want to do to the sets will give us other sets in the σ-algebra.

Definition 2.1.2 (Borel σ-algebra). *If $\Omega = \mathbb{R}$, the Borel σ-algebra on Ω, denoted by $\mathcal{B}_1 = \mathcal{B}(\mathbb{R}) \triangleq \sigma(\mathcal{A})$, is the smallest σ-algebra on Ω containing the set $\mathcal{A}$ of all open subsets (intervals (a,b)), or equivalently all closed subsets, in $\mathbb{R}$.*

Definition 2.1.3 (Measurable Space). *If $\mathcal{F}$ is a σ-algebra of subsets of Ω, then $(\Omega, \mathcal{F})$ is called a* measurable space *and any set $A \in \mathcal{F}$ is called an* event *on Ω.*

Definition 2.1.4 (Probability Measure). *Let Ω be a nonempty set, and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A probability measure $\mathbb{P}$ is a function that assigns a number in $[0, 1]$ to every set $A \in \mathcal{F}$. We call this number the* probability of *A and write $\mathbb{P}(A)$. We require:*

1. $\mathbb{P}(\Omega) = 1$, *and*
2. *Whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.*

Definition 2.1.5 (Probability Space). *The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a* probability space.

Definition 2.1.6 (Random Variable). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A* random variable *is a real-valued function X defined on Ω (i.e., $X : \Omega \to \mathbb{R}$) with the property that for every Borel subset B of $\mathbb{R}$, the subset of Ω given by $\{X \in B\} = \{\omega \in \Omega \mid X(\omega) \in B\}$ is in the σ-algebra $\mathcal{F}$.*

A random variable X is essentially a numerical quantity whose value is determined by the random experiment of choosing any $\omega \in \Omega$. The index $t \in [0, \infty)$ of the random variables $X_t \triangleq X(t)$ admits a convenient interpretation as *time*. In order to simplify notation, we may use X_t or $X(t)$ indistinctively to denote that the variable X is a function of the parameter t. This convention applies to any other variable that is dependent on any other parameter we may use throughout this work.

2.2 Information and Stochastic Processes

Definition 2.2.1 (Fitration). *A filtration* $\{\mathcal{F}_n\}$ *is a sequence of* σ*-algebras* $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n$ *with the property that* $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F}$.

The property says that each σ-algebra in the nondecreasing sequence contains all the sets contained by the previous σ-algebra. For our purposes, we use $\mathcal{F}_t$ to denote the information available at time t. Then, the set $\{\mathcal{F}_t\}_{t\geq 0}$ is called an *information filtration*. So, $E[\bullet \mid \mathcal{F}_t]$ denotes an expected value that is conditional on the information available up to time t. We usually write $E[\bullet \mid \mathcal{F}_t]$ as $E_t[\bullet]$.

Definition 2.2.2 (Stochastic Process). *A sequence of random variables* $X_1, X_2, \ldots, X_n$ *is called a* stochastic process.

For our purposes, a *stochastic process* is a mathematical model for the occurrence of a random phenomenon at each moment (time t) after a given initial time. The random nature is captured by the measurable space $(\Omega, \mathcal{F})$ on which probability measures can be placed. Thus, a stochastic process is a collection of random variables $X = \{X_t \mid 0 \leq t < \infty\}$ on $(\Omega, \mathcal{F})$ which take values in $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. For a fixed sample point $\omega \in \Omega$, the function $t \mapsto X_t(\omega), t \geq 0$, is the sample path (trajectory) of the process X associated with ω. The temporal feature of a stochastic process suggests a flow of time in which, at every moment $t \geq 0$, we can talk about a *past, present, and future* and can ask how much an observer of the process knows about it at present, as compared to how much he knew at some point in the past or will know at some point in the future. This allows to keep track of information and provides the mathematical model for a random experiment whose outcome can be observed continuously in time. For example, we think of X_t as the price of some asset at time t and $\mathcal{F}_t$ as the information obtained by watching all the prices in the market up to time t [16].

Definition 2.2.3 (**Adapted Stochastic Process**). *Let* Ω *be a nonempty sample space equipped with a filtration* $\{\mathcal{F}_t\}, 0 \leq t \leq T$. *Let* X_t *be a collection of random variables indexed by* $t \in [0, T]$. *We say* X_t *is an* adapted stochastic process*, or that* X_t *is adapted to the filtration* $\{\mathcal{F}_t\}$ *if, for each time* t*, the random variable* X_t *is* $\mathcal{F}_t$*-measurable.*

Asset prices, portfolio processes (i.e., positions), and wealth processes (i.e., values of portfolio processes) will all be adapted to a filtration that we regard as a model of the flow of public information. Intuitively, this definition says that the information available at time $t > 0$ is sufficient to evaluate the stochastic process X_t at that time. If we know the information in $\mathcal{F}_t$, then we know the value of X_t. Consequently, if we are at some time t_0, then for some other time $t > t_0$ the value of the process X_{t_0} is known but the value of the process X_t is unknown. For our purposes, it is worth noting that the no-arbitrage theory of derivative security pricing is based on contingency plans. In order to price a derivative security, we determine the initial wealth we would need to set up a hedge of a short position in the derivative security. The hedge must specify what position we will take in the underlying security at each future time contingent on how the uncertainty between the present time and that future time is resolved [16]. We must also mention that, in practice, we do not observe stock prices following continuous-variable, continuous-time processes. Stock prices are restricted to discrete values (e.g., multiples of a cent) and changes can be observed only when the exchange is open. Nevertheless, the continuous-variable, continuous-time process proves to be a useful model for many purposes [12].

Definition 2.2.4 (**$\mathbb{P}$-Martingale**). *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* T *be a fixed positive number, and let* $\mathcal{F}_t, 0 \leq t \leq T$*, be a filtration of sub-*σ*-algebras of* $\mathcal{F}$*. An adapted stochastic* *process X_t is a martingale with respect to the information filtration $\mathcal{F}_t$ and probability measure $\mathbb{P}$ if $E^P[|X_t|] < \infty$ (integrability condition) and $E^P[X_t \mid \mathcal{F}_s] = X_s$ for all $0 \le s \le t \le T$.*

Intuitively, a martingale is a stochastic process that, on average, has no tendency to rise or fall. Martingales and measures are critical elements to a risk-neutral valuation framework. As we will see, a martingale is also defined as a zero-drift stochastic process. A measure is the unit in which we value security prices [12].

2.3 Wiener Processes and Brownian Motion

Here we follow [12] to discuss particular types of stochastic processes and their importance in the realm of derivative pricing.

Definition 2.3.1 (**Markov Process and The Markov Property**). *A* Markov process *is a particular type of stochastic process where only the PV of a variable is relevant for predicting the future. That is, the past history of the variable and the way that the present has emerged from the past are irrelevant.*

Stock prices are usually assumed to follow a Markov process. Since predictions for the future are uncertain, they must be expressed in terms of probability distributions. The *Markov property* implies that the probability distribution of the price at any particular future time does not depend on the particular path followed by the price in the past.

The Markov property of stock prices is consistent with the weak form of market efficiency. This states that the present price of a stock carries all the information contained in a record of past prices. If this were not true, then financial analysts could make above-average returns by interpreting charts of the past history of stock prices, and there is no sufficient evidence that this is actually possible. Finally, it is worth noting that it is *competition in the marketplace* what tends to ensure that weak-form market efficiency holds.

Definition 2.3.2 (**Wiener Process**). *Let W be a variable that follows a Markov stochastic process. Let W_0 be its current value and let $\phi(m, v)$ represent the change in value during some time unit v, where ϕ denotes a probability distribution that is normally distributed with* mean change per unit time*, m, called* drift rate*, and* variance per unit time*, v, called* variance rate*. A Markov stochastic process with $\phi(m, v) = \phi(0, 1)$ is called a* Wiener process*. A drift rate of zero means that the expected value of W at any future time is equal to its current value. The variance rate of 1 means that the variance of the change in W in a time interval of length T equals T. A variable W follows a Wiener process if it has the two following properties:*

1. $\Delta W = \epsilon \sqrt{\Delta t}$ *(change ΔW during a small period of time Δt, where $\epsilon \sim \phi(0, 1)$)*
2. *For any two different short intervals of time, Δt, the values of ΔW are independent*

By Property 1, $\Delta W \sim N(0, \Delta t)$. By Property 2, the variable W follows a Markov process.

We use dW to consider the change in the value of the variable W during a relatively long period of time T. Hence, dW, with the above properties for ΔW in the limit as $\Delta W \to 0$, is a Wiener process.

Definition 2.3.3 (**Generalized Wiener Process**). *A generalized Wiener process for a variable $X(a, b)$ (i.e., with parameters a,b), can be defined in terms of dW as:*

$$dX = a\,dt + b\,dW \tag{2.1}$$

The $a\,dt$ term implies that W has an expected drift rate of a . The $b\,dW$ term implies that W has a variance rate of b^2 and it can be regarded as adding noise or variability to the path followed by W. The amount of variability is b times a Wiener process. Thus, by the same properties of Definition 2.3.2, it follows that $dW \sim N(at, b^2 t)$ in any time inteval $[0, t]$.

Definition 2.3.4 (**Martingale**). *A* martingale *is a zero-drift stochastic process. A variable* Θ *follows a Martingale if its process has the form:*

$$d\Theta = \sigma\,dW \tag{2.2}$$

where dW *is a Wiener process and* σ *is a parameter that may also be stochastic.*

As implied in Definition 2.2.4, a martingale has the property that its expected value at any future time T is equal to its value today. That is:

$$E[\Theta_T] = \Theta_0 \tag{2.3}$$

Definition 2.3.5 (**Brownian Motion**). *A continuous, adapted stochastic process* $B(\mu, \sigma) = \{B_t, \mathcal{F}_t \mid 0 \le t < \infty\}$*, defined on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, is said to be a Brownian motion with parameters* μ, σ *(drift rate* μ *and variance rate* σ*) if it satisfies the following properties:*

1. *For* $0 \le s < t$*, the increment* $B_t - B_s$ *is independent of* $\mathcal{F}_s$ *and is normally distributed with mean zero and variance* $t - s$*.*
2. *For fixed times* $0 = t_0 < t_1 < \cdots < t_n$*, the increments* $(B_{t_1} - B_{t_0}), \ldots, (B_{t_n} - B_{t_{n-1}})$ *are mutually independent with mean zero (i.e.,* $E[B_{t_{i+1}} - B_{t_i}] = E[dB_t] = 0$ *and variance* $t_{i+1} - t_i$*.*
3. *For any increment* $\tau > 0, (B_{t+\tau} - B_t) \sim N(\mu\tau, \sigma^2\tau)$*.*

Definition 2.3.6 (**Standard Brownian Motion**). *When* $\mu = 0$ *and* $\sigma = 1$, *the process* $B(0,1)$ *is called a* standard Brownian motion, *denoted by* W_t^{SB}. *We always assume that this process begins at zero, so that* $W_0^{SB} = 0$. *Clearly, we also have* $W_t^{SB} \sim N(0,t)$.

If $X_t \sim B(\mu,\sigma)$ with $X_0 = x$, then we can write $X_t = x + \mu t + \sigma W_t^{SB}$. Note that $E[X_t] = x + \mu t + \sigma E[W_t^{SB}] = x + \mu t$ and $Var(X_t) = \sigma^2 Var\left(W_t^{SB}\right) = \sigma^2 t$. Hence, $X_t \sim N(x + \mu t, \sigma^2 t)$.

The following standard results are admitted without proof. See [16].

Theorem 2.3.7. *Standard Brownian motion is a Markov process.*

Theorem 2.3.8. *Standard Brownian motion is a martingale.*

2.4 Geometric Brownian Motion and The Process for a Stock Price

Definition 2.4.1 (**Geometric Brownian Motion**). *A continuous, adapted stochastic process* $\{S_t, \mathcal{F}_t \mid 0 \le t < \infty\}$ *is a* geometric Brownian motion *with parameters* μ, σ, *and we write* $S_t \sim GBM(\mu,\sigma)$, *if, for all* $t \ge 0$,

$$S_t = S_0\, e^{\left(\mu - \frac{1}{2}\sigma^2\right)t \,+\, \sigma W_t^{SB}} \tag{2.4}$$

and it satisfies the following properties:

1. *If* $S_t > 0$, *then* $S_{t+s} > 0$ *for any* $s > 0$.
2. *The distribution of* $\frac{S_{t+s}}{S_t}$ *only depends on s and not on* S_t.

It can be shown that *geometric Brownian motion in differential form* is given by the following stochastic differential equation:

$$dS_t = \mu\, S_t\, dt \,+\, \sigma\, S_t\, dW_t^{SB} \tag{2.5}$$

The above properties suggest that geometric Brownian motion is a reasonable model for stock prices. This is in fact the asset-pricing model used in the Black-Scholes-Merton option pricing formula.

Definition 2.4.2 (**The Process for a Stock Price**). *Let* $W_t^{SB}(0 \leq t \leq T)$ *be a standard Brownian motion defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, with filtration* $\mathcal{F}_t$ $(0 \leq t \leq T)$ *and parameters* $\mu_t \triangleq \mu(t)$ *and* $\sigma_t \triangleq \sigma(t)$ *adapted to the filtration. The stock price model is defined to be:*

$$dS_t = \mu\, S_t\, dt \; + \; \sigma\, S_t\, dW_t^{SB} \tag{2.6}$$

Let us analyze briefly the above definition. Let S_t be the *stock price* at time t. Let μ_t be the *expected rate of return of the stock* (per unit of time) at time t, expressed in decimal form. Define the *volatility of the stock price*, σ_t, to be some measure of how uncertain we are about the future stock price movement. If the volatility of the stock price is always zero, we can express the rate of change of the stock in a short interval of time Δt as:

$$\frac{\Delta S_t}{\Delta t} = \mu_t\, S_t \Rightarrow \Delta S_t = \mu_t\, S_t\, \Delta_t \tag{2.7}$$

A reasonable assumption for the volatility σ_t is that the variability (uncertainty) of the percentage return of the stock in a short period of time Δt is the same regardless of the stock price. This suggests that the standard deviation of the change in a short period of time Δt should be proportional to the stock price S_t. Hence, in the limit, the process for a stock price can be written as $dS_t = (\mu_t\, S_t)\, dt + (\sigma_t\, S_t)\, dW_t^{SB}$. By Definition 2.4.1, this is a geometric Brownian motion in differential form which, in practice, is the most widely used model of stock price behavior. Note that the stock price model is completely general and subject only to the condition that the paths of the process are continuous [16].

Definition 2.4.3 (The Change in Wealth of an Investor). *Consider an investor who begins with non-random initial wealth X_0 and holds $\Delta_t \triangleq \Delta(t)$ shares of stock at each time t (Δ_t can be random but must be adapted). Suppose the stock is modelled by a geometric Brownian motion (in differential form) given by Definition 2.4.1. Suppose also that the investor finances his investing by borrowing or lending at interest rate r. If X_t denotes the wealth of the investor at each time t, then*

$$dX_t = r\,X_t\,dt + (\mu - r)\,\Delta_t\,S_t\,dt + \Delta_t\,S_t\,\sigma\,dW_t^{SB} \tag{2.8}$$

The three terms in this equation can be understood as follows:

- an average underlying rate of return r on the portfolio, which is the term $r\,X_t\,dt$,
- a risk premium $\mu - r$ for investing in the stock, which is the term $(\mu - r)\,\Delta_t\,S_t$, and
- a volatility term proportional to the size of the stock investment, which is reflected in the term $\Delta_t\,S_t\,\sigma\,dW_t^{SB}$.

Remark: The process for a stock price developed in Definition 2.4.2 involves two parameters: μ and σ. The parameter μ is the (annualized) expected return earned by an investor in a short period of time. Since most investors require higher expected returns to induce them to take higher risks, it follows that the value of μ should depend on the part of the risk that cannot be diversified away by the investor. Moreover, it should also depend on the level of interest rates in the economy. The higher the level of interest rates, the higher the expected return required on any given stock. The value of a derivative with a stock as underlying asset is, in general, independent of μ. In contrast, the stock price volatility σ plays a major role in the valuation of many derivatives. The stock price volatility is equal to the standard deviation of the continuously compounded return provided by the stock in 1 year. Typical values for σ for a stock are in the range of 0.15 (15%) to 0.60 (60%) [12].

2.5 Itô Process, Itô's Formula, and the Feynman-Kac Theorem

In this section, we cover stochastic processes that are more general than Brownian motion and define the integrals, and their properties, used to model the value of a portfolio that results from trading assets in continuous time.

Definition 2.5.1 (Generalized Itô Process). *A* generalized Itô process *is a generalized Wiener process (see Definition 2.3.3) in which the parameters a and b are functions of both the value of the underlying variable X and time t. Hence, it can be written as:*

$$dX = a(X,t)\,dt \;+\; b(X,t)\,dW \tag{2.9}$$

Definition 2.5.2 (Itô Process). *Let W_t^{SB} $(t \geq 0)$ be a Brownian motion with filtration $\mathcal{F}_t$, $(t \geq 0)$. An* Itô process *is a stochastic process of the form*

$$X_t = X_0 + \int_0^t \Theta_u\, u \;+\; \int_0^t \Delta_u\, dW_u^{SB} \tag{2.10}$$

where X_0 is non-random and Θ_t and Δ_t are adapted stochastic processes. In differential notation,

$$dX_t = \Theta_t\, dt \;+\; \Delta_t\, dW_t^{SB} \tag{2.11}$$

Lemma 2.5.3 (Quadratic Variation of the Itô Process). *The* quadratic variation of the Itô process*, in differential form, is $(dX_t)\,(dX_t) = \Delta_t^2\, dt$, given the differential multiplication table $(dW_t^{SB})\,(dW_t^{SB}) = dt$, $(dt)\,(dW_t^{SB}) = (dW_t^{SB})\,(dt) = 0$, and $(dt)\,(dt) = 0$.*

Theorem 2.5.4 (Itô-Doeblin Formula for an Itô Process). *Let X_t, $t \geq 0$, be an Itô process, as stated in differential form in Definition 2.5.2, with a drift rate of Θ_t and a variance rate of Δ_t^2.*

Let $F \triangleq F(t, X_t)$ be a function with defined and continuous partial derivatives $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial X_t}$, and $\frac{\partial^2 F}{\partial {X_t}^2}$. Then, for every $T \geq 0$,

$$\begin{aligned} dF &= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X_t} dX_t + \frac{1}{2} \frac{\partial^2 F}{\partial {X_t}^2} (dX_t)(dX_t) \\ &= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X_t} dX_t + \frac{1}{2} \frac{\partial^2 F}{\partial {X_t}^2} \Delta_t^2 \, dt \\ &= \left(\frac{\partial F}{\partial t} + \frac{1}{2} \Delta_t^2 \frac{\partial^2 F}{\partial {X_t}^2} \right) dt + \left(\frac{\partial F}{\partial X_t} \right) dX_t \qquad (2.12) \\ &= \mathcal{L}(F) \, dt + \left(\frac{\partial F}{\partial X_t} \right) dX_t \qquad (2.13) \end{aligned}$$

where

$$\mathcal{L}(F) \triangleq \frac{\partial F}{\partial t} + \frac{1}{2} \Delta_t^2 \frac{\partial^2 F}{\partial {X_t}^2} \qquad (2.14)$$

is called the standard pricing operator*.*

Using $dX_t = \Theta_t \, dt + \Delta_t \, dW_t^{SB}$, we also have

$$\begin{aligned} dF &= \left(\frac{\partial F}{\partial t} + \Theta_t \frac{\partial F}{\partial X_t} + \frac{1}{2} \Delta_t^2 \frac{\partial^2 F}{\partial {X_t}^2} \right) dt + \left(\Delta_t \frac{\partial F}{\partial X_t} \right) dW_t^{SB} \qquad (2.15) \\ &= \mathcal{L}^{\Theta}(F) \, dt + \left(\Delta_t \frac{\partial F}{\partial X_t} \right) dW_t^{SB} \qquad (2.16) \end{aligned}$$

which implies that F also follows an Itô process with drift rate $\mathcal{L}^{\Theta}(F) \triangleq \frac{\partial F}{\partial t} + \Theta_t \frac{\partial F}{\partial X_t} + \frac{1}{2} \Delta_t^2 \frac{\partial^2 F}{\partial {X_t}^2}$ and variance rate $\left(\Delta_t \frac{\partial F}{\partial X_t} \right)^2$.

The following important theorem relates stochastic differential equations and partial differential equations. Before stating the theorem, we first define an important concept regarding interest rates and establish an important convention for very useful notation.

Definition 2.5.5 (Compounding and Discounting Processes). *Suppose we have an adapted* interest rate process $R_t \triangleq R(t)$. *We define the following processes:*

1. *Compounding process:*

$$\Phi_{a,b}^{R(t)} \triangleq e^{\int_a^b R(u)\,du} \tag{2.17}$$

2. *Discounting process:*

$$\frac{1}{\Phi_{a,b}^{R(t)}} \triangleq e^{-\int_a^b R(u)\,du} \tag{2.18}$$

Theorem 2.5.6 (Feynman-Kac Formula). *Let X_t, $t \geq 0$, be an Itô process driven by the SDE $dX_t = \Theta(t, X_t)\,dt + \Delta(t, X_t)\,dW_t^{SB}$ with initial condition $X_0 = x$. Let $T > 0$ be a fixed parameter and let $f(t, x) \triangleq f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be the solution of:*

$$\frac{\partial f}{\partial t} + \Theta_t \frac{\partial f}{\partial X_t} + \frac{1}{2}\Delta_t^2 \frac{\partial^2 f}{\partial {X_t}^2} = q(t, x)\, f - g(t, x) \tag{2.19}$$

$$f(T, x) = h(x) \tag{2.20}$$

Then,

$$f(t, x) = E_t^Q\left[\int_t^T e^{-\int_t^s q(X_u)\,du}\, g(s, X_s)\,ds \;+\; e^{-\int_t^T q(X_u)\,du}\, h(X_T)\right] \tag{2.21}$$

$$= E_t^Q\left[\int_t^T \frac{1}{\Phi_{t,s}^{q(X_t)}}\, g(s, X_s)\,ds \;+\; \frac{1}{\Phi_{t,T}^{q(X_t)}}\, h(X_T)\right] \tag{2.22}$$

Note that Equation (2.19) corresponds to the Black-Scholes-Merton differential equation with an additional term $-g(t, x)$. If X_t represents the process for a stock price, then $h(X_T) = h(X(T))$ denotes the payoff at time T of a derivative security whose underlying asset is the geometric Brownian motion (stock price model) stated in Definition 2.4.2.

Because the stock price is Markov and the payoff is a function of the stock price alone, there is a function $f(t,x)$ such that $F(t) = f(t, X(t))$, where $F(t) \triangleq f(t,x)$ may represent the value of a derivative with underlying stock at each time t.

2.6 Some Additional Concepts of Interest in Security Pricing

2.6.1 On Interest Rates

Two important rates for derivative traders are Treasury rates and LIBOR rates. *Treasury rates* are those paid by a government in its own currency. *LIBOR rates* are short-term lending rates offered by banks in the interbank market. Derivative traders usually assume that the LIBOR rate is a risk-free rate [12].

The risk-free rate affects directly the price of a derivative. As interest rates in the economy increase, the expected return required by investors from the stock tends to increase. In addition, the PV of any future cash flow received by the holder of the derivative decreases. In general, the discounting rate that should be used for the expected cash flow at a future time T must equal at least an investor's required return on the investment. We assume the risk-free rate r is the nominal rate and not the real (effective) rate. We also assume $r > 0$. Otherwise, an investment at the risk-free rate would provide no advantages over cash.

As mentioned previously, it is also important to assume that there are some market participants, such as large investment banks, for which the following statements are true:

1. There are no arbitrage opportunities.

2. Borrowing and lending are possible at the risk-free interest rate.

3. There are no transactions costs.

4. Trading profits (net of trading losses) are not subject to tax rates.

2.6.2 Risk-Neutral Valuation (Risk-neutral World vs. Real World)

The *risk-neutral valuation principle* in derivative pricing states that we can assume the world is risk-neutral when pricing derivatives. In a *risk-neutral world*, all individuals are indifferent to risk. Investors require no compensation for risk and the expected return on all securities is the risk-free interest rate. The principle is based on one key property of the Black-Scholes-Merton differential equation that was stated in Theorem 2.5.6: the variables that appear in the equation, namely the current stock price, time, stock price volatility, and the risk-free interest rate, are all independent of the risk preferences of investors. Hence, risk preferences cannot affect the solution to the equation. Only the value of μ, the expected return on the stock, that appears in the equations (but not on their solution because the term drops out at some point) depends on the risk preferences of investors. The higher the level of risk aversion by investors, the higher μ will be for any given stock. When $\mu = r$, we have what is called a *traditional risk-neutral world*.

Any set of preferences can be used when evaluating the price of derivative. Moving from one set of risk preferences, to another is called *changing the measure*. As stated in [12], choosing a particular market price of risk is also referred to as *defining the probability measure*. The *real-world probability measure* is known as the $\mathbb{P}$-*measure*, the one we used in Definition 2.1.4 or in Definition 2.2.4 when the $\mathbb{P}$-martingale was defined . The *risk-neutral world probability measure* is referred to as the *Q-measure*. The particular measure we use will be indicated as a superscript, particularly when stating results involving expected values.

2.6.3 The Greeks: Delta of a Derivative

The delta of a stock derivative is defined to be the rate of change of the derivative price with respect to the price of the underlying asset. It is the number of units of the stock we should hold for each derivative shorted in order to create a riskless portfolio. The term appears in Theorem 2.5.4 as:

$$\frac{\partial F}{\partial X_t} \triangleq \frac{\partial V}{\partial S_t} \triangleq \Delta^s(t) \triangleq \Delta^s_t \tag{2.23}$$

where $V \triangleq V(t) \triangleq V_t$ is the value of the derivative and S_t is the price of the underlying asset (stock) at time t. The reason a riskless portfolio can be setup is that the stock price S_t and the derivative price V_t are both affected by the same underlying source of uncertainty, namely the stock price movements.

2.6.4 The Numeraire

Suppose f and g are the prices of traded securities dependent on a single source of uncertainty. As previously stated, assume that the securities pay no dividends during the period under consideration. Define $\phi \triangleq \frac{f}{g}$, which can be thought of as the relative price of f with respect to g or, most importantly, as measuring the price of the security f in units of g rather than USD. When this is the case, the security price g is referred to as the *numeraire*.

The USD money market account is a security that is worth $1 at time zero and earns the instantaneous risk-free rate r at any given time, where r may be stochastic. If we set g equal to the *money market account*, it grows at the rate r, so that $\frac{dg}{dt} = rg \Rightarrow dg = r\,g\,dt$. Note that the drift rate of g is stochastic but the volatility of g is zero.

CHAPTER 3

SINGLE CURRENCY, SINGLE ASSET ANALYSIS

3.1 Pricing by Replication without Specified Collateral Process

We first review and present with detailed proof the work done in [1] for the case of perfect collateralization, and we extend the main results to develop a more general framework to solve the derivative pricing problem including all states of collateralization.

3.1.1 The Processes for the Underlying and the Derivative

Let $V(t) \triangleq V(t, S_t),\ t \geq 0$, be the price of a collateralized derivative with underlying asset S_t, at each time t. By definition 2.4.2, assume that, under a given measure, the stock follows a process given by:

$$dS_t = \mu_t\, S_t\, dt \ + \ \sigma_t\, S_t\, dW_t^{SB} \tag{3.1}$$

Then, by Theorem 2.5.4, the price process of the derivative is given by:

$$dV(t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma_t^2\, S_t^2 \frac{\partial^2 V}{\partial {S_t}^2}\right) dt + \left(\frac{\partial V}{\partial S_t}\right) dS_t \tag{3.2}$$

$$= \mathcal{L}(V)\, dt + \Delta_t^s\, dS_t \tag{3.3}$$

where $\mathcal{L}(\bullet)$ is the standard pricing operator defined in the same theorem and Δ_t^s is the delta of the stock derivative, as defined in Section 2.6.3.

3.1.2 *Self-financing Replicating Portfolio and Black-Scholes Equation with Collateral*

To replicate the derivative, consider a self-financing replicating portfolio whose value, at each time t, is denoted by Π_t and determined by the following components:

- A cash amount, denoted by γ_t, split among the following accounts:
  - Collateral account, denoted by C_t, corresponding to the amount of cash held against the derivative at time t. The interest rate of the collateral account is referred to as the *collateral rate* and is denoted by $r_c(t)$. Generally, this rate is assumed to be the agreed overnight rate paid on the posted collateral among dealers under CSA.
  - Amount $V(t) - C_t$ (rest of the cash) that needs to be borrowed/lent unsecured from the Treasury desk, at the *short rate for unsecured funding* $r_F(t)$.
  - Amount $\Delta_t^s S_t$ that needs to be borrowed at the *repo rate* $r_R(t)$ to finance the purchase of Δ_t^s units of the underlying stock. This amount is, in fact, the *capital gain* on the stock position, which is secured by the purchased stock. The repo rate is also known as the *short rate on funding secured by the asset*.
  - Dividends paid by the stock at the rate r_D.
- A holding of Δ_t^s units of the underlying asset.

Note that, in general, it is expected that $r_c(t) \leq r_R(t) \leq r_F(t)$. Then, the value of the replicating portfolio is given by:

$$V(t) = \Pi_t = \gamma_t + \Delta_t^s S_t \tag{3.4}$$

The growth of the total cash amount γ_t can be expressed as:

$$d\gamma_t = [\, r_c(t)C_t + r_F(t)(V(t) - C_t) - r_R(t)\Delta_t^s S_t + r_D(t)\Delta_t^s S_t \,]\, dt \tag{3.5}$$

The self-financing condition requires:

$$\begin{aligned} dV(t) = d\Pi_t &= d\,(\gamma_t + \Delta_t^s S_t) \\ &= d\gamma_t + \Delta_t^s\, dS_t && (3.6) \\ &= \mathcal{L}(V)\, dt + \Delta_t^s\, dS_t && (3.7) \end{aligned}$$

which implies

$$\begin{aligned} \mathcal{L}(V)\, dt &= d\gamma_t \\ &= [\, r_c(t)C_t + r_F(t)(V(t) - C_t) - r_R(t)\Delta_t^s S_t + r_D(t)\Delta_t^s S_t \,]\, dt \\ &= [\, r_c(t)C_t + r_F(t)(V(t) - C_t) - (r_R(t) - r_D(t))\, \Delta_t^s\, S_t \,]\, dt && (3.8) \end{aligned}$$

and, hence,

$$\mathcal{L}(V) = r_c(t)C_t + r_F(t)[V(t) - C_t] - [r_R(t) - r_D(t)]\, \Delta_t^s\, S_t \tag{3.9}$$

Rearranging this equation yields:

$$\mathcal{L}(V) + [(r_R(t) - r_D(t))\, S_t]\, \Delta_t^s = r_F(t)V(t) - [r_F(t) - r_c(t)]\, C_t \tag{3.10}$$

3.1.3 Derivative Pricing Framework with Collateralization

Theorem 3.1.1. *Let $V(t) \triangleq V(t, S_t)$, $t \geq 0$, denote the price of a collateralized derivative with underlying asset S_t, at each time t. Suppose that, under the risk-neutral world probability measure (Q-measure), the stock follows the process given by Equation (3.1) and the collateralized derivative follows the process given by Equation (3.2) or Equation (3.3). Suppose it is possible to construct the self-financing replicating portfolio given by Equation (3.4) under the cash account growth and self-financing conditions given by Equations (3.5) and (3.6). Consider the discounting process notation stated in Definition 2.5.5 Then, under the specified conditions, we have the following results:*

(i) The unique solution $V(t)$ to Equation (3.10) admits the following representation:

$$V^C(t) = E_t^Q\left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}} \left[(r_F(s) - r_c(s))\, C_s\right] ds \; + \; \frac{1}{\Phi_{t,T}^{r_F(t)}}\, V(T)\right] \tag{3.11}$$

in the measure Q in which the underlying stock grows at a rate $\mu_t = r_s(t) \triangleq r_R(t) - r_D(t)$, where $V(T)$ is the payoff of the derivative at a given future time T, and with discounting process in the short rate for unsecured funding $r_F(t)$.

(ii) The solution given by Equation (3.11) has the following equivalent form:

$$V^C(t) = E_t^Q\left[-\int_t^T \frac{1}{\Phi_{t,s}^{r_c(t)}} \left[r_F(s) - r_c(s)\right]\left[V(s) - C_s\right] ds \; + \; \frac{1}{\Phi_{t,T}^{r_c(t)}}\, V(T)\right] \tag{3.12}$$

with discounting process in the collateral rate $r_c(t)$.

(iii) The rate of growth in the value of the collateralized derivative is given by:

$$E_t^Q \left[dV^C(t) \right] = [r_F(t)V(t) - s_F(t)C(t)]\, dt \tag{3.13}$$

where the rate $r_F(t)$, applied to the price of the derivative, is also referred to as the funding spread *and the rate difference $s_F(t) \triangleq r_F(t) - r_c(t)$, applied to the collateral, is called the* credit spread.

Proof.

(i) Consider the equation:

$$\mathcal{L}(V) + [(r_R(t) - r_D(t))\, S_t]\; \Delta_t^s = r_F(t)V(t) - [r_F(t) - r_c(t)]\, C_t \tag{3.14}$$

Clearly, this is the generalized Black-Scholes-Merton differential equation stated in Theorem 2.5.6 with $\Theta_t = [r_R(t) - r_D(t)]\, S_t$, $\frac{\partial V}{\partial S_t} = \Delta_t^s$, $q(t, x) = r_F(t)$, and $g(t, x) = [r_F(t) - r_c(t)]\, C_t$, subject to the terminal condition $V(T, x) = V(T)$.

Thus, by Theorem 2.5.6, the unique solution to the PDE can be written in discounting process notation as:

$$V^C(t) = E_t^Q \left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}}\; [(r_F(s) - r_c(s))\, C_s]\, ds \; + \; \frac{1}{\Phi_{t,T}^{r_F(t)}}\; V(T) \right] \tag{3.15}$$

in the measure Q in which the underlying stock grows at a rate $\mu_t = r_s(t) \triangleq r_R(t) - r_D(t)$.

Note that the solution is independent of the repo rate or of the interest rate at which the underlying stock pays dividends.

(ii) We can rearrange the right-hand side of Equation (3.14) to obtain:

$$\begin{aligned} r_F(t)V(t) - [r_F(t) - r_c(t)]\, C_t &= r_F(t)V(t) - r_F(t)C_t + r_c(t)C_t \\ &= r_F(t)V(t) - r_F(t)C_t + r_c(t)C_t + r_c(t)V(t) - r_c(t)V(t) \\ &= r_c(t)V(t) + r_F(t)[V(t) - C_t] - r_c(t)[V(t) - C_t] \\ &= r_c(t)V(t) + [r_F(t) - r_c(t)][V(t) - C_t] \end{aligned} \tag{3.16}$$

Hence, Equation (3.14) becomes

$$\mathcal{L}(V) + [(r_R(t) - r_D(t))\, S_t]\ \Delta_t^s = r_c(t)V(t) - [-[r_F(t) - r_c(t)][V(t) - C_t]] \tag{3.17}$$

Again, this is the generalized Black-Scholes-Merton differential equation stated in Theorem 2.5.6 with $\Theta_t = [r_R(t) - r_D(t)]\, S_t$, $\frac{\partial V}{\partial S_t} = \Delta_t^s$, $q(t,x) = r_c(t)$, and $g(t,x) = -[r_F(t) - r_c(t)][V(t) - C_t]$, subject to the terminal condition $V(T,x) = V(T)$.

Thus, by Theorem 2.5.6, the unique solution to the PDE can be written in discounting process notation as:

$$V^C(t) = E_t^Q\left[-\int_t^T \frac{1}{\Phi_{t,s}^{r_c(t)}}\ [r_F(s) - r_c(s)]\,[V(s) - C_s]\, ds\ +\ \frac{1}{\Phi_{t,T}^{r_c(t)}}\, V(T)\right] \tag{3.18}$$

in the same measure Q in which the underlying stock grows at a rate $r_R(t) - r_D(t)$.

(iii) Combining Equations (3.1), (3.5), and (3.6) yields:

$$\begin{aligned} dV^C(t) &= d\gamma_t + \Delta_t^s\, dS_t \\ &= [\, r_c(t)C_t + r_F(t)(V(t) - C_t) - r_R(t)\Delta_t^s S_t + r_D(t)\Delta_t^s S_t\,]\, dt + \\ &\quad \Delta_t^s\, S_t \left[[r_R(t) - r_D(t)]dt + \sigma_t\, dW_t^{SB} \right] \end{aligned} \tag{3.19}$$

which implies

$$E_t^Q\left[dV^C(t)\right] = [r_F(t)V(t) - [r_F(t) - r_c(t)]C(t)]\,dt \tag{3.20}$$

Defining $s_F(t) \triangleq r_F(t) - r_c(t)$ yields the desired result.

□

3.1.4 Derivative Pricing Formulas for Perfect,Partial, and Zero Collateralization

Theorem 3.1.2. *Consider the same conditions stated in Theorem 3.1.1. Then, we have the following results:*

(i) If the derivative is not collateralized, then its price admits the representation

$$V^{NC}(t) = E_t^Q\left[\frac{1}{\Phi_{t,T}^{r_F(t)}}\,V(T)\right] \tag{3.21}$$

and the rate of growth in the value of the non-collateralized derivative is equal to $r_F(t)$.

(ii) If the derivative is perfectly collateralized, then its price admits the representation

$$V^C(t) = E_t^Q\left[\frac{1}{\Phi_{t,T}^{r_c(t)}}\,V(T)\right] \tag{3.22}$$

and the rate of growth in the value of the non-collateralized derivative is equal to $r_c(t)$.

Proof. Setting $C_t = 0$ in Equations (3.11) and (3.13) gives Equation (3.21). Setting $C_t = V(t)$ in Equations (3.12) and (3.13) gives Equation (3.22)

□

While Equations (3.11) and (3.12) give the price $V^C(t)$ of the collateralized derivative, their practical application is limited for cases of partial collateralization. Since C_t usually depends on $V^C(t)$, we can safely assume that $C_t = \lambda V^C(t)$, $0 \leq \lambda \leq 1$. As it turns out, this is a reasonable assumption since in practice the posted amount of cash collateral is deduced directly from the value of the collateralized derivative. The additional cash flows linked to a collateral agreement depend upon the amount of collateral paid at each period. These additional cash flows are essentially interest rate differentials generated by the difference between unsecured funding and collateral rates applied to the posted collateral.

The following important theorem synthesizes the previous results and allows to determine the derivative price for all states of collateralization.

Theorem 3.1.3. *Consider the same conditions stated in Theorem 3.1.1. Suppose* $C_t = \lambda V(t)$, $0 \leq \lambda \leq 1$. *Then, the price of a collateralized derivative, at each time t, admits the following general stochastic representation:*

$$V(t) = E_t^Q \left[e^{-\int_t^T [(1-\lambda) r_F(u) + \lambda r_c(u)] \, du} \, V(T) \right] \tag{3.23}$$

- *If* $\lambda = 0$, *the derivative is not collateralized.*
- *If* $0 < \lambda < 1$, *the derivative is partially collateralized.*
- *If* $\lambda = 1$, *the derivative is perfectly collateralized.*

Proof. Consider Equation (3.10), given by

$$\mathcal{L}(V) + [(r_R(t) - r_D(t)) \, S_t] \, \Delta_t^s = r_F(t) V(t) - [r_F(t) - r_c(t)] \, C_t \tag{3.24}$$

Suppose $C_t = \lambda V(t)$, $0 \le \lambda \le 1$. We can then rearrange the right-hand side of Equation (3.10) to obtain:

$$\begin{aligned} r_F(t)V(t) - [r_F(t) - r_c(t)]\, C_t &= r_F(t)V(t) - [r_F(t) - r_c(t)]\, \lambda V(t) \\ &= [(1-\lambda)r_F(t) \;+\; \lambda r_c(t)]\, V(t) \end{aligned} \tag{3.25}$$

Hence, Equation (3.10) becomes

$$\mathcal{L}(V) + [(r_R(t) - r_D(t))\, S_t]\; \Delta_t^s = [(1-\lambda)r_F(t) \;+\; \lambda r_c(t)]\, V(t) \tag{3.26}$$

Once again, this is the generalized Black-Scholes-Merton differential equation stated in Theorem 2.5.6 with $\Theta_t = [r_R(t) - r_D(t)]\, S_t$, $\frac{\partial V}{\partial S_t} = \Delta_t^s$, $q(t,x) = (1-\lambda)r_F(t) + \lambda r_c(t)$, and $g(t,x) = 0$, subject to the terminal condition $V(T,x) = V(T)$.

Thus, by Theorem 2.5.6, the unique solution to the PDE can be written as:

$$V(t) = E_t^Q\left[e^{-\int_t^T [\,(1-\lambda)r_F(u) + \lambda r_c(u)\,]\, du}\; V(T)\right] \tag{3.27}$$

in the same measure Q in which the underlying stock grows at a rate $r_R(t) - r_D(t)$.

If $\lambda = 0$, then Equation (3.27) becomes:

$$V(t) = E_t^Q\left[e^{-\int_t^T r_F(u)\, du}\; V(T)\right] \tag{3.28}$$

This is the case when $C_t = 0$ (*zero collateralization*). This equation is consistent with Equation (3.21), which was obtained by using a different approach, as described in [1].

If $\lambda = 1$, then Equation (3.27) becomes:

$$V(t) = E_t^Q \left[e^{-\int_t^T r_c(u)\,du}\, V(T) \right] \tag{3.29}$$

This is the case when $C_t = V(t)$ (*perfect collateralization*). This equation is consistent with Equation (3.22), which was obtained by using a different approach. □

Remark 3.1.4. *When two dealers are trading with each other, the collateral is applied to the overall value of the portfolio of derivatives between them, with positive exposures on some trades offsetting negative exposures on other trades. Hence, potentially, valuation of individual trades should take into account the collateral position on the whole portfolio. As stated in [1], in the case of the collateral being a linear function of the exact value of the portfolio, the value of the portfolio is just the sum of values of individual trades (with collateral attributed to trades by the same linear function). This follows from the linearity of the pricing formula (3.11) in $V^C(t)$ and C_t.*

3.1.5 The Concept of Liquidity Value Adjustment (LVA)

Consider Equation (3.11). the unique solution $V^C(t)$ to Equation (3.10):

$$V^C(t) = E_t^Q \left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}} \, [(r_F(s) - r_c(s))\, C_s]\, ds \; + \; \frac{1}{\Phi_{t,T}^{r_F(t)}} \, V^C(T) \right] \tag{3.30}$$

We can write this equation as:

$$\begin{aligned} V^C(t) &= E_t^Q \left[\frac{1}{\Phi_{t,T}^{r_F(t)}} \, V(T) \; + \; \int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}} \, [(r_F(s) - r_c(s))\, C_s]\, ds \right] && (3.31) \\ &= E_t^Q \left[\frac{1}{\Phi_{t,T}^{r_F(t)}} \, V(T) \right] \; + \; E_t^Q \left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}} \, [(r_F(s) - r_c(s))\, C_s]\, ds \right] && (3.32) \end{aligned}$$

Note that the first term of the left-hand side of Equation (3.32) is consistent with Equation (3.21) which, in fact, corresponds to the value $V^{NC}(t)$ of a derivative contract without collateralization. That is,

$$V^{NC}(t) = E_t^Q\left[\frac{1}{\Phi_{t,T}^{r_F(t)}}\,V(T)\right] \tag{3.33}$$

This is basically the expected return of a derivative contract without collateralization, under the Q-measure.

The second term is known as *Liquidity Value Adjustment (LVA)* and is defined as:

$$LVA_t \triangleq E_t^Q\left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}}\,\left[(r_F(s) - r_c(s))\,C_s\right] ds\right] = E_t^Q\left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}}\,\left[s_F(s)\,C_s\right] ds\right] \tag{3.34}$$

Therefore,

$$V^C(t) = V^{NC}(t) + LVA_t \tag{3.35}$$

This is the continuous form of the idea that was introduced by Castagna in [6] in his binomial approach to pricing OTC derivatives with collateralization, for a particular self-financing replicating portfolio in discrete time. This equation says that the value of a collateralized derivative is equal to the value of the derivative without collateralization plus some adjustment value. More precisely, as it can be inferred from Equation (3.34), the *LVA* can be defined as the expected discounted value of the credit spread (i.e., the difference between the risk-free rate and the collateral rate paid on the collateral (from the lender's perspective (that is, the counterparty with positive PV in the contract)). As described in [6], the *LVA* is essentially the gain (or loss) corresponding to the liquidation of the PV of the derivative contract with collateralization.

3.2 Pricing by Martingales for Continuous and Perfect Collateralization

In the previous section, we used replicating portfolio and self-financing arguments to obtain the general derivative pricing formula with collateralization, given by Equation (3.11), following the work done by Piterbarg in [1]. As an extension of Piterbarg's work, we also introduced the partial collateralization constraint to obtain another form of the derivative pricing formula that yields different expressions for zero, partial, and perfect collateralization, depending on the value of the partial collateralization parameter λ. The results obtained by a different approach are consistent with Piterbarg's work.

We now show how it is possible to obtain the derivative pricing formula (3.22) for perfect collateralization by using a martingale approach. The main concept lies in the connection of stock prices with the Markov process and the Markov property, and the fact that we use stock as the underlying asset of a collateralized derivative. The idea is, then, to define a stochastic process based on Equation (3.11) and to show that such a process is a Q-martingale. Then, it can be shown that the price process of a collateralized derivative can be expressed as a function of a certain martingale process, which after some manipulation leads to the perfectly collateralized derivative pricing formula.

Theorem 3.2.1. *Let $s_F(t) \triangleq r_F(t) - r_c(t)$ and consider the discounting process notation given by Equation (2.20). Define the following stochastic process:*

$$X(t) = \frac{1}{\Phi_{0,t}^{r_F(t)}} V(t) + \int_0^t \frac{1}{\Phi_{0,s}^{r_F(t)}} s_F(s) V(s)\, ds \qquad (3.36)$$

Then, the process $X(t)$ is a Q-martingale.

Proof. We must show that, for $0 \leq \tau \leq t \leq T$, $E_t^Q[X(t) \mid \mathcal{F}_\tau] = X(\tau)$. Then,

$$
\begin{aligned}
&E_t^Q[X(t) \mid \mathcal{F}_\tau] \\
&= E_t^Q\left[\frac{1}{\Phi_{0,t}^{r_F(t)}} V(t) + \int_0^t \frac{1}{\Phi_{0,s}^{r_F(t)}} s_F(s)V(s)\,ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^Q\left[\frac{1}{\Phi_{0,t}^{r_F(t)}} V(t) \,\middle|\, \mathcal{F}_\tau\right] + E_t^Q\left[\int_0^t \frac{1}{\Phi_{0,s}^{r_F(t)}} s_F(s)V(s)\,ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^Q\left[\frac{1}{\Phi_{0,t}^{r_F(t)}} E_t^Q\left[\frac{1}{\Phi_{t,T}^{r_F(t)}} V(T) + \int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}} s_F(s)V(s)\,ds\right] \,\middle|\, \mathcal{F}_\tau\right] + E_t^Q\left[\int_0^t \frac{1}{\Phi_{0,s}^{r_F(t)}} s_F(s)V(s)\,ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^Q\left[\frac{1}{\Phi_{0,t}^{r_F(t)}} \left(\frac{1}{\Phi_{t,T}^{r_F(t)}} V(T) + \int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}} s_F(s)V(s)\,ds\right) \,\middle|\, \mathcal{F}_\tau\right] + E_t^Q\left[\int_0^t \frac{1}{\Phi_{0,s}^{r_F(t)}} s_F(s)V(s)\,ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^Q\left[\frac{1}{\Phi_{0,T}^{r_F(t)}} V(T) + \int_t^T \frac{1}{\Phi_{0,s}^{r_F(t)}} s_F(s)V(s)\,ds + \int_0^t \frac{1}{\Phi_{0,s}^{r_F(t)}} s_F(s)V(s)\,ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^Q\left[\frac{1}{\Phi_{0,T}^{r_F(t)}} V(T) + \int_0^T \frac{1}{\Phi_{0,s}^{r_F(t)}} s_F(s)V(s)\,ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^Q\left[X(T) \,\middle|\, \mathcal{F}_\tau\right]
\end{aligned}
\tag{3.37}
$$

Similarly, we also get

$$E_t^Q[X(T) \mid \mathcal{F}_\tau] = E_t^Q[X(\tau) \mid \mathcal{F}_\tau] = X(\tau) \tag{3.38}$$

Hence,

$$E_t^Q[X(t) \mid \mathcal{F}_\tau] = E_t^Q[X(T) \mid \mathcal{F}_\tau] = E_t^Q[X(\tau) \mid \mathcal{F}_\tau] = X(\tau) \tag{3.39}$$

Therefore,

$$E_t^Q[X(t) \mid \mathcal{F}_\tau] = X(\tau) \tag{3.40}$$

and it follows that the process $X(t)$ is a Q-martingale. □

Theorem 3.2.2. *Let $V(t)$ be the price of a collateralized derivative and let $dV(t)$ be its price process. Then, the price process $dV(t)$ can be expressed with a certain martingale process $M(t)$ as:*

$$dV(t) = r_c(t)V(t)\,dt \;+\; dM(t) \tag{3.41}$$

Proof. Let

$$X(t) = \frac{1}{\Phi_{0,t}^{r_F(t)}}\,V(t) \;+\; \int_0^t \frac{1}{\Phi_{0,s}^{r_F(t)}}\,s_F(s)V(s)\,ds \tag{3.42}$$

Differentiating $X(t)$ yields:

$$\begin{aligned} dX(t) &= \frac{1}{\Phi_{0,t}^{r_F(t)}}\,(-r_F(t)V(t)\,dt) \;+\; \frac{1}{\Phi_{0,t}^{r_F(t)}}\,dV(t) + \frac{1}{\Phi_{0,t}^{r_F(t)}}\,s_F(t)V(t)\,dt \\ &= \frac{1}{\Phi_{0,t}^{r_F(t)}}\,[s_F(t) - r_F(t)]\,V(t)\,dt \;+\; \frac{1}{\Phi_{0,t}^{r_F(t)}}\,dV(t) \end{aligned} \tag{3.43}$$

This implies:

$$\begin{aligned} \frac{1}{\Phi_{0,t}^{r_F(t)}}\,dV(t) &= dX(t) \;-\; \frac{1}{\Phi_{0,t}^{r_F(t)}}\,[s_F(t) - r_F(t)]\,V(t)\,dt \\ &= \frac{1}{\Phi_{0,t}^{r_F(t)}}\,r_c(t)V(t)\,dt + dX(t) \end{aligned} \tag{3.44}$$

Then,

$$\begin{aligned} dV(t) &= r_c(t)V(t)\,dt \;+\; \underbrace{\Phi_{0,t}^{r_F(t)}\,dX(t)}_{:=dM(t)} \\ &= r_c(t)V(t)\,dt \;+\; dM(t) \end{aligned} \tag{3.45}$$

where $dM(t)$ is a martingale. Therefore, the price process of the collateralized derivative $V(t)$ can be expressed with a certain martingale process $M(t)$. □

We can obtain the perfectly collateralized derivative pricing formula from Equation (3.45) as follows:

$$
\begin{aligned}
& dV(s) = r_c(s)V(s)\,ds + dM(s) \\
\Rightarrow \quad & dV(s) - r_c(s)V(s)\,ds = dM(s) \\
\Rightarrow \quad & \left(e^{-\int_0^s r_c(u)\,du}\right)[dV(s) - r_c(s)V(s)ds] = \left(e^{-\int_0^s r_c(u)\,du}\right)dM(s) \\
\Rightarrow \quad & d\left(e^{-\int_0^s r_c(u)\,du}\,V(s)\right) = \left(e^{-\int_0^s r_c(u)\,du}\right)dM(s) \\
\Rightarrow \quad & \int_t^T d\left(e^{-\int_0^s r_c(u)\,du}\,V(s)\right) = \int_t^T \left(e^{-\int_0^s r_c(u)\,du}\right)dM(s) \\
\Rightarrow \quad & e^{-\int_0^T r_c(u)\,du}\,V(T) - e^{-\int_0^t r_c(u)\,du}\,V(t) = \int_t^T \left(e^{-\int_0^s r_c(u)\,du}\right)dM(s)
\end{aligned}
\tag{3.46}
$$

Then, we have:

$$
\begin{aligned}
V(t) &= e^{\int_0^t r_c(u)\,du}\left(e^{-\int_0^T r_c(u)\,du}\,V(T) - \int_t^T e^{-\int_0^s r_c(u)\,du}\;dM(s)\right) \\
&= e^{-\int_t^T r_c(u)\,du}\,V(T) - \int_t^T e^{-\int_t^s r_c(u)\,du}\;dM(s)
\end{aligned}
\tag{3.47}
$$

Applying the conditional expectation to this equation yields:

$$
E_t^Q\left[V(t)\right] = E_t^Q\left[e^{-\int_t^T r_c(u)\,du}\,V(T) - \int_t^T e^{-\int_t^s r_c(u)\,du}\;dM(s)\right] \tag{3.48}
$$

Therefore,

$$
V(t) = E_t^Q\left[e^{-\int_t^T r_c(u)\,du}\,V(T)\right] \tag{3.49}
$$

which is the final derivative pricing formula for perfect collateralization with expectation corresponding to a collateral account used as the numeraire.

CHAPTER 4

CROSS-CURRENCY, SINGLE ASSET ANALYSIS

4.1 Pricing by Replication and LVA

We now consider the case where the single underlying asset and the derivative are in domestic currency b, but the collateral is posted in the foreign currency f. This is the case of *full collateralization* which, as stated in Section 1.5, refers to the case of perfect collateralization with collateral posted in different currency.

Let $V^{\mathsf{d}}(t) \triangleq V^{\mathsf{d}}(t, S_t)$, $t \geq 0$, denote the present value of a derivative in terms of domestic currency d, with single underlying asset S_t, at each time t. The derivative is assumed to have a payoff at time T, denoted by $V^{\mathsf{d}}(T)$, in the same domestic currency. Suppose the derivative is fully collateralized in a particular foreign currency f with *currency exchange rate* $F_x^{\mathsf{d},\mathsf{f}}(t)$ at time $t \geq 0$, which represents the number of units in currency d per unit of currency f. Then, the collateral amount posted by the counterparty in foreign currency f can be expressed as:

$$C_t^{\mathsf{f}} \triangleq \frac{V^{\mathsf{d},\mathsf{f}}(t)}{F_x^{\mathsf{d},\mathsf{f}}(t)} \tag{4.1}$$

where $V^{\mathsf{d},\mathsf{f}}(t)$ denotes the present value in domestic currency d of a fully collateralized derivative in foreign currency f, at any time $t \in [0, T)$. Note that, in general and as stated in [9], $C_t^{\mathsf{f}} \neq V^{\mathsf{d},\mathsf{f}}(t)/F_x^{\mathsf{d},\mathsf{f}}(t)$ if the derivative in domestic currency d is partially collateralized in foreign currency f. Also note that $C_t^{\mathsf{d}} \equiv V^{\mathsf{d},\mathsf{d}}(t)$ if the derivative is perfectly collateralized in domestic currency, as it was the case in Chapter 3.

Now, denote by $r_F^{\mathsf{d}}(t)$ and $r_F^{\mathsf{f}}(t)$ the short rates for unsecured domestic and foreign currency, respectively. Also, denote by $r_c^{\mathsf{f}}(t)$ the short rate of the collateral account in foreign currency f. Since the underlying stock is in domestic currency, we denote by $r_D^{\mathsf{d}}(t)$ the short rate at which the stock pays dividends, and by $r_R^{\mathsf{d}}(t)$ the short rate on funding secured by the underlying asset (i.e., repo rate).

Recall from Section 3.1.5 that the value of a collateralized derivative can be expressed as:

$$V^C = V^{NC} + LVA \tag{4.2}$$

In the absence of collateralization, V^{NC} can be expressed in domestic currency d as:

$$V^{NC\,\mathsf{d}}(t) = E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{t,T}^{r_F^{\mathsf{d}}(t)}}\, V^{\mathsf{d}}(T)\right] \tag{4.3}$$

under the domestic risk-neutral measure Q^{d} corresponding to the rate $r_F^{\mathsf{d}}(t)$.

Since the collateral is posted in a foreign currency, the LVA can be expressed as:

$$LVA_t^{\mathsf{f}} = E_t^{Q^{\mathsf{f}}}\left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F^{\mathsf{f}}(t)}}\, s_F^{\mathsf{f}}(s)\, \underbrace{\frac{V^{\mathsf{d,f}}(s)}{F_x^{\mathsf{d,f}}(s)}}_{\triangleq C_s^{\mathsf{f}}}\, ds\right] \tag{4.4}$$

in foreign currency, or equivalently as:

$$LVA_t^{\mathsf{d}} = F_x^{\mathsf{d,f}}(t)\, E_t^{Q^{\mathsf{f}}}\left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F^{\mathsf{f}}(t)}}\, s_F^{\mathsf{f}}(s) \frac{V^{\mathsf{d,f}}(s)}{F_x^{\mathsf{d,f}}(s)}\, ds\right] \tag{4.5}$$

in domestic currency, under the foreign risk-neutral measure Q^{f} corresponding to the rate $r_F^{\mathsf{f}}(t)$, where $s_F^{\mathsf{f}}(t) = r_F^{\mathsf{f}}(t) - r_c^{\mathsf{f}}(t)$.

Therefore,

$$V^{\mathsf{d,f}}(t) = E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{t,T}^{r_F^{\mathsf{d}}(t)}}\, V^{\mathsf{d}}(T)\right] + F_x^{\mathsf{d,f}}(t)\, E_t^{Q^{\mathsf{f}}}\left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F^{\mathsf{f}}(t)}}\, s_F^{\mathsf{f}}(s)\frac{V^{\mathsf{d,f}}(s)}{F_x^{\mathsf{d,f}}(s)}\, ds\right] \tag{4.6}$$

Note that the expectations in Equation (4.6) are under different measures, which is not very convenient. Aligning the measure[1] to Q^{d} yields the final cross-currency derivative pricing formula with full collateralization in foreign currency f, given by:

$$V^{\mathsf{d,f}}(t) = E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{t,T}^{r_F^{\mathsf{d}}(t)}}\, V^{\mathsf{d}}(T) + \int_t^T \frac{1}{\Phi_{t,s}^{r_F^{\mathsf{d}}(t)}}\, s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds\right] \tag{4.7}$$

where $V^{\mathsf{d,f}}(t) = F_x^{\mathsf{d,f}}(t)\, C_t^{\mathsf{f}}$.

4.2 Pricing by Martingales for Continuous and Full Collateralization

Equation (4.7) gives the general pricing formula but is not entirely convenient in practice since it is recursive in the value of the derivative. The following theorem allows us to obtain a more practical formula.

Theorem 4.2.1. *Define the following stochastic process:*

$$X(t) = \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}}\, V^{\mathsf{d,f}}(t) + \int_0^t \frac{1}{\Phi_{0,s}^{r_F^{\mathsf{d}}(t)}}\, s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds \tag{4.8}$$

Then, the process $X(t)$ is a Q^{d}-martingale.

[1] A proof of this procedure can be found in [9].

Proof. We must show that, for $0 \le \tau \le t \le T$, $E_t^{Q^{\mathsf{d}}}[X(t) \mid \mathcal{F}_\tau] = X(\tau)$. Then,

$$
\begin{aligned}
&E_t^{Q^{\mathsf{d}}}[X(t) \mid \mathcal{F}_\tau] \\
&= E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}} V^{\mathsf{d,f}}(t) + \int_0^t \frac{1}{\Phi_{0,s}^{r_F^{\mathsf{d}}(t)}} s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}} V^{\mathsf{d,f}}(t) \,\middle|\, \mathcal{F}_\tau\right] + E_t^{Q^{\mathsf{d}}}\left[\int_0^t \frac{1}{\Phi_{0,s}^{r_F^{\mathsf{d}}(t)}} s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}} E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{t,T}^{r_F^{\mathsf{d}}(t)}} V^{\mathsf{d}}(T) + \int_t^T \frac{1}{\Phi_{t,s}^{r_F^{\mathsf{d}}(t)}} s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds\right] \,\middle|\, \mathcal{F}_\tau\right] + E_t^{Q^{\mathsf{d}}}\left[\int_0^t \frac{1}{\Phi_{0,s}^{r_F^{\mathsf{d}}(t)}} s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}} \left(\frac{1}{\Phi_{t,T}^{r_F^{\mathsf{d}}(t)}} V^{\mathsf{d}}(T) + \int_t^T \frac{1}{\Phi_{t,s}^{r_F^{\mathsf{d}}(t)}} s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds\right) \,\middle|\, \mathcal{F}_\tau\right] + E_t^{Q^{\mathsf{d}}}\left[\int_0^t \frac{1}{\Phi_{0,s}^{r_F^{\mathsf{d}}(t)}} s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{0,T}^{r_F^{\mathsf{d}}(t)}} V^{\mathsf{d}}(T) + \int_t^T \frac{1}{\Phi_{0,s}^{r_F^{\mathsf{d}}(t)}} s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds + \int_0^t \frac{1}{\Phi_{0,s}^{r_F^{\mathsf{d}}(t)}} s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^{Q^{\mathsf{d}}}\left[\frac{1}{\Phi_{0,,}^{r_F^{\mathsf{d}}(t)}} V^{\mathsf{d}}(T) + \int_0^T \frac{1}{\Phi_{0,s}^{r_F^{\mathsf{d}}(t)}} s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\, ds \,\middle|\, \mathcal{F}_\tau\right] \\
&= E_t^{Q^{\mathsf{d}}}\left[X(T) \,\middle|\, \mathcal{F}_\tau\right] \qquad (4.9)
\end{aligned}
$$

Similarly, we also get

$$E_t^{Q^{\mathsf{d}}}[X(T) \mid \mathcal{F}_\tau] = E_t^{Q^{\mathsf{d}}}[X(\tau) \mid \mathcal{F}_\tau] = X(\tau) \qquad (4.10)$$

Hence,

$$E_t^{Q}[X(t) \mid \mathcal{F}_\tau] = E_t^{Q}[X(T) \mid \mathcal{F}_\tau] = E_t^{Q}[X(\tau) \mid \mathcal{F}_\tau] = X(\tau) \qquad (4.11)$$

Therefore,

$$E_t^Q\left[X(t) \mid \mathcal{F}_\tau\right] = X(\tau) \tag{4.12}$$

and it follows that the process $X(t)$ is a Q^{d}-martingale. □

Theorem 4.2.2. *Let $V^{\mathsf{d,f}}(t)$ be the value in domestic currency d of a derivative fully collateralized in foreign currency f, at any time $t \in [0, T)$. If $dV^{\mathsf{d,f}}(t)$ is the price process of the derivative, then $dV^{\mathsf{d,f}}(t)$ can be expressed with a certain martingale process $M(t)$ as:*

$$dV^{\mathsf{d,f}}(t) = \left[r_F^{\mathsf{d}}(t) - s_F^{\mathsf{f}}(t)\right] V^{\mathsf{d,f}}(t)\,dt \;+\; dM(t) \tag{4.13}$$

Proof. Let

$$X(t) = \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}}\, V^{\mathsf{d,f}}(t) \;+\; \int_0^t \frac{1}{\Phi_{0,s}^{r_F^{\mathsf{d}}(t)}}\, s_F^{\mathsf{f}}(s) V^{\mathsf{d,f}}(s)\,ds \tag{4.14}$$

Differentiating $X(t)$ yields:

$$\begin{aligned} dX(t) &= \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}} \left(-r_F^{\mathsf{d}}(t) V^{\mathsf{d,f}}(t)\,dt\right) \;+\; \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}}\, dV^{\mathsf{d,f}}(t) + \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}}\, s_F^{\mathsf{f}}(t) V^{\mathsf{d,f}}(t)\,dt \\ &= \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}} \left[s_F^{\mathsf{f}}(t) - r_F^{\mathsf{d}}(t)\right] V^{\mathsf{d,f}}(t)\,dt \;+\; \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}}\, dV^{\mathsf{d,f}}(t) \end{aligned} \tag{4.15}$$

This implies:

$$\begin{aligned} \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}}\, dV^{\mathsf{d,f}}(t) &= dX(t) \;-\; \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}} \left[s_F^{\mathsf{f}}(t) - r_F^{\mathsf{d}}(t)\right] V^{\mathsf{d,f}}(t)\,dt \\ &= \frac{1}{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}} \left[r_F^{\mathsf{d}}(t) - s_F^{\mathsf{f}}(t)\right] V^{\mathsf{d,f}}(t)\,dt + dX(t) \end{aligned} \tag{4.16}$$

Then,

$$dV^{\mathsf{d,f}}(t) = \left[r_F^{\mathsf{d}}(t) - s_F^{\mathsf{f}}(t)\right] V^{\mathsf{d,f}}(t)\, dt \; + \; \underbrace{\Phi_{0,t}^{r_F^{\mathsf{d}}(t)}\, dX(t)}_{:=dM(t)}$$

$$= \left[r_F^{\mathsf{d}}(t) - s_F^{\mathsf{f}}(t)\right] V^{\mathsf{d,f}}(t)\, dt \; + \; dM(t) \tag{4.17}$$

where $dM(t)$ is a martingale. Therefore, the price process of the fully collateralized derivative $V^{\mathsf{d,f}}(t)$ can be expressed with a certain martingale process $M(t)$. □

Following the same steps as in the last part of Section 3.2 yields the fully collateralized derivative pricing formula:

$$V^{\mathsf{d,f}}(t) = E_t^{Q^{\mathsf{d}}}\left[e^{-\int_t^T \left[r_F^{\mathsf{d}}(u) - s_F^{\mathsf{f}}(u)\right] du}\, V^{\mathsf{d}}(T)\right] \tag{4.18}$$

which can also be written as:

$$V^{\mathsf{d,f}}(t) = E_t^{Q^{\mathsf{d}}}\left[e^{-\int_t^T r_F^{\mathsf{d}}(u)\, du} \left(e^{\int_t^T s_F^{\mathsf{f}}(u)\, du}\right) V^{\mathsf{d}}(T)\right] \tag{4.19}$$

It is worth noting that if the value of the derivative and its collateral are in the same currency, then Equation (4.19) reduces to Equation (3.49) for perfect collateralization in single currency.

CHAPTER 5

SINGLE-CURRENCY, MULTI-ASSET ANALYSIS

5.1 The Processes for the Underlying Assets and the Derivative

Let $V(t) \triangleq V(t, \mathbf{S_t})$, $t \geq 0$, be the price of a collateralized derivative with a series of underlying assets denoted by $\mathbf{S_t} = (S_t^{(1)}, \ldots, S_t^{(n)})^\intercal \in \mathbb{R}^n$, at each time t. By definition 2.4.2, assume that, under a given measure, the ith stock follows a process given by:

$$dS_t^{(i)} = \mu_t^{(i)}\, S_t^{(i)}\, dt \;+\; \sum_{j=1}^{n} \sigma_t^{(i,j)}\, S_t^{(i)}\, dW_t^{SB(j)}, \quad i = 1 : n \tag{5.1}$$

Then, by Theorem 2.5.4, the price process of the derivative is given by:

$$dV(t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_t^{(i,j)} S_t^{(i)} S_t^{(j)} \frac{\partial^2 V}{\partial S_t^{(i)} \partial S_t^{(j)}} \right) dt \;+\; \sum_{i=1}^{n} \left(\frac{\partial V}{\partial S_t^{(i)}} \right) dS_t^{(i)} \tag{5.2}$$

$$= \mathcal{L}(V)\, dt + \sum_{i=1}^{n} \Delta_t^{s(i)}\, dS_t^{(i)} \tag{5.3}$$

where $\mathcal{L}(\bullet)$ is the standard two-dimensional pricing operator and $\Delta_t^{s(i)}$ is the delta of the ith stock derivative, as defined in Section 2.6.3.

5.2 Self-financing Replicating Portfolio and Black-Scholes Equation with Collateral

To replicate the derivative, consider a self-financing replicating portfolio whose value, at each time t, is denoted by Π_t and determined by the following components:

- A cash amount, denoted by γ_t, split among the following accounts:
  - Collateral account, denoted by C_t, corresponding to the amount of cash held against the derivative at time t, with *collateral rate* denoted by $r_c(t)$.
  - Amount $V(t) - C_t$ (rest of the cash) that needs to be borrowed/lent unsecured from the Treasury desk, at the *short rate for unsecured funding* $r_F(t)$.
  - Amount $\Delta_t^{s(i)} S_t^{(i)}$ that would need to be borrowed at the *repo rate* $r_R^{(i)}(t)$ to finance the purchase of $\Delta_t^{s(i)}$ units of the ith underlying stock.
  - Dividends paid by the ith stock at the rate $r_D^{(i)}$.
- A holding of $\Delta_t^{s(i)}$ units of the ith underlying asset.

Then, the value of the replicating portfolio is given by:

$$V(t) = \Pi_t = \gamma_t + \sum_{i=1}^{n} \Delta_t^{(i)} S_t^{(i)} \tag{5.4}$$

The growth of the total cash amount γ_t can be expressed as:

$$d\gamma_t = \left[r_c(t)C_t + r_F(t)(V(t) - C_t) - \sum_{i=1}^{n} r_R^{(i)}(t)\Delta_t^{(i)} S_t^{(i)} + \sum_{i=1}^{n} r_D^{(i)}(t)\Delta_t^{(i)} S_t^{(i)} \right] dt \tag{5.5}$$

The self-financing condition requires:

$$\begin{aligned} dV(t) = d\Pi_t &= d\left(\gamma_t + \sum_{i=1}^{n} \Delta_t^{(i)} S_t^{(i)} \right) \\ &= d\gamma_t + \sum_{i=1}^{n} \Delta_t^{(i)} dS_t^{(i)} \qquad (5.6) \\ &= \mathcal{L}(V)\, dt + \sum_{i=1}^{n} \Delta_t^{(i)} S_t^{(i)} \qquad (5.7) \end{aligned}$$

which implies

$$
\begin{aligned}
\mathcal{L}(V)\,dt &= d\gamma_t \\
&= \left[r_c(t)C_t + r_F(t)(V(t) - C_t) - \sum_{i=1}^{n} r_R^{(i)}(t)\Delta_t^{(i)}S_t^{(i)} + \sum_{i=1}^{n} r_D^{(i)}(t)\Delta_t^{(i)}S_t^{(i)} \right] dt \\
&= \left[r_c(t)C_t + r_F(t)(V(t) - C_t) - \sum_{i=1}^{n} (r_R^{(i)}(t) - r_R^{(i)}(t))\,\Delta_t^{s(i)}\,S_t^{(i)} \right] dt \qquad (5.8)
\end{aligned}
$$

and, hence,

$$
\mathcal{L}(V) = r_c(t)C_t + r_F(t)[V(t) - C_t] - \sum_{i=1}^{n} \left[r_R^{(i)}(t) - r_D^{(i)}(t) \right] \Delta_t^{s(i)}\,S_t^{(i)} \qquad (5.9)
$$

Rearranging this equation yields:

$$
\mathcal{L}(V) + \sum_{i=1}^{n} \left[r_R^{(i)}(t) - r_D^{(i)}(t) \right] \Delta_t^{s(i)}\,S_t^{(i)} = r_F(t)V(t) - [r_F(t) - r_c(t)]\,C_t \qquad (5.10)
$$

5.3 Derivative Pricing Framework with Collateralization

Theorem 5.3.1. *Let* $V(t) \triangleq V(t, \mathbf{S_t})$, $t \geq 0$, *be the price of a collateralized derivative with a series of underlying assets denoted by* $\mathbf{S_t} = (S_t^{(1)}, \ldots, S_t^{(n)})^\intercal \in \mathbb{R}^n$ *, at each time t. Suppose that, under the risk-neutral world probability measure (Q-measure), the ith stock follows the process given by Equation (5.1) and the collateralized derivative follows the process given by Equation (5.2) or Equation (5.3). Suppose it is possible to construct the self-financing replicating portfolio given by Equation (5.4) under the cash account growth and self-financing conditions given by Equations (5.5) and (5.6). Consider the discounting process notation stated in Definition 2.5.5.*

Then, under the specified conditions, the unique solution $V(t)$ to Equation (5.10) admits the following representation:

$$V^C(t) = E_t^Q\left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}}\,\left[(r_F(s) - r_c(s))\,C_s\right] ds \;+\; \frac{1}{\Phi_{t,T}^{r_F(t)}}\,V(T)\right] \tag{5.11}$$

in the measure Q in which the ith underlying stock grows at a rate $\mu^{(i)}(t) = r_s^{(i)}(t) \triangleq r_R^{(i)}(t) - r_D^{(i)}(t)$, where $V(T)$ is the payoff of the derivative at a given future time T, and with discounting process in the short rate for unsecured funding $r_F(t)$.

Proof. Consider the equation:

$$\mathcal{L}(V) + \sum_{i=1}^{n}\left[r_R^{(i)}(t) - r_D^{(i)}(t)\right]\Delta_t^{s(i)}\,S_t^{(i)} = r_F(t)V(t) - \left[r_F(t) - r_c(t)\right]C_t \tag{5.12}$$

Clearly, this is the generalized Black-Scholes-Merton differential equation stated in Theorem 2.5.6 with $\Theta_t = \sum_{i=1}^{n}\left[r_R^{(i)}(t) - r_D^{(i)}(t)\right]S_t^{(i)}$, $\frac{\partial V}{\partial S_t^{(i)}} = \Delta_t^{s(i)}$, $q(t,x) = r_F(t)$, and $g(t,x) = \left[r_F(t) - r_c(t)\right]C_t$, subject to the terminal condition $V(T,x) = V(T)$.

Thus, by Theorem 2.5.6, the unique solution to the PDE can be written as:

$$V^C(t) = E_t^Q\left[\int_t^T \frac{1}{\Phi_{t,s}^{r_F(t)}}\,\left[(r_F(s) - r_c(s))\,C_s\right] ds \;+\; \frac{1}{\Phi_{t,T}^{r_F(t)}}\,V(T)\right] \tag{5.13}$$

in the measure Q in which the ith underlying stock grows at a rate $\mu^{(i)}(t) = r_s^{(i)}(t) \triangleq r_R^{(i)}(t) - r_D^{(i)}(t)$. Again, as it was the case in Section 3.1, note that the solution is independent of the series of repo rates of the underlying assets or the interest rates at which each of the underlying assets pays dividends. □

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