CAN ONE HEAR…?
AN EXPLORATION INTO INVERSE EIGENVALUE PROBLEMS RELATED TO MUSICAL INSTRUMENTS

by

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B.S. Palm Beach Atlantic University, 2010

A thesis submitted in partial fulfillment of the requirements
for the degree of Master of Science
in the department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Spring Term
2013

Major Professor: M. Z. Nashed

ABSTRACT

The central theme of this thesis deals with problems related to the question, “Can one hear the shape of a drum?” first posed formally by Mark Kac in 1966. More precisely, can one determine the shape of a membrane with fixed boundary from the spectrum of the associated differential operator? For this paper, Kac received both the Lester Ford Award and the Chauvant Prize of the Mathematical Association of America. This problem has received a great deal of attention in the past forty years and has led to similar questions in completely different contexts such as “Can one hear the shape of a graph associated with the Schrödinger operator?”, “Can you hear the shape of your throat?”, “Can you feel the shape of a manifold with Brownian motion?”, “Can one hear the crack in a beam?”, “Can one hear into the sun?”, etc. Each of these topics deals with inverse eigenvalue problems or related inverse problems.

For inverse problems in general, the problem may or may not have a solution, the solution may not be unique, and the solution does not necessarily depend continuously on perturbation of the data. For example, in the case of the drum, it has been shown that the answer to Kac’s question in general is “no.” However, if we restrict the class of drums, then the answer can be yes. This is typical of inverse problems when a priori information and restriction of the class of admissible solutions and/or data are used to make the problem well-posed. This thesis provides an analysis of shapes for which the answer to Kac's question is positive and a variety of interesting questions on this problem and its variants, including cases that remain open. This thesis also provides a synopsis and perspectives of other types of “can one hear” problems mentioned above. Another part of this thesis deals with aspects of direct problems related to musical instruments.

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CHAPTER 1: INTRODUCTION

1.1: Direct and Inverse Problems

Often it is of great interest mathematically to use the properties of the materials, along with the initial position and velocity of a system to determine the system's state for times in the future. This is known as a *direct problem*.

However, over the past century, mathematicians have become increasingly more interested in the *inverse problem* of what has been studied for so long. In other words, if we know the system's state, can we determine certain properties about the system itself? This question has been studied in many different forms.

Bevilacqua, Brandao, and Bassanezi [2] assert in their paper titled *A Mathematical Approach to Plato's Problem* that one of the first inverse problems ever presented can be found in Plato's *Republic*, specifically *The Allegory of the Cave*. Plato [16] constructs the situation as such: there are several people sitting in a cave, facing the cave's back wall. They are not able to move. A fire is sparked between the people and the wall, and between the wall and the fire there is a procession of men and women and animals of all ages, shapes, and sizes pulling and pushing a variety of objects. The people sitting facing the wall of the cave are only able to watch the shadows projected on the wall; they were not able to see the procession itself. The inverse problem is this: from the shadows on the wall, the people were to determine what kind of people, animals, and objects composed the parade. The three authors state "Plato explored through this allegory the knowing process and how to reach the truth. Despite the fact that his aim was distinct he presented maybe for the first time in a well organized structure an inverse problem, specifically, pattern recognition." [2]

Inverse problems can be found in a variety of fields of study. For example, the task of the meteorologist is to forecast the weather – an inverse problem that many people rely on daily. When one visits the doctor's office, one is hoping that the medical doctor is able to diagnose and provide a cure for his illness – another common inverse problem. The field of engineering thrives on direct problems – i.e., building bridges, tunnels, constructing roads, and assembling computers. An inverse problem in engineering would be to determine why a bridge happened to collapse or why an engine will not start.

1.2: Direct Spectral Problems

A direct spectral problem is one in which the system of equations is given, and one is asked to find the spectrum. This problem can be given in many forms. Some examples include:

- Computing the eigenvalues and associated eigenvectors of a real or complex $n \times n$ matrix
- Computing the singular values and singular vectors of a $m \times n$ real or complex matrix
- Computing the singular values and vectors of a compact linear operator acting between two Hilbert spaces
- Computing the eigenvalues and eigenfunctions of a differential operator, for example, a second order Sturm-Liouville differential equation subject to homogeneous boundary conditions
- Lastly, in quantum mechanics, the time-independent, Schrödinger equation in one space variable takes the form

$$-u'' + q(x)u = Eu.$$

In this example, the direct spectral problem would be to determine the energy levels, E (that is, the sequence of eigenvalues), from the knowledge of the potential, $q(x)$.

1.3: Inverse Spectral Problems

An inverse spectral problem is one in which all of the eigenvalues and eigenfunctions are known, and one is tasked with finding the corresponding $n \times n$ matrix, for example. In the general case, one is not able to complete this task. However, there are certain restrictions in which the problem can be solved. We know that this problem has a unique solution if and only if A has n linearly independent eigenvectors. For instance, if the matrix is normal (in particular, if the matrix is real and symmetric, or if it is complex and Hermitian), then the inverse spectral problem can be solved. In this case, we have the factorization

$$A = SDS^{-1},$$

where S is the matrix whose columns are linearly independent eigenvectors, and D is a diagonal matrix whose diagonal entries are the eigenvalues associated with the eigenvectors in the order that the columns are listed in S. Other examples of inverse spectral problems include:

- Finding an $m \times n$ matrix from a singular system of the matrix (one would need two sets of eigenvectors in addition to the singular values)
- Determining the coefficients in a Sturm-Liouville differential equation from the knowledge of the spectrum

- Lastly, the quantum mechanical inverse problem associated with the one-dimensional Schrödinger equation deals with the reconstruction of the unknown potential, $q(x)$ from the energy levels.

1.4: Direct and Inverse Problems of Musical Instruments

In relation to music, it seems reasonable to presume that the shape of a drum would affect the sound that is produced. This would be the direct problem that one might consider. An example of an inverse problem in the musical realm would be one where the vibrating modes of a string are given, and one is tasked with finding the one-dimensional hyperbolic partial differential equation that produced those vibrating modes. The inverse eigenvalue problem for the vibrating string can be paraphrased as "can one hear the density of a string?" which leads into the famous question posed by Mark Kac.

In 1966, Mark Kac [11] asked the question, "Can One Hear the Shape of a Drum?" – a famous *inverse problem* in mathematics. This question has sparked the interest of many mathematicians over the past forty-six years. Many different approaches were taken, until two mathematicians were able to come up with a counterexample in 1991 for a specific class of drums – the polygonal drum. Carolyn Gordon and David Webb [7] at Washington University and Scott Wolpert at the University of Maryland were able to construct a counterexample which determined that the answer was a firm: "No. One cannot always hear the shape of a drum." Barry Cipra [4] discusses the results of these three mathematicians in an article in "What's Happening in the Mathematical Sciences" published in 1993.

1.5: Thesis Content

In Chapter 2, we consider Mark Kac's question and different approaches that may be taken towards answering his question. His question can be considered for many different categories of the "drum." In Chapter 3, we consider inverse problems that arise from musical instruments. Several instruments are discussed, as well as numerical experiments and results.

In Chapter 4, we briefly give examples of other questions that have been considered in the form "Can you hear…?" Other mathematicians have written papers that follow similar patterns. They ask questions stemming from similar inverse problems. Gopinath and Sondhi [6] ask the question in 1970, "Can One Hear the Shape of Your Throat?" In 1987, Sekii and Shibahashi [18] ask if one can hear into the sun. In 2001, Gutkin and Smilansky [9] ask the question related to the Schrödinger operator on a finite metric graph "Can One Hear the Shape of a Graph?" "Can One Hear the Crack in a Beam?" [13] was the question explored by Lin in 2004, and lastly, Steven Cox, Mark Embree, and Jeffrey Hokanson [5] pose the question, "Can One Hear the Composition of a String?" in 2012.

In chapter 5, we consider the direct eigenvalue problem as related to musical instruments, specifically the guitar string, the flute, and the clarinet. We discuss the direct problem relating to the drum, and finally, the case for a bell.

The aims of this thesis are as follows:

1. To provide some history behind the work done in the area of inverse eigenvalue problems relating to Mark Kac's paper: "Can One Hear the Shape of a Drum?"
2. To make connections and correlations between papers written in the area
3. To consider the work done in the area of musical instruments

CHAPTER 2: CAN ONE HEAR THE SHAPE OF A DRUM?

In 1966, Marc Kac gave a lecture at The Rockefeller University in New York. In this lecture, he proposed the famous question in spectral theory: "Can One Hear the Shape of a Drum?" His lecture has been the focus of many mathematicians' attention since. For the majority of this chapter, we will follow the structure that M. H. Protter constructed in his paper "Can One Hear the Shape of a Drum – Revisited." However, we begin by considering the case of a manifold. Then we will progress to the case of a circular drumhead whose boundary is fixed. Protter's Problems I, II, III, and IV will be the next few sections, and the case of the polygonal drum (introduced by Kac and solved in part by Gordon and Webb) will be the last section.

2.1: Can One Hear the Shape of a Manifold?

Before considering Kac's question, one must consider the work done in a more general case – that is, the case of the Riemannian manifold. A *manifold* is a curved surface that satisfies the following property: when one considers a neighborhood of a point on the surface, the neighborhood resembles Euclidean space. A classic example of a manifold is a donut. When one considers small portions of the donut, the portions resemble (not in every way) a small piece of the Euclidean plane. When one considers the donut as a whole, it is very different than the Euclidean plane.

A *Riemannian manifold* is a manifold in which one can measure distances and angles. Gordon and Webb write the following in their paper, "You Can't Hear the Shape of a Drum": "Any Riemannian manifold has a wave equation, so it makes sense to ask: Can one hear the shape of a Riemannian manifold? Of course, if the answer is 'yes,' this is a harder problem than the original one, since a drumhead is a special case of a Riemannian manifold; it may be, however, that the answer is 'no,' in which case the more general problem offers wider scope for seeking counterexamples." [7]

Two years before Kac's paper was published, John Milnor [14] published a paper titled "Eigenvalues of the Laplace Operator on Certain Manifolds." His paper considered a generalization of Kac's proposed question but in the case of 16-dimensional tori. In his brief paper, he proved that there exist two 16-dimensional tori such that they are isospectral but not isometric. In other words, he proved that these specific tori have the same set of eigenvalues, but are not congruent in the sense of Euclidean geometry. Therefore, Kac's question was answered in the negative for the more general case of the manifold before he had even formally posed the question.

A year after Kac's paper was published, Kneser [12] proved the existence of two 12-dimensional tori which were also isospectral but not isometric. In 1980, Marie-France Vignéras [23] was able to provide a counterexample for compact manifolds of dimension $n \geq 2$, meaning that there exists a pair of isospectral manifolds for each n such that the two manifolds are not isometric. She also proved that, for the case of $n = 2$, tori which are isospectral *must* be isometric. Therefore by 1980, Kac's question had been answered for tori of dimensions 2, 12, and 16, as well as for compact manifolds of dimension $n \geq 2$. The case of tori in the 3rd through 11th dimensions, 13th through 15th dimensions, and dimensions greater than or equal to 17 are yet to be discussed.

2.2: Can One Hear the Shape of a Circular Drum?

In Shivakumar, Wu, and Zhang's recent paper titled, "Shape of a Drum, a Constructive Approach," they consider the question at hand for a general boundary, a circle, and an ellipse [19]. Due to the heavy amount of complex analysis needed for the case of an ellipse, we will only give their result for the case of the circle. As one may presume, the answer is proven to be that one can, in fact, hear the shape of a circular drum. Let us consider the proof of this presumption.

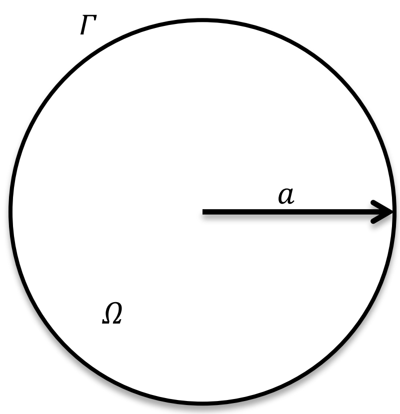

Figure 1: Circle of Radius a

Given a circular region Ω with radius a and boundary Γ, we begin with the equation

$$u_{xx} + u_{yy} + \lambda^2 u = 0 \ \text{ in } \ \Omega \ \text{ with } \ u = 0 \ \text{ on } \ \Gamma \tag{1}$$

The solution of (1) is given by

$$u = a_0 J_0\left(\lambda\sqrt{x^2 + y^2}\right)$$

where J_0 represents the Bessel function of the first kind and of order zero given by

$$J_0\left(\lambda\sqrt{x^2 + y^2}\right) = J_0(\lambda|a|) = \sum_{q=0}^{\infty}\left(-\frac{\lambda^2}{4}\right)^q \frac{|a|^{2q}}{q!\,q!}.$$

The eigenvalues are given by the zeros generated by $J_0(\lambda a) = 0$, or

$$\lambda_i = \frac{z_i}{a}, \quad i = 1, 2, \dots, \infty.$$

We now have

$$u = a_0 J_0(\lambda|a|) = \sum_{q=0}^{\infty} \left(-\frac{\lambda^2}{4}\right)^q \frac{a^{2q}}{q!\,q!} = \prod_{j=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_j^2}\right).$$

When we compare the coefficients of λ^2, we have

$$a^2 = 4 \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}.$$

This proves that we will know the radius of our circular drumhead, a, if all of the eigenvalues $\lambda_1, \lambda_2, \dots$ are known.

2.3: Can One Hear the Shape of a Drum whose Boundary is Fixed?

Now let us consider Kac's paper—which happens to be what M. H. Protter deems as Problem I in his paper, "Can One Hear the Shape of a Drum? Revisited" [17]. We begin with a membrane, call it Ω. Let us call its boundary Γ, and let us hold the membrane fixed along Γ. Kac begins his paper by discussing the displacement or movement of Ω after it is set in motion. See Figure 2 on the following page.

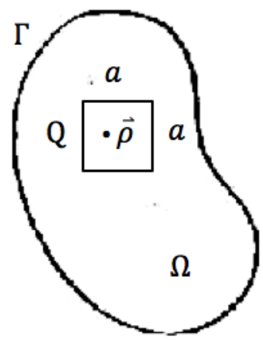

Figure 2: Membrane with Fixed Boundary

It is known that when Ω is held fixed along Γ and is then set in motion, the membrane's movement must obey the wave equation:

$$\frac{\partial^2 F}{\partial t^2} = c^2 \nabla^2 F,$$

where c is a constant that depends on properties of Ω.

Now, when Ω is held fixed by a curve Γ that is *smooth enough*, there is a sequence of numbers (eigenvalues) to each of which there exists a corresponding eigenfunction such that

$$\frac{1}{2}\nabla^2 \Psi_n + \lambda_n \Psi_n = 0$$

where λ_n are the eigenvalues, and Ψ_n are the corresponding eigenfunctions with the condition that $\Psi_n \to 0$ as our point given in Cartesian coordinates (x, y) approaches the boundary, Γ. (Note: it is usual to normalize the Ψ's, meaning that $\iint_\Omega \Psi_n^2(\vec{\rho})d\vec{\rho} = 1$, where $d\vec{\rho} \equiv dxdy$.)

Kac's goal is to determine whether the knowledge of all the eigenvalues λ_n and eigenfunctions, Ψ_n, is enough to determine the shape and the structure of Ω. We can pose the problem in the following way to aid our understanding. Let us consider two problems of the same nature:

$$\frac{1}{2}\nabla^2 U + \lambda U = 0 \text{ in } \Omega_1 \text{ with } U = 0 \text{ on } \Gamma_1 \tag{2}$$

$$\frac{1}{2}\nabla^2 V + \mu V = 0 \text{ in } \Omega_2 \text{ with } V = 0 \text{ on } \Gamma_2 \tag{3}$$

Now, assume for a moment that the solutions for Ω_1 (λ_n) are equal to the solutions for Ω_2 (μ_n). Does this imply that Ω_1 is equal, or congruent, to Ω_2?

In October 1910, H. A. Lorentz, a Dutch physicist, conjectured that "the number of sufficiently high overtones which lies between v and $v + dv$ is independent of the shape of the enclosure and is simply proportional to its volume." In other words,

$$N(\lambda) = \sum_{\lambda_n < \lambda} 1 \sim \frac{|\Omega|}{2\pi}\lambda\,, \tag{4}$$

where $N(\lambda)$ is the number of eigenvalues less than λ and $|\Omega|$ is the area of Ω. In equation (4), $\sim$ represents the equation:

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{|\Omega|}{2\pi}. \tag{5}$$

Herman Weyl was able to prove Lorentz's conjecture by using the theory of integral equations only two years after Hilbert predicted that the theorem would not be proved in his lifetime [11].

In proving this conjecture, we must begin by considering a system of particles, M, confined to a space, Ω. The particles are in equilibrium with a thermostat of temperature, T. From classical statistical mechanics, we then know that the probability of finding the specified particles at $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_M$ is

$$\frac{\exp\left[-\frac{1}{kT}V(\vec{r}_1, \dots, \vec{r}_M)\right] d\vec{r}_1 \dots d\vec{r}_M}{\int_\Omega \dots \int \exp\left[-\frac{1}{kT}V(\vec{r}_1, \dots, \vec{r}_M)\right] d\vec{r}_1 \dots d\vec{r}_M},$$

where $\overrightarrow{dr_1}, \overrightarrow{dr_2}, \dots, \overrightarrow{dr_M}$ are the volume elements, $V(\vec{r}_1, \dots, \vec{r}_M)$ is the interaction potential of the particles, and $k = R/N$. R is the "gas constant" and N is the Avogadro number. From here, we must move to the Schrödinger equation

$$\frac{\hbar^2}{2m}\nabla^2\psi - V(\vec{r}_1, \dots, \vec{r}_M)\psi = -E\psi\ ,$$

where $\hbar = \frac{\mathrm{h}}{2\pi}$ with h the Planck constant. We must also note the boundary condition that, in essence, confines the particles to Ω: $\lim \psi(\vec{r}_1, \dots, \vec{r}_M) = 0$ when at least one $\vec{r}_k$ approaches the boundary of Ω. Let the eigenvalues be such that $E_1 \leq E_2 \leq E_3 \leq \cdots$ corresponding to the normalized eigenfunctions $\psi_1,\ \psi_2, \dots$. We then know that the probability of finding specified particles at $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_M$ within our specified volume elements is

$$\frac{\sum_{s=1}^{\infty} e^{-E_s/kT}\psi_s^2(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_M)d\vec{r}_1, \dots, d\vec{r}_M}{\sum_{s=1}^{\infty} e^{E_s/kT}}\ .$$

We must now focus our attention on the smaller class of ideal gases. This means that we now have that $V(\vec{r}_1, \dots, \vec{r}_M) \equiv 0$. The probability of finding specified particles at $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_M$ with $|\Omega|$ being the volume of Ω is now

$$\frac{d\vec{r}_1, \dots, d\vec{r}_M}{|\Omega|^M}.$$

Considering only ideal gases gives us the separable Schrödinger equation

$$\frac{\hbar^2}{2m}\nabla^2\psi = -E\psi\ .$$

Switching gears from the $3M$-dimensional eigenvalue problem to simply the three-dimensional eigenvalue problem gives us

$$\frac{1}{2}\nabla^2\psi(\vec{r}) = -\lambda\psi(\vec{r})\ , \qquad \vec{r} \in \Omega,$$

$$\psi(\vec{r}) \to 0 \text{ as } \vec{r} \to \text{ the boundary of } \Omega\,.$$

Now we have that the formula for the probability of finding specified particles at $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_M$ is

$$\prod_{k=1}^{M} \frac{\sum_{n=1}^{\infty} \exp\left[-\frac{\lambda_n \hbar^2}{mkT}\right] \psi_n^2(r_k)}{\sum_{n=1}^{\infty} \exp\left[-\frac{\lambda_n \hbar^2}{mkT}\right]} d\vec{r}_k\,.$$

If we are to consider $\hbar \to 0$ or $T \to \infty$ we will have the result that

$$\tau \to 0 \quad \left[\tau = \frac{\hbar^2}{mkT}\right],$$

$$\sum_{n=1}^{\infty} e^{-\lambda_n \tau} \psi_n^2(\vec{r}) \sim \frac{1}{|\Omega|} \sum_{n=1}^{\infty} e^{-\lambda_n \tau}\,.$$

Considering this in two dimensions instead of three would mean that one can “hear” the *area* of Ω instead of the volume. Kac clarifies that this is only for $\vec{r}$ in the interior of Ω. From this work, we come to Weyl’s result. His result is the following from equations (4) and (5) above:

> The number of eigenvalues less than λ, as λ tends to infinity, is approximately the area of Ω times λ all divided by 2π. In other words, $N(\lambda) \sim \frac{|\Omega|}{2\pi}\lambda$.

All of Kac’s work is done “asymptotically.” He makes the assumption that Γ is “sufficiently regular.” He proves that

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t}\,, \qquad t \to 0.$$

(note: $\sim$ means “asymptotic to.”)

After working through and explaining his analysis of the asymptotic properties of the eigenvalues relating to this problem, he concludes his paper with the result that it is possible to “hear” the *area* of a drum—that is, one can obtain information regarding the drum’s geometry from studying its normal modes of vibration. For the case of the polygonal drum (see section 2.7), he concludes that one is able to "hear" the connectivity of a multiply connected smooth drum with r smooth holes. However, he was not able to give a direct answer on whether one is able to "hear" exactly the shape of a drum from studying the asymptotic properties of the drum's eigenvalues.

Kac's paper was published in 1966, and is a topic of interest still to this day. In 1987, M. H. Protter revisited Kac's results while describing the recent developments in the field at that time regarding this inverse problem.

Protter gives two different interpretations of Mark Kac's question. This first is this: let us assume that there is a drum being played in such a way that a person with perfect pitch (a person who can hear and identify *all* the normal modes of vibration) can only *hear* the drum – the drum cannot be seen. Is it possible for the person to determine *exactly* how the drum is shaped by knowing only the modes of vibration?

He gives his second interpretation mathematically: "If two domains in R^2 are isospectral, is it necessarily true that they are isometric?" As stated in a previous section, for two domains to be *isospectral* they must have the same set of eigenvalues. For two domains to be *isometric* they must be congruent in the sense of Euclidean geometry [17]. A good way to understand two domains being isometric is to consider two circles:

$$x^2 + y^2 = 1 \qquad \text{and}$$

$$(x-2)^2 + (y-2)^2 = 1.$$

The first circle is centered on the origin and has a radius of one. The second has a center at (2,2) and has a radius of one. These circles are called isometric. They have the exact same shape; the second circle is just translated up two units and to the right two units.

Protter sets up his paper as the proposal of four separate propositions. He calls the propositions Problems I, II, III, and IV.

Problem I (Kac's Problem) is the following, which has been stated previously: given two isospectral domains in R^2, can one know that they are isometric? In other words, we must start with the wave equation that models the vibration of a two-dimensional drum

$$\frac{\partial^2 v}{\partial t^2} = c^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right),$$

where $v = v(x, y, t)$ denotes the transverse displacement of a point (x, y) at time t. For convenience, we will assume that the constant $c^2 = 1$. After separating variables such that $v = F(t)u(x, y)$, our solution, $u(x, y)$, will be a solution in the domain Ω of the equation

$$\Delta u + \lambda u = 0.$$

We have that the boundary of our drum is attached, therefore the solution of the above equation must satisfy the boundary condition

$$u = 0 \text{ on } \partial\Omega.$$

We can assume that there is a countable sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$ tending to $+\infty$. This is known to be true for the domain Ω which is in R^n, with $n \geq 2$. In 1911 (as mentioned previously), Weyl approximated λ_k as the following:

$$\lambda_k \sim 4\pi^2 \left(\frac{k}{B_n V} \right)^{2/n}, \qquad k \to \infty.$$

V denotes the volume of Ω, and B_n denotes the volume of the unit ball in R^n.

We can also assume that there is a corresponding sequence of eigenfunctions $u_1, u_2, \ldots, u_k, \ldots$ such that each u_k satisfies the boundary condition as well as the equation

$$\Delta u_k + \lambda_k u_k = 0 \text{ in } \Omega.$$

We know that the eigenfunctions are orthogonal in $L_2(\Omega)$, and we will normalize them such that $||u_k||_{L_2(\Omega)} = 1\ \forall k.$ Now we can consider the question as stated previously: If two domains ($\Omega_1$and Ω_2) in R^2 are isospectral, is it necessarily true that they are isometric?

2.4: Can One Hear the Shape of a Drum whose Normal Derivative on the Boundary is Zero?

Problem II is the same question as is asked as in Problem I, but the boundary condition is now

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

where $\partial/\partial n$ is the directional derivative that is normal to the boundary of Ω at each point. One can think of this problem in the following way: the drum material is no longer fastened to the rim but now just resting on top of the rim. Another way of considering this problem is "whether or not two nonisometric domains Ω_1 and Ω_2 can be isospectral with respect to the solutions of $\Delta u + \lambda u = 0 \text{ in } \Omega$." [17]

2.5: Can One Hear the Shape of a Stiff Plate from Its Modes of Elastic Vibrations?

Next, Protter gives Problem III, where he considers the vibration of a stiff plate. The vibrations of a plate (which is clamped so that no lateral motion can occur at its edge and whose rim is fastened at the boundary) that span the domain Ω in R^2 are governed by the equation

$$\Delta^2 u - \nu u = 0 \text{ in } \Omega.$$

Its boundary conditions are

$$u = 0 \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

The question that arises in this problem is whether plates which have are nonisometric can be isospectral.

2.6: Can One Hear the Shape of a Three-Dimensional Drum from Its Modes of Elastic Vibrations?

Finally, Problem IV considers a domain Ω which is in R^3. He defines $\boldsymbol{u} = (u_1, u_2, u_3)$ to be the elastic displacement vector. The function which takes elements of Ω into R^3 is given by the system

$$\mu \Delta u_p + (\lambda + \mu)\left(\frac{\partial}{\partial x_p}\right)(\nabla \cdot \mathrm{u}) + \sigma \mathrm{u_p} = 0 \text{ in } \Omega, \qquad p = 1, 2, 3.$$

u must satisfy the boundary conditions

$$u_p = 0 \text{ on } \partial\Omega, \qquad p = 1, 2, 3,$$

where μ and λ the Lamé constants, and σ_k are the eigenvalues.

After defining Problems I-IV, Protter continues by studying the asymptotic properties of each of the four discrete sequences of eigenvalues, since all four of them tend to infinity. He also studies how these asymptotic properties affect the geometry of the domain. While considering the inequalities of the eigenvalues, some show to be universal inequalities. He also constructs the bounds for the first eigenvalue, or the fundamental tone in Problem I.

In the conclusion of his paper, he discusses his findings—that is, that for Problems I and II there exist domains in R^n, with $n \geq 4$, which are isospectral but not isometric. Unfortunately, the question remains unanswered for $n = 2, 3$ and in all dimensions for Problems III and IV.

2.7: Can One Hear the Shape of a Polygonal Drum?

In the last few pages of Kac's paper [11], he considers the case of a polygonal drum. He simplifies this case by only considering convex drums in which every angle is obtuse. Kac is able to prove that the constant term is the same for all simply connected drums. That is, the constant is equal to $\frac{1}{6}$. For multiply connected drums in which the holes are polygonal as well, the constant term is equal to $(1-r)\frac{1}{6}$, where r is the number of holes. This leads us to the case of a smooth drum with r smooth holes. One can then conclude that

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} + (1-r)\frac{1}{6}.$$

This shows that one can, in fact, hear the connectivity of a smooth drum with r smooth holes.

The polygonal drum is the main subject of Gordon and Webb's paper, "You Can't Hear the Shape of a Drum" [7]. The two authors were aided by the work of Scott Wolpert from the University of Maryland. They begin their paper by explaining the steps needed to solve an analogue of Kac's question for the case of a string – that is, they proved that for a single dimension one can, in fact, hear the shape of a string. They showed how to determine the length of the string from the lowest frequency of its vibrations. Their conclusion was that the length is given by ½ of the reciprocal of the lowest frequency. [7]

From there they move to situations in two dimensions, or the case of the vibrating drumhead. It is not until the last few pages of their paper that they explain their methods for constructing a counterexample for a specific case of Kac's question. They complement their work with a series of figures to illustrate their methods. Their work stems from Sunada's Theorem [3], which is stated as follows:

> Let M be a complete Riemannian manifold and let G be a finite group acting on M by isometries with at most finitely many fixed points. If H_1 and H_2 are almost conjugate subgroups of G acting freely on M, then the quotients $H_1\backslash M$ and $H_2\backslash M$ are isospectral.

For quite some time, it was thought that Sunada's Theorem would not aid in the solving of Kac's question. However, in 1989, the French mathematician Pierre Berard was able to provide a new proof of this theorem that gave the mathematical world a wider application of Sunada's Theorem. Just a year later the authors and Scott Wolpert were able to provide a counterexample to Kac's question for the case of polygonal drums. For the specifics in how they applied Sunada's Theorem and group theory to provide this counterexample, please see their paper [7]. They give two Caley graphs which depict their permutations beautifully.

Barry Cipra's article "You Can't Always Hear the Shape of a Drum," discusses the results of Carolyn Gordon and David Webb at Washington University and Scott Wolpert at the University of Maryland in 1991. They finally were able to answer the question posed by Mark Kac—the answer being "No." Cipra explains that the three mathematicians were able to find a pair of distinct shapes that, when studied mathematically as a "drum" or a "membrane," resonated at the same frequencies. Cipra stated the result simply, "if your goal is to deduce the shape of a drum merely from the sound it makes, these two drums provide an example where that goal cannot be achieved: You can't decide which drum you're listening to, because they both sound the same." [4]

2.8: Can One Hear the Shape of an Annular Drum?

In 1982, H. P. W. Gottlieb [8] considered the case of an annular drum – that is, a ring-shaped drum, or one with a smooth boundary that would be convex if it were not for a hole in the middle. He proved that the area, the total perimeter, and the drum's connectivity could all be determined given the set of eigenvalues.

2.9: Can One Hear the Fractal Dimension of a Drum?

In 2005, Walter Arrighetti and Giorgio Gerosa [1] extend Mark Kac's proposed question to arbitrary, finite-dimensional domains and to fractal sets. They set out to determine if the fractal dimension can be deduced from its countable eigenvalues. "As the 'self-similar' spectrum of the fractal is enough to compute box-counting dimension, positive answer is given to title question." [1]

CHAPTER 3: EIGENVALUES AND MUSICAL INSTRUMENTS: DIRECT AND INVERSE PROBLEMS

3.1: Can One Hear the Composition of a String?

In their article entitled, “One Can Hear the Composition of a String: Experiments with an Inverse Eigenvalue Problem,” Cox, Embree, and Hokanson [5] discuss an inverse eigenvalue problem regarding masses on a string. The mathematical question at hand stems from the title of their paper. In their paper they seek to determine if one can determine the masses and the locations of symmetrically placed beads on a string if one is given the set of eigenvalues. The title gives away the answer – that one can, in fact, hear the locations of the beads given the corresponding set of eigenvalues.

They begin with a beaded string and explore the differential equations resulting from its displacement when plucked. Then they solve for the eigenvalues and eigenvectors in the differential equations created by the plucked string. After determining the eigenvalues, they use numerical algorithms due to de Boor, Golub, and Gladwell to determine the positions of the beads and the masses on the strings (the inverse problem). They conclude their paper with the result that one *can* determine the locations and the masses of beads placed on a string when given one set of eigenvalues.

The forward problem results from the figure found on page 158 from their paper [5]. After organizing the linear equations created from the first-order approximations to the equations given in their paper, they write the differential equation in this matrix form:

$$My''(t) = -Ky(t),$$

where $y(t)$ is the state vector, M is the mass matrix, and K is the stiffness matrix. The mass and stiffness matrices are both symmetric and positive definite. The solutions to this differential equation are given by:

$$y(t) = \sum_{k=1}^{n} \gamma_k(t) \mathrm{v}_k,$$

$$\text{where } \gamma_j(t) = c_j \cos(\omega_j t) + s_j \sin(\omega_j t),$$

and where the c_j and s_j coefficients are determined by the initial pluck of the beaded string. We have that $y'(0) = 0$ which implies that $s_j = 0\ \forall j$, and $y(0) = y_0$ implies that the c_j coefficients are the expansion coefficients of y_0 in the eigenvector basis, i.e. the c_j coefficients can be found by solving $y_0 = \sum_{j=1}^{n} c_j \mathrm{v}_j$.

The inverse problem as implied from the title is: given the set of eigenvalues, can one determine the location and the masses of beads placed on a string?

When beginning to explore the inverse problem, Cox, Embree, and Hokanson first model the beaded string vibrations with their corresponding matrices:

$$A_n = M^{-1/2} K M^{-1/2} = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{bmatrix}$$

$$\text{where } a_k = \tau \frac{\left(\frac{1}{l_{k-1}} + \frac{1}{l_k}\right)}{m_k} \text{ and } b_k = -\tau / l_k \sqrt{m_k m_{k+1}} < 0.$$

They denote the nodes and weights as follows:

$$nodes: \xi_1 < \xi_2 < \cdots < \xi_n$$

$$weights: w_1, w_2, \ldots, w_n > 0,$$

and they set out to determine these weights and nodes. The authors gave the following algorithm for recovering the Jacobi matrix A_n:

1. Use the following to determine the nodes of the inner product:

$$\xi_k := \lambda_k.$$

2. Construct the weights, with $q_n'(\xi_k)$ given by the following:

$$w_k := \frac{1}{q_n'(\xi_k)\hat{q}_n(\xi_k)} = \frac{1}{\left(\prod_{\substack{j=1 \\ j \neq k}}^{n}(\xi_k - \xi_j)\right)\left(\prod_{j=1}^{n}(\xi_k - \hat{\lambda}_j)\right)}.$$

3. Determine the values of q_n and q_{n-1} at the nodes; see the following:

$$q_n(\xi_k) := 0,$$

$$q_{n-1}(\xi_k) := \frac{\hat{q}_n(\xi_k) - q_n(\xi_k)}{\sum_{j=1}^{n}(\lambda_j - \hat{\lambda}_j)} = \frac{\prod_{j=1}^{n}(\xi_k - \hat{\lambda}_j)}{\sum_{j=1}^{n}(\lambda_j - \hat{\lambda}_j)}.$$

4. Compute a_n, using the following:

$$\langle p, q \rangle = \sum_{j=1}^{n} w_j p(\xi_j) q(\xi_j)$$

$$a_n := \frac{\langle z q_{n-1}, q_{n-1} \rangle}{\langle q_{n-1}, q_{n-1} \rangle} = \frac{\sum_{j=1}^{n} w_j \xi_j q_{n-1}(\xi_j)^2}{\sum_{j=1}^{n} w_j q_{n-1}(\xi_j)^2}.$$

5. For $k = n-1, n-2, \ldots, 1$

   a. Compute $b_k = -\sqrt{b_k^2}$ via the approach described after (6.6) on page 169 [5]:

$$b_k^2 := \frac{\langle z q_k, z q_k \rangle - a_{k+1}\langle q_k, z q_k \rangle - \langle q_{k+1}, z q_k \rangle}{\langle q_k, q_k \rangle}.$$

b. Compute q_{k-1} at the nodes:

$$q_{k-1}(\xi_j) \coloneqq \frac{(\xi_j - a_{k+1})q_k(\xi_j) - q_{k+1}(\xi_j)}{b_k^2}.$$

c. Define

$$a_k \coloneqq \frac{\langle z q_{k-1}, q_{k-1} \rangle}{\langle q_{k-1}, q_{k-1} \rangle}.$$

Once A_n is defined, one is able to determine the masses and locations of the beads placed on the string by following the algorithm taken directly from their paper [5].

1. Solve $\mathbf{A}_n\mathbf{x} = \mathbf{e}_1$ and $\mathbf{A}_n\mathbf{y} = \mathbf{e}_n$ for $\mathbf{x}$ and $\mathbf{y}$.
2. $\gamma_2 \coloneqq \sum_{j=1}^{n}(\lambda_j - \hat{\lambda}_j)$ and $\gamma_1 \coloneqq (1 - y_n\gamma_2)/x_n$.
3. $\tilde{\mathbf{d}} \coloneqq \mathbf{d}/\sqrt{m_n} = \gamma_1\mathbf{x} + \gamma_2\mathbf{y}$.
4. $l_j m_n \coloneqq -\tau/(b_j\tilde{d}_j\tilde{d}_{j+1})$ for $j = 1, \dots, n-1$.

   $l_0 m_n \coloneqq \tau/(a_1\tilde{d}_1^2 - \tau/(l_1 m_n))$.

   $l_n m_n \coloneqq \tau/(a_n\tilde{d}_n^2 - \tau/(l_{n-1} m_n))$.
5. $m_n \coloneqq (\sum_{j=0}^{n} l_j m_n)/l$.

In one of the latter sections of their paper, they discuss their experiments and how well their algorithm matches the results from their experiments. They note that their algorithm returns the masses and the positions of the beads (≤ 4) with an error less than 3.5%. However, they also discuss that they discovered some challenges when six beads are placed on the string. They computed a relative error of 18% of the lengths when six beads are placed on the string.

3.2: Eigenvalues and Musical Instruments

In 2001, V. E. Howle and Lloyd N. Trefethen [10] published a paper titled, "Eigenvalues and Musical Instruments." In their paper, they analyze the frequencies that are produced by musical instruments – more specifically the string of a guitar, the flute, the clarinet, a kettledrum, and lastly, bells. Their paper studies the basis upon which musical instruments are designed – the fact that physical systems oscillate at certain frequencies. Mathematicians have studied these frequencies, and have determined the mathematical reasons for the beautiful melodies that are produced by these instruments.

The linear operator that defines the oscillations of the frequencies of linear systems like drums and strings can be given by

$$\frac{du}{dt} = \frac{1}{2\pi} Au$$

where A is a matrix or linear operator. The solution to the system, $u(t)$ can be given by

$$u(t) = e^{\lambda t/2\pi} u(0)$$

The authors explain that the imaginary parts of the eigenvalues, λ, correspond to the musical instrument's frequency and the negative real parts of the eigenvalues correspond to the decay rate of the frequency. For notation's sake, the (real) frequency of the oscillation is denoted $\mathrm{Im}\lambda$, and the decay rate would then be denoted $-\mathrm{Re}\lambda$.

After discussing the linear systems, the authors then move on to the nonlinear systems. Their methodology is this: they begin with the eigenvalues of a musical instrument. They study the eigenvalues of the particular instrument, considering the decay rate and the frequencies separately. They do this for all of the instruments listed above. Other aspects of this direct problem will be considered in Chapter 5.

3.3: A Mathematical Model of a Guitar String

In "A Mathematical Model of a Guitar String," Rasmus Storjohann [21] studies several different aspects of the vibrating guitar string. He explains first the difference between potential energy and kinetic energy in a vibrating string. How tight the string is pulled gives rise to different frequencies, which then brings about the different pitches one can hear as the musician strums the guitar. He also explains to the reader the meaning of a harmonic, what causes the vibrational frequency to decay, and what happens when it does decay.

He proposes without rigorous justification or proof the following mathematical expression that takes all of those processes into account:

$$u(x,t) = \left(\frac{1}{k(1-k)}\right) \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \sin(n\pi k)\, e^{-\alpha_n t} \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{cn\pi}{L} t\right). \quad (6)$$

Here $u(x,t)$ is the amplitude of the string at a given point x and time t. The length of the string is denoted L, and k is a fraction denoting the point of the string where it is plucked. Lastly, $e^{-\alpha_n t}$ represents the decay rate associated with the nth term.

CHAPTER 4: OTHER "CAN YOU HEAR" PROBLEMS

4.1: Can One Hear the Shape of a Graph?

In their paper, "Can One Hear the Shape of a Graph?" Boris Gutkin and Uzy Smilansky consider Kac's question as related to the Schrödinger operator on a finite metric graph [9]. They have determined that the spectrum of the Schrödinger operator will determine the connectivity and the bond lengths uniquely. The condition for which this uniqueness exists is that the connectivity is simple ("no parallel bonds between vertices and no loops connecting a vertex to itself") and that the lengths are non-proportional. In other words, one *can* hear the shape of a graph.

> For boundaries in intermediate classes of smoothness the answer is not known. The existence of isospectral systems was investigated for Laplacians on closed Riemannian manifolds and for discrete Laplacians which are formed by the connectivity matrices of graphs. In both cases, elaborate techniques were devised to identify large sets of different, but isospectral, systems. However, if the domains are analytic surfaces of revolution, the spectrum determines the manifold uniquely [9].

Several authors have considered inverse problems for differential equations on metric graphs and have shown that their spectrum determine important geometric and topological characteristics of the underlying finite metric graph. These advances are beyond the scope of this thesis.

4.2: Can One Hear the Shape of His Throat?

In 1970, B. Gopinath and M. M. Sondhi [6] had a paper published titled, “Determination of the Shape of the Human Vocal Tract from Acoustical Measurements.” In their paper, they measure the acoustics from one end of a human vocal tract.

Mermelstein and Schroeder explained a few years prior to Gopinath and Sondhi that if the area of a vocal tract is of the form

$$\log A(x) = \log A_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where l is the length and $A(x)$ is the area function, then as $a_n \to 0$ for all n, the nth eigenfrequency is given by $\lambda_n = \lambda_{0n}(1 - \frac{1}{2}a_{2n-1})$, where λ_{0n} is the nth eigenfrequency of the uniform tract ($a_n \equiv 0, n = 1, \ldots$). If one were to close the vocal tract at both ends, the approximation of the nth eigenfrequency is then given by $\mu_n = \mu_{0n}(1 - \frac{1}{2}a_{2n})$.

Gopinath and Sondhi expound on Mermelstein and Schroeder’s work by giving two noniterative methods for obtaining the area function from the acoustical data they acquire. The methods that they provide are useful in three aspects: they bring about a better understanding of the physical meaning of the problem, they clarify the mathematical characteristics of the problem, and they give solutions that enable one to explore the affects of inaccuracies in the data on the area functions. The authors conclude that when $A(x)$ has continuous first and second derivatives, its spectrum will uniquely determine $A(x)$.

4.3: Can One Hear Into the Sun?

The goal of Takashi Sekii and Hiromoto Shibahashi [18] in their paper, "Inverse Problems of Solar Oscillations" is to numerically deduce the sound velocity distribution of the interior of the sun from their gathered oscillation data. In 1984, Gough provided a method for providing the sound velocity distribution based on an asymptotic expression of eigenfunctions of p-modes. His method was validated a year later for the outer regions of the sun by Christensen-Dalsgaard's numerical approach using solar models. For the inner parts of the sun, Gough and Christensen-Dalsgaard's solutions of the integral equation were not consistent at all. This left Sekii and Shibahashi searching for a more accurate method.

In helioseismology, it is common to consider oscillating modes of vibration in terms of functions called spherical harmonics. P, f, and g modes are resonant modes of oscillation generated within the sun. The p-modes (the modes studied by Sekii and Shibahashi) are spherically harmonic in nature. One can characterize the p-modes in three terms: order, harmonic degree, and the number of planes that create vertical sections within the sun. When one considers a large number of modes at once, one can begin to have an understanding of the shape of the interior of the sun. This is the focus of Sekii and Shibahashi's paper.

4.4: Can One Hear the Crack in a Beam?

In 2003, Hai-Ping Lin [13] published a paper titled, "Direct and Inverse Methods on Free Vibration Analysis of Simply Supported Beams with a Crack." In other words, he is asking the question, "Can you hear the crack in a beam?"

Consider a beam supporting part of a parking garage. When the beam begins to develop a crack, the structure itself becomes compromised. The beam is not as stiff, thus it cannot support the mass of the cars driving overhead as it was designed. The defect in the beam can also affect the damping properties and mass distribution (the weight of the cars), and it can eventually cause detrimental issues with the structure.

The purpose of Lin's paper is to set up a method in which one can determine the position of a crack in a beam, as well as the extent of the crack, from merely measuring the natural frequencies of the cracked system. In order to answer this question, he takes an approach that is common in both optics and in acoustics. He uses an analytical transfer matrix to study the propagation of waves though a simply supported beam with a crack. This method helps him to solve both the direct and the inverse problem. He models the beam with a crack as such: he considers the beam to be two separate pieces and the crack to be a "rotational spring with sectional flexibility." He then uses the Timoshenko Beam Theory on the two "separate" beams and applies the compatibility requirements of the crack. This yields the characteristic equation for the system explicitly, a characteristic equation which is a function of the eigenvalue (or natural frequency), the location of the crack, and its sectional flexibility. Lin explains that there are four integration constants that arise from the eigenfunctions between adjacent sub-beams. The relationship between these constants can be determined by considering the compatibility requirements of the crack. Because Lin uses the transfer matrix method, the four unknown constants can be determined by satisfying the four boundary conditions on the system.

In order to determine the location of the crack, as well as the sectional flexibility of the beam, one must measure any two natural frequencies of the cracked system and use the characteristic equation compute the information. Lin uses a crack-disturbance function $f(x, z)$ to model the cracked region with local flexibility. The location of the crack and the beam's sectional flexibility has been determined. Therefore, the size of the crack can then be determined.

The author not only gives his theoretical work, but he also provides experimental measurements. He gives the measurements of his simply supported cracked beam, and he explains how he measures the frequencies.

The relationship between the position of the crack and the natural frequencies from the identification equation is given.

CHAPTER 5: DIRECT EIGENVALUE PROBLEMS OF MUSICAL INSTRUMENTS

In V. E. Howle and Lloyd N. Trefethen's paper titled "Eigenvalues and Musical Instruments," they detail direct eigenvalue problems associated with five different instruments: the guitar (string), the flute, the clarinet, the drum, and bells [10]. The purpose of this chapter is to make note of the marked similarities and differences in these eigenvalue problems. We must first make sure that we have a proper understanding of the way that these systems behave. One of the defining characteristics of (most) musical instruments is that they are designed to produce distinguishing oscillations at varying frequencies. When considering the eigenvalue problem that results from the equation that describes the motion of the system, one will find notice two things. The first is that the imaginary parts of these eigenvalues represent the frequency of oscillation. So, given an eigenvalue $\lambda = \alpha \pm \beta i$, one will have a solution similar to

$$y(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)).$$

The second thing you will notice is that the real part of the eigenvalue determines how quickly (or slowly) the instrument's oscillation will decay.

Howle and Trefethen have structured their paper around the graphs of these eigenvalues, which have been determined by experiments run in a lab. For some of the instruments tested, much can be said about the behavior of the eigenvalues. However, this is not the case for the drums and the bells. For each of these cases, the authors use "idealized" instruments for their experiments. The results will have slight variations when actual instruments are used.

5.1: The Guitar String

As we are now well aware, the motion of a guitar string is governed by the second order wave equation $u_{tt} = c^2 u_{xx}$, $c^2 = \frac{T}{\rho S}$, where c represents the wave speed, T represents the tension, and S represents the cross-sectional area of the string. The eigenvalues of this problem are $\lambda_n = \frac{n\pi}{L}$, and the corresponding eigenfunctions are $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$. (See section 3.3 for details.) The authors note that for every positive frequency, there will always be a corresponding negative frequency of the same magnitude, thus only the upper half of the plane is needed to gain an understanding of what is going on in the figures.

There are three different factors that one must take into consideration when studying the energy losses in a guitar string. Damping, which (graphically speaking) shifts the eigenvalues to the left half of the complex plane, can occur from non-rigid end supports, air viscosity, and internal losses. The non-rigid end supports "couple the string to the soundboard," thus providing audible sound volume. The damping caused by the end supports is important to note, but does not play a major role in the energy losses in the strings. Second, there exists a damping caused by air viscosity. When considering both the damping caused by air viscosity and the damping caused by internal losses (the third case), one can see that the damping caused by air viscosity has a greater impact on steel string than nylon strings in relation to the damping caused by internal losses. If you were just to study the damping caused by air viscosity in the steel and nylon strings, the damping is much greater for the nylon string. It depends on which perspective you take.

The damping from the viscosity of the air can be modeled by the following equation, according to Howle and Trefethen:

$$\alpha_a = \frac{\pi \rho_a f\left(2\sqrt{2}M + 1\right)}{\rho M^2},$$

where ρ is the density of the string material, ρ_a is the density of the air, f is the frequency, S is the cross-sectional area of the string, r is the radius of the cross-section of the string $\left(r = \sqrt{\frac{S}{\pi}}\right)$, η_a is the kinematic viscosity of the air, and $M = \left(\frac{r}{2}\right)\sqrt{f/\eta_a}$. Since the decay is a function of the frequency, we have manipulated the variables to show that $\alpha(f)$ is indeed a square root function, as it appears from Figure 3 of Howle and Trefethen's paper. We have

$$\alpha(f) = \frac{4\sqrt{2}\pi \rho_a \sqrt{f\eta_a}}{\rho r} + \frac{4\pi \rho_a \eta_a}{\rho r^2}.$$

We must keep in mind that all of the variables except f are constants. It is most interesting to note that the losses are much greater for nylon strings than for steel strings. This is due to the fact that nylon strings are less dense than steel strings, thus the constant multiplied by $\sqrt{f}$ will be larger.

Internal losses are modeled by a different function given by

$$\alpha_i = \pi f \frac{Q_2}{Q_1},$$

where f is the frequency as before and $Q_1 + iQ_2$ is the complex Young's modulus.

In the last two figures of the authors' section on guitar strings, they plot the decay rates from the viscosity of the air and internal losses on the same graph. One will notice that for nylon strings, the internal losses play a much bigger role than damping due to air viscosity. The overall decay rate for nylon strings is approximately proportional to the frequency.

For steel strings, there is only a slight difference between the plot of the damping due to the viscosity of the air and the internal damping. Overall, the decay rate for steel strings is approximately proportional to the square root of the frequency.

5.2: The Flute

For the flute, we know that “plane pressure waves obey the same equation as transverse waves on a string: $u_{tt} = c^2 u_{xx}$, where c is speed of sound in the air under normal conditions” [10]. It is important to note that the eigenvalues and eigenfunctions are exactly the same as those for an ideal string fixed at both ends $\left(\lambda_n = \frac{n\pi}{L}\right.$ and $\left.X_n(x) = \sin\left(\frac{n\pi x}{L}\right)\right)$. Thus the eigenvalues for the flute behave in a very similar manner to the eigenvalues for the guitar string.

Just like in the case of the guitar string, there are important factors in a flute that cause damping, or energy losses. The first factor happens when frictional and thermal energy is transferred to the wall of the flute. The authors call the damping that results “wall losses.” The losses that take place in this category have the greatest affect on the eigenvalues, and they have represented the decay rate as $\alpha = 2A\omega^{1/2}c/r$. A is a constant that is determined based on the material of the walls and the surface condition, ω is the angular frequency, and c is the speed of sound in the air. In this decay rate, one can see that α is a function of ω, making this a square root function like in the case of the guitar string.

An approximation to the damping rate due to sound radiation is $\alpha = \frac{\pi}{4}\left(\frac{r}{l}\right)^2 (2n-1)\omega$. Here, r is the radius of the bore, l is the length of the flute, and n is the node number. ω represents the angular frequency, as before. One will notice that the decay rate due to sound radiation is proportional to the frequency.

5.3: The Clarinet

Flutes and clarinets are very similar in their structure. However, there is one distinct difference: that is, a reed drives the oscillations in a clarinet. Another difference between a flute and a clarinet is that the tone holes (relative to the size of the bore) are smaller in a clarinet than a flute. This means that the tone holes do not actually cut off the tube when pressed by the musician's fingers.

Very similar to the flute, the clarinet experiences energy losses from wall losses and sound radiation. When observing the graph of the eigenvalues for the ideal clarinet, one will notice that the fundamental frequency is an octave lower than that of an ideal flute of the same length. One will also notice that the harmonics are all odd multiples of the fundamental.

5.4: The Drum

The authors give a simple definition for a drum: "an ideal circular membrane with clamped edges." The equation of motion for a circular membrane is given by $\nabla^2\eta = \frac{1}{c^2}\eta_{tt}$, $c^2 = \frac{T}{\sigma}$. One should note that $\eta(r,\theta)$ is the displacement of the membrane from its equilibrium at the point (r,θ) on its surface, T is the tension, and σ is the mass density per unit area. The solution of the above equation contains a Bessel function. We will derive the solution here.

First, we must start with the partial differential equation:

$$\nabla^2\eta = \frac{1}{c^2}\eta_{tt}. \qquad (16)$$

We then assume a solution of the form:

$$\eta(r,\theta,t) = R(r)\Theta(\theta)e^{-i\omega t}.$$

Substituting the assumed solution into our PDE (16) yields:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \eta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2}+\frac{\partial^2 \eta}{\partial t^2}=\frac{1}{c^2}\left(R(r)\Theta(\theta)-\omega^2 e^{-i\omega t}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rR'(r)\Theta(\theta)e^{-i\omega t}\right)+\frac{1}{r^2}\frac{\partial \eta}{\partial \theta}\left(\Theta'(\theta)R(r)e^{-i\omega t}\right)+\frac{\partial \eta}{\partial t}\left(R(r)\Theta(\theta)(-i\omega)e^{-i\omega t}\right)$$

$$=\frac{-\omega^2}{c^2}\left(R(r)\Theta(\theta)e^{-i\omega t}\right)$$

$$\frac{1}{r}\left(R'(r)\Theta(\theta)e^{-i\omega t}+rR''(r)\Theta(\theta)e^{-i\omega t}\right)+\frac{1}{r^2}\left(\Theta''(\theta)R(r)e^{-i\omega t}\right)+\left(R(r)\Theta(\theta)(-\omega^2)e^{-i\omega t}\right)$$

$$=\frac{-\omega^2}{c^2}\left(R(r)\Theta(\theta)e^{-i\omega t}\right)$$

$$\frac{1}{r}R'(r)\Theta(\theta)+R''(r)\Theta(\theta)+\frac{1}{r^2}\Theta''(\theta)R(r)-\omega^2 R(r)\Theta(\theta)=\frac{-\omega^2}{c^2}R(r)\Theta(\theta)$$

$$\frac{1}{r}R'+R''+\frac{1}{r^2}\frac{\Theta''}{\Theta}R-\omega^2 R=\frac{-\omega^2}{c^2}R\Theta$$

$$\frac{1}{r}\frac{R'}{R}+\frac{R''}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}-\omega^2=\frac{-\omega^2}{c^2}$$

$$\frac{r^2R''+rR'+\left(\frac{\omega^2 r^2}{c^2}-\omega^2 r^2\right)R}{R}=-\frac{\Theta''}{\Theta}=\lambda^2$$

$$\begin{cases}\dfrac{r^2R''+rR'+\left(\frac{\omega^2 r^2}{c^2}-\omega^2 r^2\right)R}{R}=\lambda^2\\ -\dfrac{\Theta''}{\Theta}=\lambda^2\end{cases}$$

$$\begin{cases}r^2R''+rR'+\left(\dfrac{\omega^2 r^2}{c^2}-\omega^2 r^2\right)R=\lambda^2 R\\ -\Theta''=\lambda^2\Theta\end{cases}$$

$$\begin{cases}r^2R''+rR'+\left(\dfrac{\omega^2 r^2}{c^2}-\omega^2 r^2-\lambda^2\right)R=0\\ \Theta''+\lambda^2\Theta=0\end{cases}$$

$$\begin{cases} r^2R'' + rR' + \left(r^2\left(\frac{\omega^2}{c^2} - \omega^2\right) - \lambda^2\right)R = 0 & (17) \\ \Theta'' + \lambda^2\Theta = 0 & (18) \end{cases}$$

The solution of (18) is of the form:

$$\Theta(\theta) = \begin{cases} A + B\theta & \lambda = 0 \\ Dcos(\lambda\theta) + Esin(\lambda\theta) & \lambda \neq 0 \end{cases}$$

However, $R(r)$ in equation (17) is a Bessel-type equation. We may seek a Frobenius series solution for $R(r)$ of the form:

$$R(r) = \sum_{k=0}^{\infty} a_k r^{k+m}\,.$$

This leads to

$$R(r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+m)!}\left(\frac{\omega r}{2c}\right)^{2k+m}.$$

If we let $z = \frac{\omega r}{c}$, then we obtain

$$R(r) = J_m(z),$$

the Bessel function of the first kind of order m. Thus,

$$\eta(r,\theta) = AJ_m(z)\cos(m\theta)\,, \qquad \text{where } z = \frac{\omega r}{c}\,.$$

When studying the eigenvalues of an ideal drum, one must consider two important properties: the membrane moving through the air and the presence of the kettle. The purpose of the kettle is to tune the modes that are the least damped.

From the graphs of the eigenvalues given, one can see that an ideal drum does not have any distinctive or definite pitch. This comes from the fact that the zeros of the Bessel function are not all harmonically related.

5.5: The Bell

A bell constructed at random will almost assuredly not sound musical. After hundreds of years of work, many people have contributed to the adjusting of the eigenvalues so that they will have a musical sound. For the authors' experiments for this musical instrument, they decided to use an actual $A_4^{\#}$ minor-third bell. They do not go into much discussion. We will conclude with this quote: "For our purposes, it is enough to note the astonishingly satisfying imaginary parts of the first six eigenvalues in [Figure 19]. These six noted line up like a chord played on a piano, and with decay rates as low as about half an e-folding per second, you can almost hear this clean bell ring."

REFERENCES

[1] W. ARRIGHETTI AND G. GEROSA, *Can You Hear the Fractal Dimension of a Drum?*, Applied and Industrial Mathematics in Italy, 1 (2005), pp. 1-11.

[2] L. BEVILACQUA AND A. J. V. BRANDAO AND R. C. BASSANEZI, *A Mathematical Approach to the Plato's Problem*, Journal of Mathematical Modeling and Application, 1 (2009), pp. 2-17.

[3] P. BUSER, *Geometry and Spectra of Compact Riemann Surfaces: Sunada's Theorem*, Birkhäuser Boston (2010), pp. 283-310.

[4] B. CIPRA, *You Can't Always Hear the Shape of a Drum*, What's Happening in the Mathematical Sciences, 1 (1993), pp. 12-16.

[5] S. J. COX AND M. EMBREE AND J. M. HOKANSON, *One Can Hear the Composition of a String: Experiments with an Inverse Eigenvalue Problem*, SIAM Review, 54 (2012), pp. 157-178.

[6] B. GOPINATH AND M. M. SONDHI, *Determination of the Shape of the Human Vocal Tract from Acoustical Measurements*, The Bell System Technical Journal, 49 (1970), pp. 1195-1214.

[7] C. GORDON AND D. WEBB, *You Can't Hear the Shape of a Drum*, 84 (1996), pp. 46-55.

[8] H. P. W. GOTTLIEB, *Hearing the Shape of an Annular Drum*, Journal of the Australian Mathematical Society, Series B, 24 (1983), pp. 435-438.

[9] B. GUTKIN AND U. SMILANSKY, *Can One Hear the Shape of a Graph?*, Journal of Physics A: Mathematical and General, 34 (2001), pp. 6061-6068.

[10] V. E. HOWLE AND L. N. TREFETHEN, *Eigenvalues and Musical Instruments*, Journal of Computational and Applied Mathematics, 135 (2001), pp. 23-40.

[11] M. KAC, *Can One Hear the Shape of a Drum?*, American Mathematical Monthly, 73 (1966), pp. 1-23.

[12] M. KNESER, *Lineare Relationen zwischen Darstellungsanzahlen quadratischer Formen*, Math. Annalen, 168 (1967), pp. 31-39.

[13] H.-P. LIN, *Direct and Inverse Methods on Free Vibration Analysis of Simply Supported Beams with a Crack*, Eng. Structures, 26 (2004), pp. 427-436.

[14] J. MILNOR, *Eigenvalues of the Laplace Operator on Certain Manifolds*, Proceedings of the National Academy of Sciences, 51 (1964), 542.

[15] M. A. PINSKY, *Can You Feel the Shape of a Manifold with Brownian Motion?*, Expositiones Mathematicae, 2 (1984), pp. 263-271.

[16] PLATO, *The Republic*, (circa 385 BCE), Book VII, 514a-517c.

[17] M. H. PROTTER, *Can One Hear the Shape of a Drum? Revisited*, SIAM Review, 29 (1987), pp. 185-197.

[18] T. SEKII AND H. SHIBAHASHI, *Inverse Problem of Solar Oscillations*, in Stellar Pulsation, Lecture Notes in Physics 274, Springer, New York, 1987, pp. 322-325.

[19] P. N. SHIVAKUMAR, YAN WU, AND YANG ZHANG, *Shape of a Drum, a Constructive Approach*, WSEAS Transactions on Mathematics, 10 (2011), pp. 21-31.

[20] K. STEWARTSON AND R. T. WAECHTER, *On Hearing the Shape of a Drum: Further Results*, Mathematical Proceedings of the Cambridge Philosophical Society, 69 (1971), pp. 441-456.

[21] R. STORJOHANN, *A Mathematical Model of a Guitar String*, preprint, 5 pages.

[22] H. URAKAWA, *Bounded Domains which are Isospectral but not Congruent*, Annales Scientifiques de l'École Normale Supérieure, 15 (1982), pp. 21-32.

[23] M.-F. Vignéras, *Variétés Riemanniennes Isospectrales et non Isométriques*, Annals of Mathematics, 112 (1980), pp. 21-32.