

# INTEGRAL REPRESENTATIONS OF POSITIVE LINEAR FUNCTIONALS

by

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## ABSTRACT

In this dissertation we obtain integral representations for positive linear functionals on commutative algebras with involution and semigroups with involution. We prove Bochner and Plancherel type theorems for representations of positive functionals and show that, under some conditions, the Bochner and Plancherel representations are equivalent. We also consider the extension of positive linear functionals on a Banach algebra into a space of pseudoquotients and give under conditions in which the space of pseudoquotients can be identified with all Radon measures on the structure space. In the final chapter we consider a system of integrated Cauchy functional equations on a semigroup, which generalizes a result of Ressel and offers a different approach to the proof.

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# INTRODUCTION

At the core of this dissertation is Bochner's theorem. The classical Bochner's theorem characterizes the Fourier transforms of finite positive Radon measures on the real line as positive definite functions. Since Bochner proved this theorem, it has been generalized and modified in many different ways. Common to these generalizations is identification of a space of positive definite functions on semigroups, groups, or algebras with a space of Radon measures on a set of characters. In this dissertation we prove some known results using new, simpler, and more elegant techniques. We also obtain some new results. Our approach can be characterized by a direct construction of the measure rather than using Choquet's theory or Gelfand's theory.

In Chapter 1 we prove an integral representation of a positive definite functional on a commutative semigroup with involution. This type of result was first proved by Berg-Maserick in [10]. Then it was generalized by Atanasiu in [1] and [4]. Here we present a new proof of Atanasiu's result.

In Chapter 2 we study semigroup representations using a direct sum of  $L^2$  spaces without using Gelfand's theory. Instead, we use the integral representation of positive definite functionals from Chapter 1.

In Chapter 3 we consider a system of integrated Cauchy functional equations on a semigroup. The original paper of Deny considers the equation  $\sigma = \mu * \sigma$  on locally compact

commutative groups. Ressel generalized this result in [24]. Here we prove an extension of Ressel's result. Our proof is simpler and more elementary than that of Ressel.

In Chapter 4 we find Bochner and Plancherel type integral representations of positive functionals on a commutative algebra with involution. We also show that, under some conditions, Plancherel representation is equivalent to a Bochner representation.

In Chapter 5 we consider pseudoquotient extensions of positive linear functionals on a commutative Banach algebra and give conditions under which the constructed space of pseudoquotients can be identified with all Radon measures on the structure space.

# CHAPTER 1

## A BERG-MASERICK TYPE THEOREM

### 1.1 Introduction

Let  $(S, \cdot)$  be a commutative semigroup with involution  $*$  and neutral element  $e$ .

**Definition 1.1.1.** An *involution* on a semigroup is a map  $x \mapsto x^*$  from  $S$  to  $S$  such that the following hold for all  $x, y \in S$ ,

$$e^* = e \quad (xy)^* = y^*x^*, \quad (x^*)^* = x$$

**Definition 1.1.2.** A function  $S \rightarrow \mathbb{C}$  is *positive definite* if

$$\sum_{i,j=1}^n c_i \bar{c}_j \varphi(s_i s_j^*) \geq 0.$$

for all  $\{c_k\}_{k=1}^n \in \mathbb{C}$ ,  $\{a_k\}_{k=1}^n \in S$ , and  $n \in \mathbb{N}$ .

**Definition 1.1.3.** A function  $v : S \rightarrow \mathbb{R}_+$  is called an *absolute value* if

- (i)  $v(e) = 1$ ,
- (ii)  $v(st) \leq v(s)v(t)$  for all  $s, t \in S$
- (iii)  $v(s^*) = v(s)$  for all  $s \in S$



**Definition 1.1.4.** A function  $\varphi : S \rightarrow \mathbb{C}$  is called *v-bounded* if there exists a constant  $C > 0$  such that

$$|\varphi(s)| \leq Cv(s) \text{ for all } s \in S.$$

The following lemma is Proposition 1.12 on p. 90 in [9].

**Lemma 1.1.5.** *Let  $\varphi$  be a v-bounded positive definite functional on  $S$ . Then*

$$|\varphi(s)| \leq \varphi(e)v(s)$$

for all  $s \in S$ .

Define the set of characters on  $S$  by

$$\hat{S} = \{\rho : S \rightarrow \mathbb{C} \mid \rho(e) = 1, \rho(s^*) = \overline{\rho(s)}, \rho(st) = \rho(s)\rho(t), s, t \in S\}.$$

Consider on the set of characters the topology of pointwise convergence.

We will consider the following subset of  $\hat{S}$ ,

$$\mathcal{V} = \{\rho \in \hat{S} \mid |\rho(s)| \leq v(s), s \in S\}$$

which is compact by Tychonoff's theorem.

Berg-Maserick proved the following theorem in [10], see also Theorem 4.2.5 in [9] on p. 93.

**Theorem 1.1.6.** *A function  $\varphi : S \rightarrow \mathbb{C}$  has an integral representation of the form*

$$\varphi(s) = \int_{\mathcal{V}} \rho(s) d\mu(\rho)$$

if and only if  $\varphi$  is positive definite and v-bounded.

Theorem 1.2.4, proved in the next section, is a generalization of the Berg-Maserick theorem. Theorem 1.2.4 was proved in [1], [4]. The proof presented here constructs the measure which represents the positive definite function using the technique in [19].

Let  $E$  be a locally convex Hausdorff topological vector space. Let  $A$  be a subset of  $E$ . Then a point  $a \in A$  is called an extreme point if for all  $x, y \in A$  and  $\lambda \in (0, 1)$

$$\lambda x + (1 - \lambda)y = a \text{ implies } x = y = a.$$

The set of extreme points of  $A$  will be denoted  $\text{ext}(A)$ . Let  $B$  be a subset of  $E$ . The convex hull of  $B$ ,  $\text{conv}(B)$ , is the set of all the convex linear combinations of  $B$ , i.e.

$$\text{conv}(B) = \left\{ \sum_{k=1}^n \lambda_k b_k \mid \sum_{k=1}^n \lambda_k = 1, n \in \mathbb{N}, b_k \in B, \right\}.$$

The closure of  $\text{conv}(B)$  will be denoted  $\overline{\text{conv}}(B)$ . The Krein-Milman theorem uses extreme points and convex hulls to classify compact convex sets. The following is a statement of the Krein-Milman theorem from [9] on p. 57.

**Theorem 1.1.7** (Krein-Milman). *Every compact convex set,  $X$ , in a locally convex Hausdorff topological vector space is the closed convex hull of its extreme points, i.e*

$$X = \overline{\text{conv}}(\text{ext}(X)).$$

Let  $X$  be a locally compact Hausdorff space. We denote the algebra of continuous complex-valued functions which vanish at infinity on  $X$  by  $\mathcal{C}_0(X)$ . The following is the Stone-Weirstrass theorem as stated in [12] on p. 293.

**Theorem 1.1.8** (Stone-Weirstrass theorem). *Let  $A$  a subalgebra of  $\mathcal{C}_0(X)$ . If for every point of  $X$  the subalgebra  $A$  contains a function which not vanish there,  $A$  separates points of  $X$ , and is closed under complex conjugation, then  $A$  is dense in  $\mathcal{C}_0(X)$ .*

## 1.2 Berg-Maserick type theorem

Let  $(a_{\gamma,t})_{\gamma \in \Gamma, t \in S}$  be a family of complex numbers such that for every  $\gamma$  in a set  $\Gamma$  there are finite number of  $t \in S$  such that  $a_{\gamma,t} \neq 0$ .

For a function  $\varphi : S \rightarrow \mathbb{C}$  and  $\gamma \in \Gamma$  define the function  $\varphi_\gamma : S \rightarrow \mathbb{C}$  by

$$\varphi_\gamma(s) = \sum_{t \in S} a_{\gamma,t} \varphi(ts).$$

Let  $v$  be an absolute value on  $S$ . Define the set

$$\mathcal{P} = \{\varphi : S \rightarrow \mathbb{C} \mid \varphi, (\varphi_\gamma)_{\gamma \in \Gamma} \text{ are positive definite, } \varphi \text{ } v\text{-bounded}\}.$$

**Lemma 1.2.1.** *Let  $\varphi : S \rightarrow \mathbb{C}$  be positive definite and  $v$ -bounded. For arbitrary  $\{c_1, \dots, c_n\} \subset \mathbb{C}$ ,  $\{x_1, \dots, x_n\} \subset S$  define the function*

$$\psi(s) = \sum_{i,j=1}^n c_i \bar{c}_j \varphi(x_i x_j^* s)$$

*Then  $\psi$  is positive definite and  $v$ -bounded.*

*Proof.* Let  $\{d_1, \dots, d_m\} \subset \mathbb{C}$ ,  $\{y_1, \dots, y_m\} \subset S$  then

$$\sum_{k,\ell=1}^m d_k \bar{d}_\ell \sum_{i,j=1}^n c_i \bar{c}_j \varphi(x_i x_j^* y_k y_\ell^*) = \sum_{k,\ell=1}^m \sum_{i,j=1}^n (c_i d_k) (\bar{c}_j \bar{d}_\ell) \varphi((x_i y_k)(x_j y_\ell)^*) \geq 0$$

Thus  $\psi$  is positive definite.

Note that

$$\begin{aligned}
|\psi(s)| &= \left| \sum_{i,j=1}^n c_i \bar{c}_j \varphi(x_i x_j^* s) \right| \\
&\leq \sum_{i,j=1}^n |c_i \bar{c}_j| |\varphi(x_i x_j^* s)| \\
&\leq \sum_{i,j=1}^n |c_i \bar{c}_j| v(x_i x_j^* s) \\
&\leq v(s) \sum_{i,j=1}^n |c_i \bar{c}_j| v(x_i x_j^*),
\end{aligned}$$

thus  $\psi$  is  $v$ -bounded. □

For a function  $\varphi : S \rightarrow \mathbb{C}$ , complex number  $\tau$ , and  $a \in S$  define the function  $\varphi_{\tau,a} : S \rightarrow \mathbb{C}$  by

$$\varphi_{\tau,a}(s) = \varphi(aa^*s) + \tau\varphi(as) + \bar{\tau}\varphi(a^*s) + \varphi(s).$$

**Lemma 1.2.2.** *Let  $\varphi : S \rightarrow \mathbb{C}$  be positive definite and  $v$ -bounded. For  $\tau \in \{1, -1, i, -i\}$ ,  $\varphi_{\tau,a}$  is positive definite and  $v$ -bounded.*

*Proof.* We will use Lemma 1.2.1. Fix  $n = 2$ ,  $c_1 = \tau$ ,  $c_2 = 1$ ,  $x_1 = a$ ,  $x_2 = e$  in  $\psi$ . Then we have

$$\psi(s) = \varphi(aa^*s) + \tau\varphi(as) + \bar{\tau}\varphi(a^*s) + \varphi(s) = \varphi_{\tau,a}(s)$$

Thus  $\varphi_{\tau,a}$  positive definite and  $v$ -bounded. □

**Lemma 1.2.3.** *Let  $\varphi \in \mathcal{P}$ . If  $\tau \in \{1, -1, i, -i\}$  then*

$$\varphi_{\tau,a} \text{ and } (v(a) + 1)^2\varphi - \varphi_{\tau,a} \in \mathcal{P}.$$

*Proof.* Let  $\varphi \in \mathcal{P}$ . Suppose that for  $\tau \in \{1, -1, i, -i\}$ . Note that

$$\begin{aligned}
(\varphi_{\tau,a})_{\gamma}(s) &= \sum_{t \in S} a_{\gamma,t} \varphi_{\tau,a}(ts) \\
&= \sum_{t \in S} a_{\gamma,t} (\varphi(aa^*st) + \tau\varphi(ast) + \bar{\tau}\varphi(a^*st) + \varphi(st)) \\
&= \sum_{t \in S} a_{\gamma,t} \varphi(aa^*st) + \sum_{t \in S} a_{\gamma,t} \tau \varphi(ast) + \sum_{t \in S} a_{\gamma,t} \bar{\tau} \varphi(a^*st) + \sum_{t \in S} a_{\gamma,t} \varphi(st) \\
&= \varphi_{\gamma}(aa^*s) + \tau \varphi_{\gamma}(as) + \bar{\tau} \varphi_{\gamma}(a^*s) + \varphi_{\gamma}(s) \\
&= (\varphi_{\gamma})_{\tau,a}(s).
\end{aligned}$$

Therefore by Lemma 1.2.2  $(\varphi_{\tau,a})_{\gamma}$  are positive definite for all  $\gamma \in \Gamma$ ,  $\varphi_{\tau,a}$  is  $v$ -bounded. Thus

$\varphi_{\tau,a} \in \mathcal{P}$ .

Next we consider the function  $(v(a)+1)^2\varphi - \varphi_{\tau,a}$  for  $\tau \in \{1, -1, i, -i\}$ . Let  $\{c_1, \dots, c_n\} \subset \mathbb{C}$ ,  $\{x_1, \dots, x_n\} \subset S$  be arbitrary. By Lemma 1.2.1

$$\psi(s) = \sum_{i,j=1}^n c_i \bar{c}_j \varphi(x_i x_j^* s)$$

is positive definite and  $v$ -bounded. Therefore by Lemma 1.1.5

$$|\psi(s)| \leq \psi(e)v(s).$$

We want to show that

$$\sum_{i,j=1}^n c_i \bar{c}_j \varphi_{\tau,a}(x_i x_j^*) \leq \sum_{i,j=1}^n c_i \bar{c}_j (v(a) + 1)^2 \varphi(x_i x_j^*)$$

Since  $\varphi_{\tau,a}$  is positive definite, we have

$$\sum_{i,j=1}^n c_i \bar{c}_j \varphi_{\tau,a}(x_i x_j^*)$$

$$\begin{aligned}
&= \left| \sum_{i,j=1}^n c_i \bar{c}_j \varphi_{\tau,a}(x_i x_j^*) \right| \\
&= \left| \sum_{i,j=1}^n c_i \bar{c}_j \varphi(aa^* x_i x_j^*) + \tau \sum_{i,j=1}^n c_i \bar{c}_j \varphi(ax_i x_j^*) + \bar{\tau} \sum_{i,j=1}^n c_i \bar{c}_j \varphi(a^* x_i x_j^*) + \sum_{i,j=1}^n c_i \bar{c}_j \varphi_{\tau,a}(x_i x_j^*) \right| \\
&= |\psi(aa^*) + \tau\psi(a) + \bar{\tau}\psi(a^*) + \psi(e)| \\
&\leq |\psi(aa^*)| + |\psi(a)| + |\psi(a^*)| + |\psi(e)| \\
&\leq \psi(e)v(aa^*) + \psi(e)v(a) + \psi(e)v(a^*) + \psi(e)v(e) \\
&= \sum_{i,j=1}^n c_i \bar{c}_j (v(a))^2 \varphi(x_i x_j^*) + \sum_{i,j=1}^n c_i \bar{c}_j 2v(a) \varphi(x_i x_j^*) + \sum_{i,j=1}^n c_i \bar{c}_j \varphi(x_i x_j^*) \\
&= \sum_{i,j=1}^n c_i \bar{c}_j (v(a) + 1)^2 \varphi(x_i x_j^*).
\end{aligned}$$

Therefore  $(v(a) + 1)^2 \varphi - \varphi_{\tau,a}$  is positive definite for  $\tau \in \{1, -1, i, -i\}$ . Similarly  $((v(a) + 1)^2 \varphi - \varphi_{\tau,a})_\gamma$  are positive definite for all  $\gamma \in \Gamma$ . Thus  $(v(a) + 1)^2 \varphi - \varphi_{\tau,a} \in \mathcal{P}$ .  $\square$

Let

$$\mathcal{M} = \{\rho \in \mathcal{V} \mid \rho_\gamma(e) \geq 0, \gamma \in \Gamma\}.$$

**Theorem 1.2.4.** *Let  $(S, \cdot)$  be a commutative semigroup with involution  $*$  and neutral element  $e$ . If  $\varphi \in \mathcal{P}$ , then there exists a unique positive Radon measure  $\mu$  on  $\mathcal{M}$  such that*

$$\varphi(s) = \int_{\mathcal{M}} \rho(s) d\mu(\rho),$$

for all  $s \in S$ .

*Proof.* We will find the integral representation for elements of  $\mathcal{P}$  in the set

$$A = \{\varphi \in \mathcal{P} \mid \varphi(e) = 1\}.$$

If  $\varphi(e) \neq 1$  then find the measure  $\mu$  that corresponds to  $\frac{1}{\varphi(e)}\varphi$ . Then  $\varphi$  corresponds to  $\varphi(e)\mu$ .

First we will show that the extreme points of  $A$  are in  $\mathcal{M}$ . Let  $\rho$  be an extreme point of  $A$ . Suppose that  $\rho_{\tau,a}(e) \neq 0$  and  $((v(a) + 1)^2\rho - \rho_{\tau,a})(e) \neq 0$ . Then by Lemma 1.2.3

$$\frac{\rho_{\tau,a}}{\rho_{\tau,a}(e)}, \frac{(v(a) + 1)^2\rho - \rho_{\tau,a}}{((v(a) + 1)^2\rho - \rho_{\tau,a})(e)} \in A.$$

Note that

$$\rho = \frac{\rho_{\tau,a}(e)}{(v(a) + 1)^2} \frac{\rho_{\tau,a}}{\rho_{\tau,a}(e)} + \frac{((v(a) + 1)^2\rho - \rho_{\tau,a})(e)}{(v(a) + 1)^2} \frac{(v(a) + 1)^2\rho - \rho_{\tau,a}}{((v(a) + 1)^2\rho - \rho_{\tau,a})(e)}.$$

Since  $\rho$  is an extreme point  $\rho = \frac{\rho_{\tau,a}}{\rho_{\tau,a}(e)}$ , that is

$$\rho_{\tau,a}(e)\rho = \rho_{\tau,a}.$$

Since  $\rho_{\tau,a}$  is  $v$ -bounded,

$$|\rho_{\tau,a}(s)| \leq \rho_{\tau,a}(e)v(s).$$

If  $\rho_{\tau,a}(e) = 0$ , then  $\rho_{\tau,a} = 0$ . So,

$$\rho_{\tau,a}(e)\rho = \rho_{\tau,a}.$$

Since  $(v(a) + 1)^2\rho - \rho_{\tau,a}$  is  $v$ -bounded,

$$((v(a) + 1)^2\rho - \rho_{\tau,a})(s) \leq ((v(a) + 1)^2\rho - \rho_{\tau,a})(e)v(s).$$

If  $((v(a) + 1)^2\rho - \rho_{\tau,a})(e) = 0$ , then  $((v(a) + 1)^2\rho - \rho_{\tau,a})(s) = 0$  for all  $s \in S$ . Therefore,

$$\rho_{\tau,a}(e)\rho = (v(a) + 1)^2\rho(e)\rho = (v(a) + 1)^2\rho = \rho_{\tau,a}.$$

For all  $a, s \in S$ ,

$$4\rho(as) = \sum_{\tau \in \{1, -1, i, -i\}} \bar{\tau} \rho_{\tau, a}(s).$$

Hence,

$$\begin{aligned} \rho(as) &= \frac{1}{4} \sum_{\tau \in \{1, -1, i, -i\}} \bar{\tau} \rho_{\tau, a}(s) \\ &= \frac{1}{4} \sum_{\tau \in \{1, -1, i, -i\}} \bar{\tau} \rho_{\tau, a}(e) \rho(s) \\ &= \rho(s) \frac{1}{4} \sum_{\tau \in \{1, -1, i, -i\}} \bar{\tau} \rho_{\tau, a}(e) \\ &= \rho(s) \rho(a). \end{aligned}$$

Therefore  $\rho$  is a character on  $S$  and  $\rho \in \mathcal{M}$ , that is  $\text{ext}(A) \subset \mathcal{M}$ .

The other inclusion  $\mathcal{M} \subset \text{ext}(A)$  can be shown using Corollary 2.5.12 in [9] on p. 60. Thus  $\text{ext}(A) = \mathcal{M}$ .

By the Krein-Milman theorem  $A = \overline{\text{conv}}(\text{ext}(A))$ . We will construct a Radon measure on  $\text{ext}(A)$ . Let  $\varphi \in A = \overline{\text{conv}}(\text{ext}(A))$ . Then there exists a net  $(\psi_\alpha)$  in  $\text{conv}(\text{ext}(A))$  such that

$$\varphi(s) = \lim_{\alpha} \psi_\alpha(s) \text{ for all } s \in S.$$

Since

$$\psi_\alpha = \sum_{k=1}^{n_\alpha} d_k^\alpha \rho_k^\alpha,$$

for some  $d_k^\alpha \geq 0$ ,  $\sum_{k=1}^{n_\alpha} d_k^\alpha = 1$ ,  $\rho_k^\alpha \in \text{ext}(A)$  and  $n_\alpha \in \mathbb{N}$ ,

$$\varphi(s) = \lim_{\alpha} \sum_{k=1}^{n_\alpha} d_k^\alpha \rho_k^\alpha(s)$$



Then we have for  $\{c_1, \dots, c_n\} \subset \mathbb{C}$  and  $\{s_1, \dots, s_n\} \subset S$

$$\begin{aligned}
\left| \sum_{j=1}^n c_j \varphi(s_j) \right| &= \left| \sum_{j=1}^n c_j \lim_{\alpha} \sum_{k=1}^{n_{\alpha}} d_k^{\alpha} \rho_k^{\alpha}(s_j) \right| \\
&= \left| \lim_{\alpha} \sum_{k=1}^{n_{\alpha}} d_k^{\alpha} \sum_{j=1}^n c_j \rho_k^{\alpha}(s_j) \right| \\
&\leq \lim_{\alpha} \sum_{k=1}^{n_{\alpha}} d_k^{\alpha} \sup_{\rho \in \text{ext}(A)} \left| \sum_{j=1}^n c_j \rho(s_j) \right| \\
&= \sup_{\rho \in \text{ext}(A)} \left| \sum_{j=1}^n c_j \rho(s_j) \right|.
\end{aligned}$$

Define  $\hat{s} : \text{ext}(A) \rightarrow \mathbb{C}$  by  $\hat{s}(\rho) = \rho(s)$ . Note for  $s_1, s_2 \in S$ ,  $\widehat{s_1 s_2}(\rho) = \rho(s_1 s_2) = \rho(s_1) \rho(s_2) = \hat{s}_1 \hat{s}_2(\rho)$ . Now we will construct a function on the set of functions,

$$B = \left\{ \sum_{j=1}^n c_j \hat{s}_j \mid c_j \in \mathbb{C}, s_j \in S, n \in \mathbb{N} \right\}$$

The set  $B$  contains the constant functions and separates points thus by the Stone-Weierstrass theorem  $B$  is dense in continuous functions from  $\text{ext}(A)$  to  $\mathbb{C}$ ,  $\mathcal{C}(\text{ext}(A))$ . Define

$$F \left( \sum_{j=1}^n c_j \hat{s}_j \right) = \sum_{j=1}^n c_j \varphi(s_j).$$

Note if  $\sum_{j=1}^n c_j \hat{s}_j = \sum_{k=1}^m d_k \hat{s}_k$  then from the above inequality

$$\left| \sum_{j=1}^n c_j \varphi(s_j) - \sum_{k=1}^m d_k \varphi(s_k) \right| \leq \sup_{\rho \in \text{ext}(A)} \left| \sum_{j=1}^n c_j \rho(s_j) - \sum_{k=1}^m d_k \rho(s_k) \right| = 0.$$

Thus  $F$  is well defined. We extend  $F$  to  $\mathcal{C}(\text{ext}(A))$ . Let  $f \in \mathcal{C}(\text{ext}(A))$ , then  $f = \lim_{\beta} \sum_{j=1}^{n_{\beta}} c_j^{\beta} \hat{s}_j^{\beta}$ .

Then we define

$$F(f) = \lim_{\beta} F \left( \sum_{j=1}^{n_{\beta}} c_j^{\beta} \hat{s}_j^{\beta} \right)$$

$$= \lim_{\beta} \sum_{j=1}^{n_{\beta}} c_j^{\beta} \varphi \left( s_j^{\beta} \right)$$

Thus  $F$  is a Radon measure on  $\text{ext}(A)$ .

Next we will show that  $F$  is positive. Let  $f \in \mathcal{C}(\text{ext}(A))$  such that  $f \geq 0$ . Then  $\sqrt{f} \in \mathcal{C}(\text{ext}(A))$ . So there exists  $\sum_{k=1}^{n_{\alpha}} c_k^{\alpha} \hat{s}_k^{\alpha}$  in  $A$  such that  $\lim_{\alpha} \sum_{k=1}^{n_{\alpha}} c_k^{\alpha} \hat{s}_k^{\alpha} \rightarrow \sqrt{f}$ .

Hence,  $\lim_{\alpha} \sum_{k=1}^{n_{\alpha}} \overline{c_k^{\alpha} \hat{s}_k^{\alpha}} \rightarrow \sqrt{f}$ . Therefore  $\lim_{\alpha} \sum_{k,j=1}^{n_{\alpha}} c_k^{\alpha} \overline{c_j^{\alpha}} \hat{s}_k^{\alpha} \overline{\hat{s}_j^{\alpha}} \rightarrow f$ . Since  $\varphi$  is positive definite

$$\begin{aligned} F(f) &= F \left( \lim_{\alpha} \sum_{k,j=1}^{n_{\alpha}} c_k^{\alpha} \overline{c_j^{\alpha}} \hat{s}_k^{\alpha} \overline{\hat{s}_j^{\alpha}} \right) \\ &= F \left( \lim_{\alpha} \sum_{k,j=1}^{n_{\alpha}} c_k^{\alpha} \overline{c_j^{\alpha}} \hat{s}_k^{\alpha} (\widehat{s_j^{\alpha}})^* \right) \\ &= \lim_{\alpha} F \left( \sum_{k,j=1}^{n_{\alpha}} c_k^{\alpha} \overline{c_j^{\alpha}} s_k^{\alpha} (\widehat{s_j^{\alpha}})^* \right) \\ &= \lim_{\alpha} \sum_{k,j=1}^{n_{\alpha}} c_k^{\alpha} \overline{c_j^{\alpha}} \varphi(s_k^{\alpha} (s_j^{\alpha})^*) \\ &\geq 0. \end{aligned}$$

Thus  $F$  is positive.

We can write  $F$  as an integral with a unique positive Radon measure  $\mu$  on  $\text{ext}(A)$ , i.e.

$$F(f) = \int_{\text{ext}(A)} f(\rho) d\mu(\rho).$$

Thus

$$\varphi(s) = F(\hat{s}) = \int_{\text{ext}(A)} \rho(s) d\mu(\rho) = \int_{\mathcal{M}} \rho(s) d\mu(\rho).$$

This finishes the proof. □

The set  $\Gamma$  lets us control the set  $\mathcal{M}$ . For example we suppose  $S$  is an commutative algebra with involution. Let  $\varphi$  be a positive definite linear functional on  $S$  that is  $v$ -bounded.

We will show that

$$\varphi(s) = \int_{\mathcal{M}} \rho(s) d\mu(\rho),$$

where  $\mathcal{M}$  are linear characters on  $S$ .

Let  $a_1, a_2 \in \mathbb{C}$  and  $t_1, t_2 \in S$  be arbitrary. Define  $t_3 = a_1 t_1 + a_2 t_2 \in S$ . We consider the following for  $\Gamma = \{1, 2\}$ ,

$$a_{1,t_1} = -a_1, \quad a_{1,t_2} = -a_2, \quad a_{1,t_3} = 1,$$

$$a_{2,t_1} = a_1, \quad a_{2,t_2} = a_2, \quad a_{2,t_3} = -1.$$

Then

$$\begin{aligned} \varphi_1(s) &= a_{1,t_1} \varphi(t_1 s) + a_{1,t_2} \varphi(t_2 s) + a_{1,t_3} \varphi(t_3 s) \\ &= -a_1 \varphi(t_1 s) - a_2 \varphi(t_2 s) + \varphi((a_1 t_1 + a_2 t_2) s) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \varphi_2(s) &= a_{2,t_1} \varphi(t_1 s) + a_{2,t_2} \varphi(t_2 s) + a_{2,t_3} \varphi(t_3 s) \\ &= a_1 \varphi(t_1 s) + a_2 \varphi(t_2 s) - \varphi((a_1 t_1 + a_2 t_2) s) \\ &= 0. \end{aligned}$$

Thus both  $\varphi_1$  and  $\varphi_2$  are positive definite.

By Theorem 1.2.4

$$\varphi(s) = \int_{\mathcal{M}} \rho(s) d\mu(\rho),$$

where  $\mathcal{M} = \{\rho \in \mathcal{V} \mid \rho_\gamma(e) \geq 0, \gamma \in \Gamma\}$ . That is

$$a_1\rho(t_1) + a_2\rho(t_2) \leq \rho(a_1t_1 + a_2t_2)$$

and

$$a_1\rho(t_1) + a_2\rho(t_2) \geq \rho(a_1t_1 + a_2t_2).$$

So

$$a_1\rho(t_1) + a_2\rho(t_2) = \rho(a_1t_1 + a_2t_2).$$

Since  $a_1, a_2$  and  $t_1, t_2$  where arbitrary,  $\mu$  is concentrated on the set of linear characters on  $S$ .

Therefore this theory on semigroups can be applied to algebras.

## CHAPTER 2

### SEMIGROUP REPRESENTATIONS

#### 2.1 Introduction

In this chapter  $H$  is a Hilbert space and  $\mathcal{B}(H)$  is the space of bounded operators on  $H$ . Let  $(S, \cdot)$  be a commutative semigroup with an identity  $e$  and an involution  $*$ . We will study functions of the type  $\mathcal{U} : S \rightarrow \mathcal{B}(H)$  such that

$$\mathcal{U}(e) = 1, \mathcal{U}(s^*) = \mathcal{U}(s)^*, \mathcal{U}(st) = \mathcal{U}(s)\mathcal{U}(t),$$

for all  $s, t \in S$ . Such a function is called  $*$ -representation.

This kind of representation was studied in [27] using spectral measures without using the theory of  $C^*$ -algebras or more general Banach algebra arguments. We will use  $L^2$  theory to study this representation instead of spectral measures. We will use the result from Chapter 1 and therefore, as in [27], we will not use Gelfand theory or Banach algebras.

## 2.2 Semigroup Representations

We begin by showing that for a fixed  $a \in H$ , such that  $a \neq 0$  the function  $\varphi_a(s) = \langle \mathcal{U}(s)a, a \rangle$  is positive definite.

**Lemma 2.2.1.** *The function  $\varphi_a : S \rightarrow \mathbb{C}$  defined by*

$$\varphi_a(s) = \langle \mathcal{U}(s)a, a \rangle$$

*is positive definite.*

*Proof.* Let  $\{c_1, \dots, c_n\} \subset \mathbb{C}$ ,  $\{x_1, \dots, x_n\} \subset S$  be arbitrary. Then

$$\begin{aligned} \sum_{i,j=1}^n c_i \bar{c}_j \varphi_a(x_i x_j^*) &= \sum_{i,j=1}^n c_i \bar{c}_j \langle \mathcal{U}(x_i x_j^*)a, a \rangle \\ &= \sum_{i,j=1}^n c_i \bar{c}_j \langle \mathcal{U}(x_i) \mathcal{U}(x_j)^* a, a \rangle \\ &= \sum_{i,j=1}^n c_i \bar{c}_j \langle (\mathcal{U}(x_j))^* a, (\mathcal{U}(x_i))^* a \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \langle (\mathcal{U}(x_j))^* a, (\mathcal{U}(x_j))^* a \rangle \\ &= \left\langle \sum_{i=1}^n c_i (\mathcal{U}(x_i))^* a, \sum_{j=1}^n c_j (\mathcal{U}(x_j))^* a \right\rangle \\ &= \left\| \sum_{i=1}^n c_i (\mathcal{U}(x_i))^* a \right\|^2 \\ &\geq 0. \end{aligned}$$

So,  $\varphi_a$  is positive definite. □

**Lemma 2.2.2.** *The function  $\alpha(s) = \|\mathcal{U}(s)\|$  is an absolute value on  $S$ .*

*Proof.* Note that the following,

$$\alpha(e) = \|\mathcal{U}(e)\| = \|1_H\| = 1,$$

$$\alpha(st) = \|\mathcal{U}(st)\| = \|\mathcal{U}(s)\mathcal{U}(t)\| \leq \|\mathcal{U}(s)\| \|\mathcal{U}(t)\|,$$

$$\alpha(s^*) = \|\mathcal{U}(s^*)\| = \|(\mathcal{U}(s))^*\| = \|\mathcal{U}(s)\| = \alpha(s) \text{ for all } s, t \in S. \quad \square$$

**Lemma 2.2.3.** *The function  $\varphi_a$  is  $\alpha$ -bounded.*

*Proof.* Note that

$$|\varphi_a(s)| = |\langle \mathcal{U}(s)a, a \rangle| \leq \|\mathcal{U}(s)a\| \|a\| \leq \|\mathcal{U}(s)\| \|a\|^2.$$

Therefore  $\varphi_a$  is  $\alpha$ -bounded.  $\square$

Let  $(a_{\gamma,t})_{\gamma \in \Gamma, t \in S}$  be a family of complex numbers such that for every  $\gamma$  in a set  $\Gamma$  there are finite number of  $t \in S$  such that  $a_{\gamma,t} \neq 0$ . We will use Theorem 1.2.4. We will rewrite the sets  $\mathcal{V}$  and  $\mathcal{M}$  from Chapter 1 using the setting of this section.

$$\mathcal{V} = \{\rho \in \hat{S} \mid |\rho(s)| \leq \alpha(s), s \in S\} = \{\rho \in \hat{S} \mid |\rho(s)| \leq \|\mathcal{U}(s)\|, s \in S\}$$

and

$$\mathcal{M} = \{\rho \in \mathcal{V} \mid \rho_\gamma(e) \geq 0, \gamma \in \Gamma\}.$$

**Lemma 2.2.4.** *We suppose  $(\varphi_a)_\gamma$  are positive definite for all  $\gamma \in \Gamma$ . There exists a positive Radon measure on  $\mathcal{M}$  such that*

$$\varphi_a(s) = \int_{\mathcal{M}} \rho(s) d\mu_a(\rho),$$

for all  $s \in S$ .

*Proof.* By Lemma 2.2.4  $\varphi_a$  is positive definite and by Lemma 2.2.3  $\varphi_a$  is  $\alpha$ -bounded. Also, by assumption  $(\varphi_a)_\gamma$  are positive definite for all  $\gamma \in \Gamma$ . Therefore by Theorem 1.2.4 there exists a unique positive Radon measure,  $\mu_a$ , on  $\mathcal{M}$  such that for all  $s \in S$ ,

$$\varphi_a(s) = \int_{\mathcal{M}} \rho(s) d\mu_a(\rho).$$

□

Define

$$K_a = \text{supp } \mu_a.$$

**Lemma 2.2.5.** *Let  $\{c_1, \dots, c_n\}, \{d_1, \dots, d_m\} \subset \mathbb{C}$ ,  $\{s_1, \dots, s_n\}, \{t_1, \dots, t_m\} \subset S$  be arbitrary. Then*

$$\left\langle \sum_{j=1}^n c_j \mathcal{U}(s_j)a, \sum_{k=1}^m d_k \mathcal{U}(t_k)a \right\rangle = \int_{K_a} \sum_{j=1}^n c_j \rho(s_j) \sum_{k=1}^m \overline{d_k \rho(t_k)} d\mu_a(\rho).$$

*Proof.* Let  $\{c_1, \dots, c_n\}, \{d_1, \dots, d_m\} \subset \mathbb{C}$ ,  $\{s_1, \dots, s_n\}, \{t_1, \dots, t_m\} \subset S$  be arbitrary. Then

$$\begin{aligned} \left\langle \sum_{j=1}^n c_j \mathcal{U}(s_j)a, \sum_{k=1}^m d_k \mathcal{U}(t_k)a \right\rangle &= \sum_{j=1}^n c_j \sum_{k=1}^m \overline{d_k} \langle \mathcal{U}(s_j)a, \mathcal{U}(t_k)a \rangle \\ &= \sum_{j=1}^n c_j \sum_{k=1}^m \overline{d_k} \langle (\mathcal{U}(t_k))^* \mathcal{U}(s_j)a, a \rangle \\ &= \sum_{j=1}^n c_j \sum_{k=1}^m \overline{d_k} \langle \mathcal{U}(t_k^* s_j)a, a \rangle \\ &= \sum_{j=1}^n c_j \sum_{k=1}^m \overline{d_k} \int_{K_a} \rho(s_j t_k^*) d\mu_a(\rho) \\ &= \sum_{j=1}^n c_j \sum_{k=1}^m \overline{d_k} \int_{K_a} \rho(s_j) \overline{\rho(t_k)} d\mu_a(\rho) \end{aligned}$$



$$= \int_{K_a} \sum_{j=1}^n c_j \rho(s_j) \sum_{k=1}^m \overline{d_k \rho(t_k)} d\mu_a(\rho).$$

□

Define  $\hat{s} : K_a \rightarrow \mathbb{C}$  by

$$\hat{s}(\rho) = \rho(s).$$

**Lemma 2.2.6.** *The set  $\left\{ \sum_{j=1}^n c_j \hat{s}_j \mid c_j \in \mathbb{C}, s_j \in S, n \in \mathbb{N} \right\}$  is dense in  $L^2(K_a, \mu_a)$ .*

*Proof.* By the Stone-Weierstrass the set of functions  $\sum_{j=1}^n c_j \hat{s}_j$  is dense in the set on all continuous on  $K_a$  with uniform norm. Since the measure in the compact set  $K_a$  is finite the set of functions  $\sum_{j=1}^n c_j \hat{s}_j$  are dense in the set on all continuous on  $K_a$  with the  $L^2$  norm. The continuous functions are dense in  $L^2$  with  $L^2$  norm. Thus  $\left\{ \sum_{j=1}^n c_j \hat{s}_j \mid c_j \in \mathbb{C}, s_j \in S, n \in \mathbb{N} \right\}$  is dense in  $L^2(K_a, \mu_a)$ . □

Define the function

$$\xi_a : \left\{ \sum_{j=1}^n c_j \mathcal{U}(s_j) a \mid c_j \in \mathbb{C}, s_j \in S, n \in \mathbb{N} \right\} \rightarrow \left\{ \sum_{j=1}^n c_j \hat{s}_j \mid c_j \in \mathbb{C}, s_j \in S, n \in \mathbb{N} \right\}$$

by

$$\xi_a \left( \sum_{j=1}^n c_j \mathcal{U}(s_j) a \right) = \sum_{j=1}^n c_j \hat{s}_j.$$

Now we will show that  $\xi_a$  is well defined. Suppose that  $\sum_{j=1}^n c_j \mathcal{U}(s_j) a = \sum_{k=1}^m d_k \mathcal{U}(t_k) a$ .

Then

$$\left\| \sum_{j=1}^n c_j \hat{s}_j - \sum_{k=1}^m d_k \hat{t}_k \right\|_{L^2(K_a, \mu_a)}^2 = \int_{K_a} \left| \sum_{j=1}^n c_j \rho(s_j) - \sum_{k=1}^m d_k \rho(t_k) \right|^2 d\mu_a(\rho)$$

$$\begin{aligned}
&= \left\langle \sum_{j=1}^n c_j \mathcal{U}(s_j) a - \sum_{k=1}^m d_k \mathcal{U}(t_k) a, \sum_{j=1}^n c_j \mathcal{U}(s_j) a - \sum_{k=1}^m d_k \mathcal{U}(t_k) a \right\rangle \\
&= \left\| \sum_{j=1}^n c_j \mathcal{U}(s_j) a - \sum_{k=1}^m d_k \mathcal{U}(t_k) a \right\|_H^2 \\
&= 0.
\end{aligned}$$

Since  $K_a$  is the support of  $\mu_a$ ,

$$\sum_{j=1}^n c_j \hat{s}_j = \sum_{k=1}^m d_k \hat{t}_k$$

and  $\xi_a$  is well defined. From the definition we note that  $\xi_a$  is linear. Next we will show that  $\xi_a$  is isometric and thus continuous. By Lemma 2.2.5

$$\left\langle \sum_{j=1}^n c_j \mathcal{U}(s_j) a, \sum_{j=1}^n c_j \mathcal{U}(s_k) a \right\rangle = \int_{K_a} \left| \sum_{j=1}^n c_j \rho(s_j) \right|^2 d\mu_a(\rho).$$

Thus,

$$\begin{aligned}
\left\| \xi_a \left( \sum_{j=1}^n c_j \mathcal{U}(s_j) a \right) \right\|_{L^2(K_a, \mu_a)}^2 &= \left\| \sum_{j=1}^n c_j \hat{s}_j \right\|_{L^2(K_a, \mu_a)}^2 \\
&= \int_K \left| \sum_{j=1}^n c_j \rho(s_j) \right|^2 d\mu_a(\rho) \\
&= \left\langle \sum_{j=1}^n c_j \mathcal{U}(s_j) a, \sum_{j=1}^n c_j \mathcal{U}(s_k) a \right\rangle \\
&= \left\| \sum_{j=1}^n c_j \mathcal{U}(s_k) a \right\|_H^2.
\end{aligned}$$

Since  $\xi_a$  is continuous, we can extend  $\xi_a$  to be a map from  $H_a$  to  $L^2(K, \mu_a)$ , where

$$H_a = \overline{\left\{ \sum_{j=1}^n c_j \mathcal{U}(s_j) a \mid c_j \in \mathbb{C}, s_j \in S, n \in \mathbb{N} \right\}} \subset H.$$

**Theorem 2.2.7.** *The spaces  $H_a$  and  $L^2(K_a, \mu_a)$  are isomorphic Hilbert spaces.*

*Proof.* We will show that  $\xi_a$  is an isomorphism from  $H_a$  to  $L^2(K_a, \mu_a)$

First we show that  $\xi_a$  is surjective. Let  $f \in L^2(K_a, \mu_a)$ . Then there exists a sequence

$\left(\sum_{j=1}^{n_k} c_{j,k} \hat{s}_{j,k}\right)_{k=1}^{\infty}$  such that

$$f = \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \hat{s}_{j,k}.$$

Therefore,

$$\begin{aligned} \xi_a \left( \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \mathcal{U}(s_{j,k})a \right) &= \lim_{k \rightarrow \infty} \xi_a \left( \sum_{j=1}^{n_k} c_{j,k} \mathcal{U}(s_{j,k})a \right) \\ &= \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \hat{s}_{j,k} \\ &= f. \end{aligned}$$

Therefore  $\xi_a$  is surjective.

Since  $\xi_a$  is linear, surjective, and

$$\left\langle \sum_{j=1}^n c_j \mathcal{U}(s_j)a, \sum_{k=1}^m d_k \mathcal{U}(t_k)a \right\rangle = \int_K \sum_{j=1}^n c_j \rho(s_j) \sum_{k=1}^m \overline{d_k \rho(t_k)} d\mu_a(\rho),$$

$\xi_a$  is an isomorphism from  $H_a$  to  $L^2(K_a, \mu_a)$ . □

Let  $M_g$  where  $g \in L^\infty(K_a)$  be the multiplication operator on  $L^2(K_a, \mu_a)$ , that is for  $f \in L^2(K_a, \mu_a)$

$$M_g f = gf.$$

**Lemma 2.2.8.**

1.  $H_a$  is  $\mathcal{U}(s)$  invariant for all  $s \in S$ .

2.  $\mathcal{U}(s)$  corresponds to  $M_{\hat{s}}$  on  $L^2(K_a, \mu_a)$ .

*Proof.* Proof of 1.

Let  $h \in H_a$ . Then

$$h = \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \mathcal{U}(s_{j,k}) a,$$

for  $n_k \in \mathbb{N}$ ,  $c_{j,k} \in \mathbb{C}$  and  $s_{j,k} \in S$ . Then for  $s \in S$ ,

$$\begin{aligned} \mathcal{U}(s)h &= \mathcal{U}(s) \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \mathcal{U}(s_{j,k}) a \\ &= \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \mathcal{U}(s) \mathcal{U}(s_{j,k}) a \\ &= \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \mathcal{U}(ss_{j,k}) a. \end{aligned}$$

Since  $\lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \mathcal{U}(ss_{j,k}) a \in H_a$ ,  $H_a$  is  $\mathcal{U}(s)$  invariant for all  $s \in S$ .

Proof of 2.

$$\begin{aligned} \xi_a(\mathcal{U}(s)h) &= \xi_a \left( \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \mathcal{U}(ss_{j,k}) a \right) \\ &= \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \widehat{SS}_{j,k} \\ &= \hat{s} \lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} c_{j,k} \widehat{S}_{j,k} \\ &= \hat{s} \xi_a(h) \\ &= M_{\hat{s}} \xi_a(h). \end{aligned}$$

□

**Theorem 2.2.9.** *If  $H$  is separable, then there exist  $a_1, a_2, \dots \in H$  such that*

$$H = \bigoplus_{n=1}^{\infty} H_{a_n} \simeq \bigoplus_{n=1}^{\infty} L^2(K_{a_n}, \mu_{a_n})$$

*Proof.* Fix  $a_1 \in H$ . Then we construct  $H_{a_1}$ . If  $H_{a_1} = H$ , then  $H = H_{a_1} \simeq L^2(K_{a_1}, \mu_{a_1})$ . Suppose that  $H_{a_1} \neq H$ . Let  $a_2 \in H_{a_1}^\perp$ , where  $H_{a_1}^\perp$  is the orthogonal complement of  $H_{a_1}$ . Then consider  $H_{a_1} \oplus H_{a_2}$ . If  $H_{a_1} \oplus H_{a_2} \neq H$ , then let  $a_3 \in (H_{a_1} \oplus H_{a_2})^\perp$ . Since  $H$  is separable, we continue to find a countable number of  $a_n \in H$  such that  $H = \bigoplus_{n=1}^{\infty} H_{a_n}$ . From Theorem 2.2.7 we have

$$H = \bigoplus_{n=1}^{\infty} H_{a_n} \simeq \bigoplus_{n=1}^{\infty} L^2(K_{a_n}, \mu_{a_n}).$$

□

## 2.3 Examples

**Example 2.3.1.** Let  $H$  be a separable Hilbert space and  $A$  a positive operator on  $H$ . This means that we have  $\langle Ah, h \rangle \geq 0$ .

Let  $S$  be the semigroup  $(\mathbb{N}_0, +)$  with involution  $n^* = n$ . Consider  $\mathcal{U} : S \rightarrow \mathcal{B}(H)$  defined by

$$\mathcal{U}(n) = A^n$$

Take  $a \in H$ . The functions  $n \mapsto \langle A^n a, a \rangle$  and  $n \mapsto \langle A^{n+1} a, a \rangle$  are positive definite. This implies that there is a positive Radon measure  $\mu_a$  on  $K = [0, \|A\|]$  such that

$$\langle A^n a, a \rangle = \int_K x^n d\mu_a(x) = \int_{K_a} x^n d\mu_a(x).$$

where  $K_a$  is the support of the measure  $\mu_a$ .

This is because a character on  $\mathbb{N}_0$  is of the form  $n \mapsto x^n$  and can be identified with the real number  $x$ .

We have

$$H_a = \overline{\left\{ \sum_{j=0}^n c_j A^j a \mid c_j \in \mathbb{R}, n \in \mathbb{N}_0 \right\}}$$

and

$$H = \bigoplus_{n=1}^{\infty} H_{a_n} \simeq \bigoplus_{n=1}^{\infty} L^2(K_{a_n}, \mu_{a_n}). \quad (2.3.1)$$

Denote by  $\xi_a$  the isomorphism

$$H_a \rightarrow L^2(K_a, \mu_a),$$

we have by Lemma 2.2.8

$$\xi_a(A^n h) = M_{\widehat{n}} \xi_a(h), \quad (2.3.2)$$

where the function  $\widehat{n} : K_a \rightarrow \mathbb{R}$  is defined by  $\widehat{n}(x) = x^n$ .

The equality (2.3.2) is equivalent to

$$A^n h = \xi_a^{-1} M_{\widehat{n}} \xi_a(h), \text{ for all } h \in H_a$$

We consider the operator  $M_{\sqrt{\widehat{1}}}$  on  $L^2(K_a, \mu_a)$ . We have

$$\xi_a^{-1} M_{\sqrt{\widehat{1}}} \xi_a \xi_a^{-1} M_{\sqrt{\widehat{1}}} \xi_a(h) = \xi_a^{-1} M_{\widehat{1}} \xi_a(h) = Ah, \text{ for all } h \in H_a$$

Now using (2.3.1) we have proved that a positive operator has a square root.

**Example 2.3.2.** Let  $H$  be a separable Hilbert space and  $T$  a normal operator on  $H$ . Let  $S$  be the semigroup  $(\mathbb{N}_0^2, +)$  with involution  $(m, n)^* = (n, m)$ . Consider  $\mathcal{U} : S \rightarrow \mathcal{B}(H)$  defined

by

$$\mathcal{U}(m, n) = T^m(T^*)^n.$$

Fix  $a \in H$ . We suppose the functions

$$(m, n) \mapsto \langle T^{m+1}(T^*)^n a + T^m(T^*)^{n+1} a, a \rangle$$

and

$$(m, n) \mapsto \frac{1}{i} \langle T^{m+1}(T^*)^n a - T^m(T^*)^{n+1} a, a \rangle$$

are positive definite. This implies that there is a positive Radon measure  $\mu_a$  on

$$K = \{z \in \mathbb{C} \mid |z| \leq \|T\|, \Re z \geq 0, \Im z \geq 0\}$$

such that

$$\langle T^m(T^*)^n a, a \rangle = \int_K z^n \bar{z}^m d\mu_a(z) = \int_{K_a} z^n \bar{z}^m d\mu_a(z).$$

where  $K_a$  is the support of the measure  $\mu_a$ .

This is because a character on  $(\mathbb{N}_0^2, +)$  is of the form  $(n, m) \mapsto z^n \bar{z}^m$  and can be identified with the complex number  $z$ .

Denote by  $\xi_a$  the isomorphism

$$H_a \rightarrow L^2(K_a, \mu_a).$$

We have by Lemma 2.2.8

$$\xi_a(T^m(T^*)^n h) = M_{\widehat{(m, n)}} \xi_a(h) \tag{2.3.3}$$

where the function  $\widehat{(m, n)} : K_a \rightarrow \mathbb{C}$  is defined by  $\widehat{(m, n)}(z) = z^n \bar{z}^m$ .

The equality (2.3.5) is equivalent to

$$T^m(T^*)^n h = \xi_a^{-1} M_{\widehat{(m, n)}} \xi_a(h), \text{ for all } h \in H_a$$

This yields

$$Th = \xi_a^{-1} M_{\widehat{(1,0)}} \xi_a(h), \text{ for all } h \in H_a$$

**Example 2.3.3.** Let  $H$  be a separable Hilbert space and  $A, B$  and  $C$  commuting selfadjoint operators on  $H$ . Let  $S$  be the semigroup  $(\mathbb{N}_0^3, +)$  with involution  $(m, n, p)^* = (n, m, p)$ . Consider  $\mathcal{U} : S \rightarrow \mathcal{B}(H)$  defined by

$$\mathcal{U}(m, n, p) = A^m B^n C^p$$

Fix  $a \in H$ . We suppose the function

$$(m, n, p) \mapsto \langle (A^{m+2} B^n C^p + A^m B^{n+2} C^p - A^m B^n C^{p+1})a, a \rangle$$

is positive definite.

This implies that there is a positive Radon measure  $\mu_a$  on

$$K = \{(x, y, z) \in \mathbb{R}^3 \mid |x| \leq \|A\|, |y| \leq \|B\|, |z| \leq \|C\|, x^2 + y^2 - z \geq 0\}$$

such that

$$\langle A^m B^n C^p a, a \rangle = \int_K x^m y^n z^p d\mu_a(x, y, z) = \int_{K_a} x^m y^n z^p d\mu_a(x, y, z).$$

where  $K_a$  is the support of the measure  $\mu_a$ .

This is because a character on  $(\mathbb{N}_0^3, +)$  is of the form  $(n, m, p) \mapsto x^n y^m z^p$  and can be identified with the element  $(x, y, z) \in \mathbb{R}^3$ .

Denote by  $\xi_a$  the isomorphism

$$H_a \rightarrow L^2(K_a, \mu_a).$$



We have by Lemma 2.2.8

$$\xi_a(A^m B^n C^p h) = M_{(\widehat{m, n, p})} \xi_a(h) \quad (2.3.4)$$

where the function  $(\widehat{m, n, p}) : K_a \rightarrow \mathbb{R}$  is defined by  $(\widehat{m, n, p})(x, y, z) = x^m y^n z^p$ .

**Example 2.3.4.** Let  $H$  be a separable Hilbert space . Let  $S$  be a commutative normed algebra with involution and unity  $e$ . Consider a linear  $*$ -representation  $\mathcal{U} : S \rightarrow \mathcal{B}(H)$ .

Fix  $a \in H$ . We suppose the functions

$$s \mapsto \langle (\|t\|^2 \mathcal{U}(s) - \mathcal{U}(tt^* s))a, a \rangle$$

are positive definite for every  $t$ .

This implies that there is a positive Radon measure  $\mu_a$  on

$$K = \{\rho : S \rightarrow \mathbb{C} \mid \rho(e) = 1, \rho \text{ linear}, \rho(s^*) = \overline{\rho(s)}, |\rho(s)| \leq \|s\|, \rho(st) = \rho(s)\rho(t), s, t \in S\}.$$

such that

$$\langle \mathcal{U}(s)a, a \rangle = \int_K \rho(s) d\mu_a(\rho) = \int_{K_a} \rho(s) d\mu_a(z),$$

where  $K_a$  is the support of the measure  $\mu_a$ .

Denote by  $\xi_a$  the isomorphism

$$H_a \rightarrow L^2(K_a, \mu_a),$$

we have by Lemma 2.2.8

$$\xi_a \mathcal{U}(s)h = M_{\widehat{s}} \xi_a(h) \quad (2.3.5)$$

where the function  $\widehat{s} : K_a \rightarrow \mathbb{C}$  is defined by  $\widehat{s}(\rho) = \rho(s)$ .

The equality (2.3.5) is equivalent to

$$\mathcal{U}(s)h = \xi_a^{-1} M_{\widehat{s}\xi_a}(h), h \in H_a.$$

# CHAPTER 3

## INTEGRATED CAUCHY FUNCTIONAL EQUATION ON COMMUTATIVE SEMIGROUPS

### 3.1 Introduction

In this chapter we are considering an expansion of a theorem by Ressel which will be introduced later. The original idea of this theorem comes from a theorem by DeFinetti about probability measures.

**Theorem 3.1.1.** *[DeFinetti 1931] Let  $X_1, X_2, \dots$  be  $\{0, 1\}$  valued random variables such that*

$$P(X_1 = x_1, \dots, X_n = x_n) = P(X_1 = x_{\sigma(1)}, \dots, X_n = x_{\sigma(n)})$$

*(exchangeable sequence) for all  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in \{0, 1\}$  and all permutations  $\sigma$  of  $\{1, \dots, n\}$ .*

*Then for some unique probability measure  $\mu$  on  $[0, 1]$  we have*

$$P(X_1 = x_1, \dots, X_n = x_n) = \int_{[0,1]} p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i} d\mu(p).$$

We now look at this from a different angle by considering the following function.

Define the function  $\varphi : \mathbb{N}_0^2 \rightarrow [0, 1]$  by

$$\varphi \left( \sum_{i=1}^n x_i, n - \sum_{i=1}^n x_i \right) = P(X_1 = x_1, \dots, X_n = x_n),$$

for all  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in \{0, 1\}$ . We will show that  $\varphi$  is well defined on  $\mathbb{N}_0^2$ . For  $(s, t) \in \mathbb{N}_0^2$ , we use  $(s, t) = (s, (t + s) - s)$ . Therefore  $n = t + s$  and  $\sum_{i=1}^n x_i = s$ . So

$$\varphi(s, t) = P(X_1 = 1, \dots, X_s = 1, X_{s+1} = 0, \dots, X_{s+t} = 0),$$

which is well defined since  $X_1, X_2, \dots$  is an exchangeable sequence. Also,

$$\varphi(s, t) \geq 0$$

because  $P$  is a probability. Note that

$$\varphi(0, 0) = 1$$

and

$$\begin{aligned} \varphi(s+1, t) + \varphi(s, t+1) &= P(X_1 = 1, \dots, X_s = 1, X_{s+1} = 0, \dots, X_{s+t} = 0, X_{s+t+1} = 1) \\ &\quad + P(X_1 = 1, \dots, X_s = 1, X_{s+1} = 0, \dots, X_{s+t} = 0, X_{s+t+1} = 0) \\ &= P(X_1 = 1, \dots, X_s = 1, X_{s+1} = 0, \dots, X_{s+t} = 0) \\ &= \varphi(s, t). \end{aligned}$$

This leads to the following generalization about functional equations on semigroups.

**Theorem 3.1.2** (Cauchy functional equation on the semigroup  $(\mathbb{N}_0^2, +)$ ). *For a function  $\varphi : \mathbb{N}_0^2 \rightarrow [0, \infty)$  the following conditions are equivalent:*

1. *The function,  $\varphi$ , satisfies*

$$\varphi(0, 0) = 1$$

and

$$\varphi(s + 1, t) + \varphi(s, t + 1) = \varphi(s, t).$$

2. *There is a unique probability measure  $\mu$  on  $[0, 1]$  such that*

$$\varphi(l, m) = \int_{[0,1]} p^l (1 - p)^m d\mu(p).$$

Theorem 3.1.2 is a consequence of Theorem 3.2.1 which was proven by Ressel in [25] (see also [26]).

For the rest of this chapter we need a few preliminary ideas.

## 3.2 Definitions and Preliminary Concepts

We consider a commutative semigroup  $(S, +)$  with neutral element 0. A set  $G \subset S$  is a generator set for  $S$  if every element  $s \in S \setminus \{0\}$  is a finite sum of elements from  $G$ .

Recall that a function  $\rho : S \rightarrow \mathbb{R}$  is a *character* of  $S$  if it satisfies  $\rho(0) = 1$  and

$$\rho(s + t) = \rho(s)\rho(t), \quad s, t \in S.$$

Let  $\hat{S}$  be the set of characters on the semigroup  $S$ . The topology on  $\hat{S}$  is the topology of pointwise convergence which is locally convex.

We will generalize the following result, from [25], to a system of equations.

**Theorem 3.2.1.** *Let  $S$  be a commutative semigroup, with a countable generator set  $G \subset S$  and let  $\beta : G \rightarrow (0, \infty)$  be a function.*

*A function  $\varphi : S \rightarrow [0, \infty)$  is a solution of the equation*

$$\varphi(s) = \sum_{a \in G} \beta(a) \varphi(a + s)$$

*if and only if there exists a positive Radon measure  $\mu$  concentrated on  $K$  such that*

$$\varphi(s) = \int_K \rho(s) d\mu(\rho),$$

$$\text{where } K = \left\{ \rho \in \hat{S} \mid \rho(s) \geq 0, \sum_{a \in G} \beta(a) \rho(a) = 1 \right\}.$$

Recall a function  $\varphi : S \rightarrow \mathbb{R}$  is called *positive definite* if

$$\sum_{j,k=1}^n c_j c_k \varphi(s_j + s_k) \geq 0,$$

for all  $n \in \mathbb{N}$ ,  $\{s_1, \dots, s_n\} \subset S$ ,  $\{c_1, \dots, c_n\} \subset \mathbb{R}$ .

Define the operator  $E_s$  that maps the set of functionals from  $S$  to  $\mathbb{R}$  to itself by,

$$E_s(\varphi)(t) = \varphi(t + s)$$

where  $\varphi : S \rightarrow \mathbb{R}$ . The operators  $(E_s)_{s \in S}$  generate an algebra.

### 3.3 System of Cauchy Functional Equations

**Theorem 3.3.1.** *Let  $S$  be a countable commutative semigroup,  $G \subset S$  be a generator set and let  $\{\beta_i : G \rightarrow [0, \infty)\}_{i \in I}$  be a family of functions where  $I$  is an indexing set. We suppose that for every  $a \in G$  there is an  $i \in I$  such that  $\beta_i(a) > 0$ .*

*A function  $\varphi : S \rightarrow [0, \infty)$  is a solution of the system*

$$\varphi(s) = \sum_{a \in G} \beta_i(a) \varphi(a + s), i \in I$$

*if and only if there exists a positive Radon measure  $\mu$  concentrated on  $K$  such that*

$$\varphi(s) = \int_K \rho(s) d\mu(\rho),$$

*where  $K = \left\{ \rho \in \hat{S} \mid \rho(s) \geq 0, \sum_{a \in G} \beta_i(a) \rho(a) = 1, i \in I \right\}$ . That is, solutions to the system can be identified with positive Radon measures concentrated on  $K$ .*

*Proof.* First we will state an inequality that will be used throughout the proof. Suppose  $\varphi$  is a solution to the system of functional equations,  $\varphi(s) = \sum_{a \in G} \beta_i(a) \varphi(a + s), i \in I$ . Then, for  $s \in S, a \in G$  and  $i \in I$ ,

$$\varphi(s) \geq \beta_i(a) \varphi(s + a).$$

Fix  $s \in S$ , then  $s = \sum_{k=1}^n a_k$  for  $a_k \in G$  and  $n \in \mathbb{N}$ . Therefore, for every  $a_k$  there exists a  $\beta_{i_k}$  such that  $\beta_{i_k}(a_k) > 0$ . Hence,

$$\varphi(s) = \varphi \left( \sum_{k=1}^n a_k \right)$$

$$\begin{aligned}
&= \varphi \left( \sum_{k=1}^{n-1} a_k + a_n \right) \\
&\leq (\beta_{i_n}(a_n))^{-1} \varphi \left( \sum_{k=1}^{n-1} a_k \right) \\
&\vdots \\
&\leq \prod_{k=1}^n (\beta_{i_k}(a_k))^{-1} \varphi(0).
\end{aligned}$$

Therefore, for every  $s \in S$  there exists  $\gamma_s$  such that

$$\varphi(s) \leq \gamma_s \varphi(0), \quad (3.3.1)$$

where  $\gamma_s = \prod_{k=1}^n (\beta_{i_k}(a_k))^{-1}$ . Therefore, if  $\varphi(0) = 0$  then  $\varphi \equiv 0$ . The zero function is identified with the zero measure on  $K$ . If  $\varphi(0) = c > 0$ , then we will find the representation of  $\frac{1}{c}\varphi$  then multiply the measure by  $c$ . So, without loss of generality we will consider functions  $\varphi$  such that  $\varphi(0) = 1$ . That is the set,

$$\mathcal{P} = \left\{ \varphi : S \rightarrow [0, \infty) \mid \varphi(0) = 1, \varphi(s) = \sum_{a \in G} \beta_i(a) \varphi(s+a), i \in I \right\}.$$

However,  $\mathcal{P}$  is not necessarily closed and therefore not necessarily compact. We would like to use the Krein-Milman theorem thus we will find a compact convex set that contains  $\mathcal{P}$ .

Consider

$$\mathcal{P}_1 = \left\{ \varphi : S \rightarrow [0, \infty) \mid \prod_{a \in F} (E_0 - \beta_{i_a}(a) E_a) \varphi(s) \geq 0, F \subset G, F \text{ finite}, \{i_a\} \subset I, \varphi(0) = 1 \right\}.$$

We will show that  $\mathcal{P}_1$  is convex. Let  $\varphi, \psi \in \mathcal{P}_1$  and  $\lambda \in (0, 1)$ . We will show that  $\lambda\varphi + (1 - \lambda)\psi \in \mathcal{P}_1$ . Note  $\lambda\varphi(0) + (1 - \lambda)\psi(0) = \lambda + (1 - \lambda) = 1$ . Let  $F \subset G$  such that  $F$



is finite and  $\{i_a\} \subset I$ . Then

$$\begin{aligned} & \prod_{a \in F} (E_0 - \beta_{i_a}(a)E_a)(\lambda\varphi(s) + (1 - \lambda)\psi(s)) \\ &= \lambda \prod_{a \in F} (E_0 - \beta_{i_a}(a)E_a)\varphi(s) + (1 - \lambda) \prod_{a \in F} (E_0 - \beta_{i_a}(a)E_a)\psi(s) \geq 0. \end{aligned}$$

Hence,  $\mathcal{P}_1$  is convex.

Note that  $\mathcal{P}_1$  is closed in the topology of pointwise convergence. We will show that  $\mathcal{P}_1$  is a subset of a compact set. First consider  $F$  to be the singleton  $\{a\}$  and  $\varphi \in \mathcal{P}_1$ . Then the condition of  $\mathcal{P}_1$  is  $(E_0 - \beta_{i_a}(a)E_a)\varphi(s) \geq 0$ , i.e.  $\varphi(s) \geq \beta_{i_a}(a)\varphi(s+a)$ . Thus, inequality (3.3.1) holds for  $\varphi \in \mathcal{P}_1$ . Since  $\varphi \in \mathcal{P}_1$ , inequality (3.3.1) simplifies to

$$\varphi(s) \leq \gamma_s.$$

Thus  $\mathcal{P}_1$  is a closed subset of  $\prod_{s \in S} [0, \gamma_s]$ . By Tychonoff theorem  $\prod_{s \in S} [0, \gamma_s]$  is compact. Hence  $\mathcal{P}_1$  is compact. So  $\mathcal{P}_1$  is a compact convex set and by Krein-Milman theorem,  $\mathcal{P}_1 = \overline{\text{conv}}(\text{ext}(\mathcal{P}_1))$ .

We will show that  $\mathcal{P} \subset \mathcal{P}_1$ . Let  $\varphi \in \mathcal{P}$ ,  $F \subset G$  such that  $F$  is finite, and  $\{i_a\} \subset I$ .

Then

$$(E_0 - \beta_{i_1}(b_1)E_{b_1})\varphi(s) = \sum_{\substack{a \in G \\ a \neq b_1}} \beta_{i_1}(a)\varphi(s+a) \geq 0$$

and

$$(E_0 - \beta_{i_2}(b_2)E_{b_2})(E_0 - \beta_{i_1}(b_1)E_{b_1})\varphi(s) = \sum_{\substack{a^{(1)} \in G \\ a^{(1)} \neq b_1}} \beta_{i_1}(a^{(1)}) \sum_{\substack{a^{(2)} \in G \\ a^{(2)} \neq b_2}} \beta_{i_2}(a^{(2)}) \varphi(s+a^{(1)}+a^{(2)}) \geq 0.$$

Continuing this process we get

$$\begin{aligned}
& \prod_{k=1}^n (E_0 - \beta_{i_k}(b_k)E_{b_k})\varphi(s) \\
&= \sum_{\substack{a^{(1)} \in G \\ a^{(1)} \neq b_1}} \beta_{i_1}(a^{(1)}) \sum_{\substack{a^{(2)} \in G \\ a^{(2)} \neq b_2}} \beta_{i_2}(a^{(2)}) \cdots \sum_{\substack{a^{(n)} \in G \\ a^{(n)} \neq b_n}} \beta_{i_n}(a^{(n)}) \varphi\left(s + \sum_{k=1}^{n-1} a^{(k)}\right) \\
&\geq 0.
\end{aligned}$$

Thus  $\mathcal{P} \subset \mathcal{P}_1$ .

Let  $\varphi \in \text{ext}(\mathcal{P}_1)$ . We will now show that  $\varphi$  is a character. Case 1, suppose  $b \in G$  such that  $\varphi(b) > 0$ ,  $\beta_i(b) > 0$  and  $1 - \beta_i(b)\varphi(b) > 0$ .

Define

$$\psi_1(s) = \varphi(s + b) \text{ and } \psi_2(s) = \varphi(s) - \beta_i(b)\varphi(s + b).$$

Note that  $\frac{\psi_1}{\varphi(b)}, \frac{\psi_2}{1 - \beta_i(b)\varphi(b)} \in \mathcal{P}$  and

$$\varphi(s) = \beta_i(b)\varphi(b)\frac{\psi_1(s)}{\varphi(b)} + (1 - \beta_i(b)\varphi(b))\frac{\psi_2(s)}{1 - \beta_i(b)\varphi(b)}.$$

Since  $\varphi$  is extreme, this implies  $\varphi(s) = \frac{\psi_1(s)}{\varphi(b)}$ , i.e.

$$\varphi(s + b) = \varphi(s)\varphi(b).$$

For the last 2 cases we will use the following. If  $\psi$  satisfies

$$\prod_{a \in F} (E_0 - \beta_i(a)E_a)\psi(s) \geq 0, F \subset G, F \text{ finite}, i \in I \quad (3.3.2)$$

then for all  $s \in S$  there exists  $\gamma_s > 0$  such that  $\gamma_s \psi(0) \geq \psi(s)$ . Note  $\psi_1$  and  $\psi_2$  satisfy condition (3.3.2).

Case 2, suppose  $b \in G$  such that  $\varphi(b) = 0$ . Then  $0 \leq \varphi(s+b) = \psi_1(s) \leq \gamma_s \psi_1(0) = \gamma_s \varphi(b) = 0$ . Thus  $\varphi(s+b) = 0 = \varphi(s)\varphi(b)$ .

Case 3, suppose  $b \in G$  such that  $1 - \beta_i(b)\varphi(b) = 0$ . Then  $\varphi(b) = (\beta_i(b))^{-1}$ . Thus  $\varphi(s) - \beta_i(b)\varphi(s+b) = \psi_2(s) \leq \gamma_s \psi_2(0) = \gamma_s(1 - \beta_i(b)\varphi(b)) = 0$ . Hence  $\varphi(s+b) = \varphi(s)(\beta_i(b))^{-1} = \varphi(s)\varphi(b)$ .

Therefore, for all  $a \in G$  and  $s \in S$ ,  $\varphi(s+a) = \varphi(s)\varphi(a)$ . This implies for all  $s, t \in S$ ,  $\varphi(s+t) = \varphi(s)\varphi(t)$ . Thus  $\text{ext}\mathcal{P}_1 \subset \hat{S} \cap \mathcal{P}_1$ .

We will construct a Radon measure on  $\text{ext}(\mathcal{P}_1)$  as in Chapter 1 which is inspired by [19]. Let  $\varphi \in \mathcal{P}_1 = \overline{\text{conv}}(\text{ext}\mathcal{P}_1)$ . Then there exists a net  $(\psi_\alpha)$  in  $\text{conv}(\text{ext}(\mathcal{P}_1))$  such that

$$\varphi(s) = \lim_{\alpha} \psi_{\alpha}(s) \text{ for all } s \in S.$$

Since

$$\psi_{\alpha} = \sum_{k=1}^{n_{\alpha}} d_k^{\alpha} \rho_k^{\alpha},$$

for some  $d_k^{\alpha} \geq 0$ ,  $\sum_{k=1}^{n_{\alpha}} d_k^{\alpha} = 1$ ,  $\rho_k^{\alpha} \in \text{ext}(\mathcal{P}_1)$  and  $n_{\alpha} \in \mathbb{N}$ ,

$$\varphi(s) = \lim_{\alpha} \sum_{k=1}^{n_{\alpha}} d_k^{\alpha} \rho_k^{\alpha}(s).$$

Then we have for  $\{c_1, \dots, c_n\} \subset \mathbb{R}$  and  $\{s_1, \dots, s_n\} \subset S$

$$\left| \sum_{j=1}^n c_j \varphi(s_j) \right| = \left| \sum_{j=1}^n c_j \lim_{\alpha} \sum_{k=1}^{n_{\alpha}} d_k^{\alpha} \rho_k^{\alpha}(s_j) \right|$$

$$\begin{aligned}
&= \left| \lim_{\alpha} \sum_{k=1}^{n_{\alpha}} d_k^{\alpha} \sum_{j=1}^n c_j \rho_k^{\alpha}(s_j) \right| \\
&\leq \lim_{\alpha} \sum_{k=1}^{n_{\alpha}} d_k^{\alpha} \sup_{\rho \in \text{ext}(\mathcal{P}_1)} \left| \sum_{j=1}^n c_j \rho(s_j) \right| \\
&= \sup_{\rho \in \text{ext}(\mathcal{P}_1)} \left| \sum_{j=1}^n c_j \rho(s_j) \right|.
\end{aligned}$$

Define  $\hat{s} : \text{ext}(\mathcal{P}_1) \rightarrow [0, \infty)$  by  $\hat{s}(\rho) = \rho(s)$ . Note for  $s_1, s_2 \in S$ ,  $\widehat{s_1 + s_2}(\rho) = \rho(s_1 + s_2) = \rho(s_1)\rho(s_2) = \hat{s}_1\hat{s}_2(\rho)$ . Now we will construct a function on the set of functions,

$$A = \left\{ \sum_{j=1}^n c_j \hat{s}_j \mid c_j \in \mathbb{R}, s_j \in S, n \in \mathbb{N} \right\}.$$

The set  $A$  contains the constant functions and separates points thus, by the Stone-Weierstrass theorem,  $A$  is dense in continuous functions from  $\text{ext}(\mathcal{P}_1)$  to  $\mathbb{R}$ ,  $\mathcal{C}(\text{ext}(\mathcal{P}_1))$ . Define

$$F \left( \sum_{j=1}^n c_j \hat{s}_j \right) = \sum_{j=1}^n c_j \varphi(s_j).$$

Note if  $\sum_{j=1}^n c_j \hat{s}_j = \sum_{k=1}^m d_k \hat{s}_k$  then from the above inequality

$$\left| \sum_{j=1}^n c_j \varphi(s_j) - \sum_{k=1}^m d_k \varphi(s_k) \right| \leq \sup_{\rho \in \text{ext}(\mathcal{P}_1)} \left| \sum_{j=1}^n c_j \rho(s_j) - \sum_{k=1}^m d_k \rho(s_k) \right| = 0.$$

Thus  $F$  is well defined. We extend  $F$  to  $\mathcal{C}(\text{ext}(\mathcal{P}_1))$ . Let  $f \in \mathcal{C}(\text{ext}(\mathcal{P}_1))$ , then  $f =$

$\lim_{\beta} \sum_{j=1}^{n_{\beta}} c_j^{\beta} \hat{s}_j^{\beta}$ . Then we define

$$\begin{aligned}
F(f) &= \lim_{\beta} F \left( \sum_{j=1}^{n_{\beta}} c_j^{\beta} \hat{s}_j^{\beta} \right) \\
&= \lim_{\beta} \sum_{j=1}^{n_{\beta}} c_j^{\beta} \varphi(s_j^{\beta}).
\end{aligned}$$

Thus  $F$  is a Radon measure on  $\text{ext}(\mathcal{P}_1)$ .

Next we will show that  $F$  is positive. Let  $f \in \mathcal{C}(\text{ext}(\mathcal{P}_1))$  such that  $f \geq 0$ . Then  $\sqrt{f} \in \mathcal{C}(\text{ext}(\mathcal{P}_1))$ . So there exists  $\sum_{k=1}^{n_\alpha} c_k^\alpha \hat{s}_k^\alpha$  in  $A$  such that  $\lim_\alpha \sum_{k=1}^{n_\alpha} c_k^\alpha \hat{s}_k^\alpha \rightarrow \sqrt{f}$ . Therefore  $\lim_\alpha \sum_{k,j=1}^{n_\alpha} c_k^\alpha c_j^\alpha \hat{s}_k^\alpha \hat{s}_j^\alpha \rightarrow f$ . So it suffices to show that

$$\begin{aligned} F \left( \sum_{k,j=1}^{n_\alpha} c_k^\alpha c_j^\alpha \hat{s}_k^\alpha \hat{s}_j^\alpha \right) &= F \left( \sum_{k,j=1}^{n_\alpha} c_k^\alpha c_j^\alpha (\widehat{s_k^\alpha + s_j^\alpha}) \right) \\ &= \sum_{k,j=1}^{n_\alpha} c_k^\alpha c_j^\alpha \varphi(s_k^\alpha + s_j^\alpha) \\ &\geq 0, \end{aligned}$$

for all  $c_k^\alpha \in \mathbb{R}$  and  $s_k^\alpha \in S$ . We will show  $\varphi$  is positive definite using the same method as in [3], which simplifies the method used in [20]. Let  $\{c_1, \dots, c_n\} \subset \mathbb{R}$  and  $\{s_1, \dots, s_n\} \subset S$ . Then  $\sum_{k,j=1}^n c_k c_j \varphi(s_k + s_j) = \left( \sum_{k=1}^n c_k E_{s_k} \right)^2 \varphi(0)$ . We will split the sum into the parts where  $c_k$  are positive and  $c_k$  are negative

$$\left( \sum_{k=1}^n c_k E_{s_k} \right)^2 = \left( \sum_{c_k > 0} c_k E_{s_k} - \sum_{c_k < 0} (-c_k) E_{s_k} \right)^2.$$

Let  $X = \sum_{c_k > 0} c_k E_{s_k}$  and  $Y = \sum_{c_k < 0} (-c_k) E_{s_k}$ . It suffices to show that

$$(X - Y)^2 \varphi(0) \geq 0.$$

Note that for  $s, t \in S$ , where  $t = \sum_{j=1}^m a_j$ ,  $a_j \in G$  and  $\beta_{i_j}(a_j) > 0$ ,

$$\begin{aligned} \varphi(s + t) &= \varphi \left( s + \sum_{j=1}^m a_j \right) \\ &= \varphi \left( s + \sum_{j=1}^{n-1} a_j + a_m \right) \end{aligned}$$

$$\begin{aligned}
&\leq (\beta_{i_m}(a_m))^{-1} \varphi \left( s + \sum_{j=1}^{m-1} a_j \right) \\
&\vdots \\
&\leq \prod_{j=1}^m (\beta_{i_j}(a_j))^{-1} \varphi(s).
\end{aligned}$$

Let  $\gamma_t = \prod_{j=1}^m (\beta_{i_j}(a_j))^{-1}$ . Then we have  $\varphi(s+t) \leq \gamma_t \varphi(s)$ . Thus

$$\begin{aligned}
X\varphi(s) &= \left( \sum_{c_k > 0} c_k E_{s_k} \right) \varphi(s) \\
&= \sum_{c_k > 0} c_k \varphi(s + s_k) \\
&\leq n \max_{c_k > 0} \{c_k \gamma_{s_k}\} \varphi(s).
\end{aligned}$$

Hence for  $M = n \max_{c_k > 0} \{c_k \gamma_{s_k}\} > 0$ ,  $U = ME_0 - X \geq 0$ . Similarly there exists  $Q > 0$  such that  $V = QE_0 - Y \geq 0$ .

We will show that  $U$  and  $V$  are greater than or equal to a finite sum with positive coefficients of products of the form

$$\prod_{k=1}^{m_1} d_k E_{a_k} \prod_{j=1}^{m_2} (E_0 - \beta_{i_j}(a_j) E_{a_j}). \tag{3.3.3}$$

First suppose  $X = cE_t$  then  $M = c\gamma_t$ . So

$$\begin{aligned}
ME_0 - X &= c\gamma_t E_0 - cE_t \\
&= c\gamma_t (E_0 - \gamma_t^{-1} E_t)
\end{aligned}$$

$$\begin{aligned}
&= c\gamma_t \left( E_0 - \prod_{j=1}^m \beta_{i_j}(a_j) E_{a_j} \right) \\
&= c\gamma_t \left( E_0 - \beta_{i_1} E_{a_1} (E_0 - \beta_{i_2} E_{a_2}) + \cdots + \prod_{j=1}^{m-1} (\beta_{i_j}(a_j) E_{a_j}) (E_0 - \beta_{i_m}(a_m) E_{a_m}) \right).
\end{aligned}$$

Next suppose  $X = \sum_{k=1}^n E_{t_k}$  then  $M = n \max \gamma_{t_k}$ . Then

$$\begin{aligned}
ME_0 - X &= n \max\{\gamma_{t_k}\} - \sum_{k=1}^n E_{t_k} \\
&\geq \sum_{k=1}^n \gamma_{t_k} E_0 - \sum_{k=1}^n E_{t_k} \\
&= \sum_{k=1}^n (\gamma_{t_k} E_0 - E_{t_k}).
\end{aligned}$$

From above each  $\gamma_{t_k} E_0 - E_{t_k}$  can be bounded below by a sum of products of the form (3.3.3). From the combination of these two cases we have that  $U$  has the desired lower bound. Similarly we can show that  $V$  has the same type of lower bound. Since  $\varphi \in \mathcal{P}_1$ , we have that products of  $U, V, X, Y$  are bounded below by product of the form (3.3.3) and therefore are nonnegative. Let

$$C(n, j, k) = \binom{n}{j} \binom{n}{k} X^j U^{n-j} Y^k V^{n-k}.$$

Now consider the following for  $n \geq 2$ .

$$Z_n = \sum_{j,k=1}^n \left[ \frac{j^2 M^2 + k^2 Q^2}{n(n-1)} - 2 \frac{jkMQ}{n^2} \right] C(n, j, k)$$

Note that  $Z_n \geq 0$  and

$$Z_n = \sum_{j,k=1}^n \left[ \frac{j(j-1)M^2}{n(n-1)} + \frac{k(k-1)Q^2}{n(n-1)} - 2 \frac{jM}{n} \frac{kQ}{n} + \frac{jM^2}{n(n-1)} + \frac{kQ^2}{n(n-1)} \right] C(n, j, k)$$

$$= (X - Y)^2 M^n Q^n + \frac{X}{n-1} M^{n+1} Q^n + \frac{Y}{n-1} M^n Q^{n+1}.$$

Therefore  $(X - Y)^2 \varphi(0) + \frac{XM}{n-1} \varphi(0) + \frac{YQ}{n-1} \varphi(0) \geq 0$ , for all  $n \geq 2$ . Hence  $(X - Y)^2 \varphi(0) \geq 0$ .

Thus  $\varphi$  is positive definite and  $F$  is positive.

We can write  $F$  as an integral with a unique positive Radon measure  $\mu$  on  $\text{ext}(\mathcal{P}_1)$ ,

i.e.

$$F(f) = \int_{\text{ext}(\mathcal{P}_1)} f(\rho) d\mu(\rho).$$

Thus

$$\varphi(s) = F(\hat{s}) = \int_{\text{ext}(\mathcal{P}_1)} \rho(s) d\mu(\rho).$$

Now consider  $\varphi \in \mathcal{P}$ . Then for each  $i \in I$  we have

$$\begin{aligned} \int_{\text{ext}(\mathcal{P}_1)} \rho(s) d\mu(\rho) &= \varphi(s) \\ &= \sum_{a \in G} \beta_i(a) \varphi(a + s) \\ &= \int_{\text{ext}(\mathcal{P}_1)} \sum_{a \in G} \beta_i(a) \rho(a + s) d\mu(\rho) \\ &= \int_{\text{ext}(\mathcal{P}_1)} \rho(s) \sum_{a \in G} \beta_i(a) \rho(a) d\mu(\rho). \end{aligned}$$

Thus for  $\varphi \in \mathcal{P}$ ,  $\mu$  is concentrated on the set

$$K = \left\{ \rho \in \hat{S} \mid \rho(s) \geq 0, \sum_{a \in G} \beta_i(a) \rho(a) = 1, i \in I \right\}.$$

□



### 3.4 Examples

The following are some examples that use Theorem 3.3.1.

**Example 3.4.1.** We consider the semigroup  $(\mathbb{N}_0^3, +)$  and the system of Cauchy functional equations

$$\varphi(m, n, p) = \varphi(m + 1, n, p) + \varphi(m, n, p + 1)$$

$$\varphi(m, n, p) = \varphi(m, n + 1, p) + \varphi(m, n, p + 1)$$

where  $\varphi : \mathbb{N}_0^3 \rightarrow [0, \infty)$ .

By Theorem 3.3.1 solutions of this system are

$$\varphi(m, n, p) = \int_K \rho(m, n, p) d\mu(\rho),$$

where  $K = \left\{ \rho \in \widehat{\mathbb{N}_0^3} \mid \rho \geq 0, \rho(1, 0, 0) + \rho(0, 0, 1) = 1, \rho(0, 1, 0) + \rho(0, 0, 1) = 1 \right\}$  and  $\mu$  is a positive Radon measure. That is, solutions of the system can be identified with positive Radon Measures on  $K$ .

To simplify  $K$ , we consider what it means to be a character on  $\mathbb{N}_0^3$ . If  $\rho$  is a character on  $\mathbb{N}_0^3$ , then

$$\begin{aligned} \rho(m, n, q) &= \rho((m, 0, 0) + (0, n, 0) + (0, 0, q)) \\ &= \rho(m, 0, 0)\rho(0, n, 0)\rho(0, 0, q) \\ &= \rho(m(1, 0, 0))\rho(n(0, 1, 0))\rho(q(0, 0, 1)) \\ &= [\rho(1, 0, 0)]^m [\rho(0, 1, 0)]^n [\rho(0, 0, 1)]^q. \end{aligned}$$

So,  $\rho$  is determined entirely by its values on  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . Thus for every character  $\rho$  on  $\mathbb{N}_0^3$  there exists  $x, y, z \in \mathbb{R}$  such that

$$\rho(m, n, q) = x^m y^n z^q.$$

The conditions  $\rho(1, 0, 0) + \rho(0, 0, 1) = 1$ ,  $\rho(0, 1, 0) + \rho(0, 0, 1) = 1$ , and  $\rho \geq 0$  become  $x + z = 1$ ,  $y + z = 1$ , and  $x, y, z, \in \mathbb{R}_+$ , respectively.

Thus the solutions of this system are

$$\varphi(m, n, p) = \int_K x^m y^n z^p d\mu(x, y, z).$$

Where  $K$  is the set

$$\{(x, y, z) \in \mathbb{R}_+^3 \mid x + z = 1, y + z = 1\}$$

and  $\mu$  is a positive Radon measure.

**Example 3.4.2.** We consider the semigroup  $(\mathbb{N}_0^2, +)$  and the equation

$$\varphi(m, q) = \sum_{k=0}^n \binom{n}{k} \varphi((m + n - k, q + k)) \quad (3.4.1)$$

where  $\varphi : \mathbb{N}_0^2 \rightarrow [0, \infty)$ .

Consider the semigroup,  $S$ , generated by

$$\{(n, 0), (n - 1, 1), (n - 2, 2), \dots, (0, n)\} \subset \mathbb{N}_0^2.$$

The equation (3.4.1) can be rewritten as

$$\varphi(m, q) = \sum_{k=0}^n \binom{n}{k} \varphi((n - k, k) + (m, q)).$$

Thus by Theorem 3.3.1 the solutions are

$$\varphi(m, q) = \int_K \rho(m, q) d\mu(\rho)$$

where  $\mu$  is a positive Radon measure,  $(m, q) \in S$ ,  $K = \left\{ \rho \in \hat{S} \mid \rho \geq 0, \sum_{k=0}^n \binom{n}{k} \rho(n-k, k) = 1 \right\}$ .

Let  $\rho \in K$ . We will show that  $\rho$  is completely determined by its values on  $(n, 0)$  and  $(0, n)$ . First we consider  $\rho$  on an element of the generator set, then

$$\begin{aligned} [\rho(n-k, k)]^n &= \rho(n(n-k), nk) \\ &= [\rho(n, 0)]^{n-k} [\rho(0, n)]^k. \end{aligned}$$

Hence,

$$\rho(n-k, k) = [(\rho(n, 0))^{1/n}]^{n-k} [(\rho(0, n))^{1/n}]^k.$$

Note that,

$$\begin{aligned} 1 &= \sum_{k=0}^n \binom{n}{k} \rho(n-k, k) \\ &= \sum_{k=0}^n \binom{n}{k} [(\rho(n, 0))^{1/n}]^{n-k} [(\rho(0, n))^{1/n}]^k \\ &= [(\rho(n, 0))^{1/n} + (\rho(0, n))^{1/n}]^n. \end{aligned}$$

Therefore,  $(\rho(n, 0))^{1/n} + (\rho(0, n))^{1/n} = 1$ . Let  $p = (\rho(n, 0))^{1/n}$ , thus  $(1-p) = (\rho(0, n))^{1/n}$  and

$$\rho(n-k, k) = p^{n-k} (1-p)^k,$$

where  $p \in [0, 1]$ .

For  $(m, q) \in S$ ,  $(m, q) = \sum_{i=1}^{\ell} (n-k_i, k_i)$ . Thus  $m = \sum_{i=1}^{\ell} (n-k_i)$  and  $q = \sum_{i=1}^{\ell} k_i$ .

Hence,

$$\begin{aligned}
\rho(m, q) &= \rho\left(n\ell - \sum_{i=1}^{\ell}(n - k_i), \sum_{i=1}^{\ell} k_i\right) \\
&= \prod_{i=1}^{\ell} \rho((n - k_i), k_i) \\
&= \prod_{i=1}^{\ell} p^{n-k_i} (1 - p)^{k_i} \\
&= p^m (1 - p)^q.
\end{aligned}$$

So,  $\rho$  can be identified with  $p$  and therefore identify  $K$  with  $[0, 1]$ . Hence for  $(m, q) \in S$  we have

$$\begin{aligned}
\varphi(m, q) &= \int_K \rho(m, q) d\mu(\rho) \\
&= \int_{[0,1]} p^m (1 - p)^q d\mu(p).
\end{aligned}$$

We extend this representation to all of  $\mathbb{N}_0^2$  to get for  $(m, q) \in \mathbb{N}_0^2$

$$\varphi(m, q) = \int_{[0,1]} p^m (1 - p)^q d\mu(p).$$

**CHAPTER 4**  
**BOCHNER-PLANCHEREL THEOREMS IN ALGEBRAS**  
**WITH INVOLUTION**

**4.1 Definitions and notation**

Let  $\mathcal{A}$  be a commutative algebra with involution  $*$ . We will state some definitions for algebras that are similar to some of the definitions on semigroups.

**Definition 4.1.1.** An *involution* on an algebra is a map  $x \mapsto x^*$  from  $\mathcal{A}$  to  $\mathcal{A}$  such that the following hold for all  $x, y \in \mathcal{A}$  and  $\lambda \in \mathbb{C}$ ,

$$(x + y)^* = x^* + y^*, \quad (\lambda x)^* = \bar{\lambda}x^*, \quad (xy)^* = y^*x^*, \quad (x^*)^* = x.$$

**Definition 4.1.2.** A linear functional  $f : \mathcal{A} \rightarrow \mathbb{C}$  is called *positive*, if

$$f(xx^*) \geq 0, \text{ for all } x \in \mathcal{A}.$$

**Definition 4.1.3.** A seminorm  $p$  is called *multiplicative* if  $p(x^*) = p(x)$  and  $p(xy) \leq p(x)p(y)$  for all  $x, y \in \mathcal{A}$ .

**Definition 4.1.4.** A net  $(e_i)_{i \in I}$  is a *p-bounded approximate identity* if  $p(e_i) \leq 1$ ,  $i \in I$  and  $\lim_{i \in I} p(x - xe_i) = 0$  for every  $x \in \mathcal{A}$ . If the topology of  $\mathcal{A}$  is defined by a family of seminorms  $\{p_k\}$ , then  $(e_i)$  is *bounded approximate identity* provided  $(e_i)$  is  $p_k$ -bounded for every  $k$ .

**Definition 4.1.5.** A *multiplicative functional* on  $\mathcal{A}$  is a nonzero homomorphism from  $\mathcal{A}$  to  $\mathbb{C}$ .

The set of nonzero multiplicative functions will be denoted  $\hat{\mathcal{A}}$  and is called the set of *characters* of  $\mathcal{A}$ . We have

$$\hat{\mathcal{A}} = \{\rho : \mathcal{A} \rightarrow \mathbb{C} \mid \rho \text{ linear, } \rho \not\equiv 0, \rho(xy) = \rho(x)\rho(y), x, y \in \mathcal{A}\}.$$

If  $x \in \mathcal{A}$ , we define the function  $\hat{x}$  on  $\hat{\mathcal{A}}$  by

$$\hat{x}(\rho) = \rho(x).$$

The map  $x \mapsto \hat{x}$  from  $\mathcal{A}$  to  $\hat{\mathcal{A}}$  is called the *Gelfand transform* on  $\mathcal{A}$ . Let

$$\Gamma(A) = \{\hat{x} \mid x \in A\}.$$

In this chapter we denote the set

$$K = \{\rho \in \hat{\mathcal{A}} \mid \rho(x^*) = \overline{\rho(x)}, |\rho(x)| \leq p(x)\},$$

where  $p$  is a multiplicative seminorm.

The set  $K$  is a locally compact space with pointwise topology. Let  $(\rho_i)$  be a net in  $K$  such that  $\rho_i \rightarrow \rho$ . Note that  $\rho$  could be identical to zero, thus  $\rho$  is not necessarily in  $K$ .

Therefore the set  $K$  is not necessarily closed. However, if  $\mathcal{A}$  has a unity  $e$  and the seminorm  $p$  satisfies  $p(e) = 1$ , then  $\rho(e) = 1$  for all  $\rho \in K$ . Thus  $K$  is a closed subset of the compact set  $\prod_{x \in \mathcal{A}} [-p(x), p(x)]$  and therefore compact.

**Definition 4.1.6.** We say that a function  $g : \mathcal{A} \rightarrow \mathbb{C}$  admits a Bochner representation if there exists a positive Radon measure  $\mu$  on  $K$  such that the functions  $\Gamma(\mathcal{A})$  are  $\mu$  integrable and we have

$$g(x) = \int_K \rho(x) d\mu(\rho), \quad x \in \mathcal{A}.$$

Note that every function which admits a Bochner representation is linear.

We denote by  $B$  the linear subspace of  $\mathcal{A}$  generated by the products  $xy$ , such that  $x, y \in \mathcal{A}$ .

**Definition 4.1.7.** We say that a linear function  $g : B \rightarrow \mathbb{C}$  admits a Plancherel representation, if there exists a positive Radon measure  $\mu$  on  $K$  such that the functions  $\Gamma(\mathcal{A})$  are in  $L^2(K)$  and we have

$$g(xy) = \int_K \rho(xy) d\mu(\rho), \quad x, y \in \mathcal{A}.$$

**Lemma 4.1.8.** If  $g : \mathcal{A} \rightarrow \mathbb{C}$  admits a Plancherel representation then  $g$  is positive.

*Proof.* Assume that  $g$  admits a Plancherel representation. Then there exist a positive Radon measure  $\mu$  on  $K$  such that for all  $x, y \in \mathcal{A}$ ,

$$g(xy) = \int_K \rho(xy) d\mu(\rho).$$

Therefore,

$$\begin{aligned}g(xx^*) &= \int_K \rho(xx^*)d\mu(\rho) \\ &= \int_K \rho(x)\rho(x^*)d\mu(\rho) \\ &= \int_K |\rho(x)|^2d\mu(\rho) \\ &\geq 0.\end{aligned}$$

So  $g$  is positive. □

We note that a Bochner representation is a Plancherel representation. Thus a function that admits a Bochner representation is positive.

## 4.2 Bochner-type integral representations

In this section we will show an alternative proof of Theorem 2.1 in [14], p. 42 (see also [15], p. 344), which is stated as Theorem 4.2.8. In [14]  $\Gamma$ -Lumer systems (see [18]) were used to prove this theorem.

**Theorem 4.2.1.** *Let  $\mathcal{A}$  be an commutative algebra with involution  $*$  and unity  $e$ . Let  $p$  be a multiplicative seminorm on  $A$  such that  $p(e) = 1$ . A function  $g : \mathcal{A} \rightarrow \mathbb{C}$  admits a Bochner representation if and only if the following conditions are satisfied,*

(a)  *$g$  is a positive linear function*



(b)  $|g(x)| \leq Cp(x)$  where  $C$  is a positive real number not depending on  $x$ .

*Proof.* This theorem is Theorem 4.2.5 in [9], where we consider our algebra as semigroup with the multiplication operation. □

We will now extend Theorem 4.2.1 to an algebra without unity. But in doing so we will need to add an extra condition to the function  $g$ . In Theorem 4.2.6 we will add an extra condition on to the algebra  $\mathcal{A}$  instead of the function  $g$ .

Let  $\mathcal{A}$  be an algebra without unity. We will embed  $\mathcal{A}$  into an algebra with unity  $\mathcal{A}_e$ . Let  $\mathcal{A}_e$  denote the set of all pairs  $(x, \lambda)$ ,  $x \in \mathcal{A}$ ,  $\lambda \in \mathbb{C}$ . Then  $\mathcal{A}_e$  is an algebra with operators.

$$(x, \lambda) + (y, \mu) = (x + y, \lambda + \mu), \mu(x, \lambda) = (\mu x, \mu \lambda)$$

and

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda \mu)$$

for  $x, y \in \mathcal{A}$  and  $\lambda, \mu \in \mathbb{C}$ . The identity element for  $\mathcal{A}_e$  is  $e = (0, 1) \in \mathcal{A}_e$ . As  $\mathcal{A}$  is commutative so is  $\mathcal{A}_e$ . The mapping  $x \mapsto (x, 0)$  is an algebra isomorphism of  $\mathcal{A}$  onto an ideal of codimension one in  $\mathcal{A}_e$ . Since  $(x, \lambda) = (x, 0) + \lambda(0, 1)$ , it is customary to write the elements  $(x, \lambda)$  as  $x + \lambda e$ . We call  $\mathcal{A}_e$  the unitization of  $\mathcal{A}$ . If  $\mathcal{A}$  has an involution, the involution on  $\mathcal{A}$  can be extended to an involution on  $\mathcal{A}_e$  by  $(x + \lambda e)^* = x^* + \bar{\lambda}e$ .

The following lemma is Proposition 21.7 in [12] on p. 59,

**Lemma 4.2.2.** *Let  $\mathcal{A}$  be an algebra with involution and  $\mathcal{A}_e$  the unitization of  $\mathcal{A}$ . Let  $g$  be a positive functional on  $\mathcal{A}$ . Then  $g$  can be extended to a positive functional on  $\mathcal{A}_e$  if and only if:*

- $g(x^*) = \overline{g(x)}$
- *There is a finite  $k \geq 0$  such that  $|g(x)|^2 \leq kg(xx^*)$  for all  $x \in \mathcal{A}$ .*

*An extension denoted  $g_e$  is defined by  $g_e(x + \lambda e) = g(x) + k\lambda$ .*

Let  $\mathcal{A}$  be an algebra without unity and  $p$  be a multiplicative seminorm on  $\mathcal{A}$ . Let  $K$  denote the set of linear functions  $\rho : \mathcal{A} \rightarrow \mathbb{C}$  such that the following hold

- (a)  $\rho \neq 0$
- (b)  $\rho(xy) = \rho(x)\rho(y)$
- (c)  $\rho(x^*) = \overline{\rho(x)}$
- (d)  $|\rho(x)| \leq p_e(x)$ .

Let  $\mathcal{A}_e$  be the unitization of  $\mathcal{A}$ . We extend the seminorm  $p$  to a seminorm on  $\mathcal{A}_e$  by

$$p_e(x + \lambda e) = p(x) + |\lambda|.$$

Let  $K_e$  denote the set of linear functions  $\rho : \mathcal{A}_e \rightarrow \mathbb{C}$  such that the following hold

- (a)  $\rho(e) = 1$

$$(b) \rho((x + \lambda e)(y + \gamma e)) = \rho(x + \lambda e)\rho(y + \gamma e)$$

$$(c) \rho((x + \lambda e)^*) = \overline{\rho((x + \lambda e))}$$

$$(d) |\rho((x + \lambda e))| \leq p((x + \lambda e)).$$

**Lemma 4.2.3.** *The set  $K_e$  can be identified with  $K \cup \{\theta\}$  where  $\theta$  is the function identical to 0 on  $\mathcal{A}$ .*

*Proof.* Let  $\eta \in K$ . Therefore  $|\eta(x)|^2 = |\eta(xx^*)| \leq p(xx^*)$ . Thus by Lemma 4.2.2,  $\eta$  can be extended to  $\mathcal{A}_e$  by  $\eta_e(x + \lambda e) = \eta(x) + \lambda$ . It is easily shown that  $\eta_e \in K_e$ . Thus  $K$  can be identified with a subset of  $K_e$ .

Let  $\rho \in K_e$ , then  $\rho(x + \lambda e) = \rho(x) + \lambda\rho(e) = \rho(x) + \lambda$ . First we suppose that  $\rho$  is identical to 0 on  $\mathcal{A}$ . Then  $\rho(x + \lambda e) = \rho(x) + \lambda = \lambda$ . Thus there is exactly one function in  $K_e$  identical to 0 on  $\mathcal{A}$ . Denote that function by  $\theta$ , that is  $\theta(x + \lambda e) = \lambda$ .

Now suppose  $\rho$  is not identical to 0 on  $\mathcal{A}$ . Then  $\rho|_{\mathcal{A}} \in K$ . Also,

$$(\rho|_{\mathcal{A}})_e(x + \lambda e) = \rho|_{\mathcal{A}}(x) + \lambda = \rho(x + \lambda e).$$

Thus,  $K_e$  can be identified with  $K \cup \{\theta\}$ .

□

**Theorem 4.2.4.** *Let  $\mathcal{A}$  be a commutative algebra with involution  $*$  and without unity. A function  $g : \mathcal{A} \rightarrow \mathbb{C}$  admits a Bochner representation if and only if the following conditions are satisfied,*

(a)  $g$  is a positive linear function

(b)  $|g(x)| \leq Cp(x)$  for every  $x \in \mathcal{A}$  where  $C$  is a positive real number not depending on  $x$

(c)  $|g(x)|^2 \leq Mg(xx^*)$  for every  $x \in \mathcal{A}$  where  $M$  is a positive real number not depending on  $x$ .

*Proof.* We are going to use Theorem 4.2.1 in this proof. By Lemma 4.2.2 the function  $g$  can be extended to linear positive functional  $g_e$  on  $\mathcal{A}_e$  by

$$g_e(x + \lambda e) = g(x) + M\lambda.$$

We also define

$$p_e(x + \lambda e) = p(x) + |\lambda|.$$

It is easy to verify that  $p_e$  is a multiplicative seminorm on  $\mathcal{A}_e$ .

We have

$$|g_e(x + \lambda e)| \leq |g(x)| + |\lambda|M \leq \max\{C, M\}p_e(x + \lambda e).$$

Let  $K_e$  denote the set of linear functions  $\rho : \mathcal{A}_e \rightarrow \mathbb{C}$  such that the following hold

(a)  $\rho(e) = 1$

(b)  $\rho((x + \lambda e)(y + \gamma e)) = \rho(x + \lambda e)\rho(y + \gamma e)$

(c)  $\rho((x + \lambda e)^*) = \overline{\rho((x + \lambda e))}$

(d)  $|\rho((x + \lambda e))| \leq p_e((x + \lambda e)).$

By Theorem 4.2.1, there is a Radon measure  $\nu$  on  $K_e$  such that for every  $x + \lambda e \in \mathcal{A}_e$  we have

$$g_e(x + \lambda e) = \int_{K_e} \rho(x + \lambda e) d\nu(\rho).$$

By Lemma 4.2.3  $K_e$  can be identified with  $K \cup \{\theta\}$ . If we restrict to elements of the algebra  $\mathcal{A}$ , we have

$$g(x) = g_e(x) = \int_{K_e} \rho(x) d\nu(\rho) = \int_{K \cup \{\theta\}} \rho(x) d\nu(\rho) = \int_K \rho(x) d\mu(\rho),$$

where  $\mu$  is the restriction of  $\nu$  to  $K$ . □

In Theorem 4.2.6 we will replace a condition for the function  $g$  with a condition to the algebra  $\mathcal{A}$ . We will need the next lemma for the proof. The following lemma is 1.6 on p. 88 in [9].

**Lemma 4.2.5.** *For any positive functional  $\varphi$  we have*

$$|\varphi(x^*y)|^2 \leq \varphi(x^*x)\varphi(y^*y)$$

for all  $x, y \in \mathcal{A}$ .

**Theorem 4.2.6.** *Let  $\mathcal{A}$  be a commutative algebra with involution  $*$  and without unity. Let  $p$  be a multiplicative seminorm on  $\mathcal{A}$  which admits a  $p$ -bounded approximate identity  $(e_i)_{i \in I}$ , where  $I$  is an indexing set. A function  $g : \mathcal{A} \rightarrow \mathbb{C}$  admits a Bochner representation if and only if the following conditions are satisfied,*

(a)  *$g$  is a positive linear function*

(b)  $|g(x)| \leq Cp(x)$  for every  $x \in \mathcal{A}$  where  $C$  is a real number not depending on  $x$ .

*Proof.* We have the inequality  $|g(x - xe_i)| \leq Cp(x - xe_i)$ ,  $i \in I$ . Thus,  $\lim_{i \in I} g(xe_i) = g(x)$ .

By Lemma 4.2.5 we have

$$|g(xe_i)|^2 \leq g(e_i e_i^*)g(xx^*) \leq Cp(e_i e_i^*)g(xx^*) \leq Cg(xx^*).$$

Therefore, we take the limit in  $I$  and

$$|g(x)|^2 = \lim_{i \in I} |g(xe_i)|^2 \leq Cg(xx^*).$$

We finish the proof using Theorem 4.2.4. □

*Remark 4.2.7.* It is enough to suppose that the net  $(e_i)_{i \in I}$  is such so  $p(e_i) \leq 1$  and  $\lim_{i \in I} g(x - xe_i) = 0$ .

As a consequence of Theorem 4.2.6 we obtain the following result which is the main result, from Theorem 2.1 in [14], p. 42 (see also [15], p. 344).

**Theorem 4.2.8.** *Let  $\mathcal{A}$  be a commutative topological algebra with involution  $*$ , whose topology is defined by a family of nonzero multiplicative seminorms with a bounded approximate identity. A linear continuous function  $g : \mathcal{A} \rightarrow \mathbb{C}$  admits a Bochner representation if and only if  $g$  is positive.*

*Proof.* Assume that  $g$  admits a Bochner representation then, by Lemma 4.1.8,  $g$  is positive.

Now assume that  $g : \mathcal{A} \rightarrow \mathbb{C}$  is linear continuous and positive. Let  $\{p_k\}_{k \in K}$  be the set of multiplicative seminorms on  $\mathcal{A}$ . Since  $g$  is continuous, there exists  $p \in \{p_k\}_{k \in K}$  such

that  $|g(x)| \leq Cp(x)$ . We will use Theorem 4.2.6 and  $p$  will be the needed multiplicative seminorm. Let  $(e_i)_{i \in I}$  be a bounded approximate identity in  $\mathcal{A}$ . Then for every  $k \in K$  there exists  $C_k \in \mathbb{R}$  such that  $p_k(e_i) \leq C_k$ , for every  $i \in I$ . Therefore  $(e_i)_{i \in I}$  is also a  $p$ -bounded approximate identity. By Theorem 4.2.6,  $g$  admits a Bochner representation.  $\square$

### 4.3 A Plancherel-type integral representation

In this section we prove a Plancherel-type integral representation. For results related to our representation see [2], [11] and [28].

**Theorem 4.3.1.** *Let  $\mathcal{A}$  be a commutative algebra with involution  $*$  and without unity. A linear function  $g : B \rightarrow \mathbb{C}$  admits a Plancherel representation if and only if the following conditions are satisfied,*

(a) *The function  $g$  is positive*

(b) *There is a family  $(C_x)_{x \in \mathcal{A}}$  of positive real numbers such that  $|g(yxx^*)| \leq C_x p(y)$ , for every  $x, y \in \mathcal{A}$*

(c) *For every  $x \in \mathcal{A}$  and every  $\epsilon > 0$  there are  $y$  and  $z$  in  $\mathcal{A}$  such that*

$$g((x - yz)(x^* - y^*z^*)) < \epsilon.$$

*Proof.* For  $x \in \mathcal{A}$  we consider function  $g_x : y \mapsto g(yxx^*)$ . Then  $g_x$  is positive for all  $x \in \mathcal{A}$ .

By Lemma 4.2.5,

$$|g_x(y)|^2 = |g(yxx^*)|^2 = |g((yx)x^*)|^2 \leq g(xx^*)g(yy^*xx^*) = g(xx^*)g_x(yy^*).$$

Therefore by Lemma 4.2.2, the function  $g_x$  can be extended to linear positive functional on  $\mathcal{A}_e$  by

$$(g_x)_e : y + \lambda e \mapsto g(yxx^*) + \lambda g(xx^*).$$

Consequently, we have

$$|(g_x)_e(y)| = |g(yxx^*)| \leq C_x p(y).$$

The seminorm  $p$  can be extended to a seminorm on  $\mathcal{A}_e$  by  $p_e(x + \lambda e) = p(x) + |\lambda|$ . We will show that  $(g_x)_e$  satisfies the conditions for Theorem 4.2.1. Note that

$$\begin{aligned} |(g_x)_e(y + \lambda e)| &\leq |g(yxx^*)| + |\lambda g(xx^*)| \\ &\leq C_x p(y) + |\lambda g(xx^*)| \\ &\leq \max\{C_x, g(xx^*)\}(p(y) + |\lambda|) \\ &\leq \max\{C_x, g(xx^*)\}p(y + \lambda e). \end{aligned}$$

Hence,  $(g_x)_e$  satisfies the conditions from Theorem 4.2.1. Thus for every  $x \in \mathcal{A}$  there exists a positive radon measure  $\nu_x$  on  $K_e$  such that

$$(g_x)_e(y + \lambda e) = \int_{K_e} \rho(y + \lambda e) d\nu_x(\rho).$$



By Lemma 4.2.3,  $K_e$  can be identified with  $K \cup \{\theta\}$ , where  $\theta$  is the function identical to 0 on  $\mathcal{A}$ . Therefore, for all  $x, y \in \mathcal{A}$  we have the following,

$$g(yxx^*) = (g_x)_e(y) = \int_{K_e} \rho(y) d\nu_x(\rho) = \int_{K \cup \{\theta\}} \rho(y) d\nu_x(\rho), \text{ for all } x, y \in \mathcal{A}.$$

and

$$g(xx^*) = (g_x)_e(e) = \int_{K \cup \{\theta\}} \rho(e) d\nu_x(\rho) = \int_{K \cup \{\theta\}} d\nu_x(\rho), \text{ for all } x \in \mathcal{A}.$$

Thus for  $x, y, z \in \mathcal{A}$

$$\int_{K \cup \{\theta\}} \rho(z) |\rho(x)|^2 d\nu_y = g(zxx^*yy^*) = \int_{K \cup \{\theta\}} \rho(z) |\rho(y)|^2 d\nu_x. \quad (4.3.1)$$

For  $z \in \mathcal{A}$  we define the function  $\hat{z} : K \cup \{\theta\} \rightarrow \mathbb{C}$  by  $\hat{z}(\rho) = \rho(z)$ . By the Stone-Weierstrass theorem,  $\Gamma(\mathcal{A})$  is dense in  $\mathcal{C}_0(K)$ , the continuous functions which tend to 0 at infinity. Since the ‘‘infinity point’’ for  $K$  is  $\theta$ ,  $\mathcal{C}_0(K) = \mathcal{C}_0(K \cup \{\theta\})$ . Therefore  $\Gamma(\mathcal{A})$  is dense in  $\mathcal{C}_0(K \cup \{\theta\})$ . Since  $\nu_y, \nu_x$  are Radon measures on  $K \cup \{\theta\}$ , by (4.3.1) we have

$$h_x \cdot \nu_y = h_y \cdot \nu_x,$$

where  $h_x$  is the function that maps  $\rho \mapsto |\rho(x)|^2$ .

We show now that we can define a unique measure  $\nu$  on  $K$  such that

$$h_x|_K \cdot \nu = \nu_x|_K$$

If  $K_x = \{\rho \in K | h_x(\rho) > 0\}$  we can define

$$\nu|_{K_x} = (1/h_x)|_{K_x} \cdot \nu_x|_{K_x}.$$

Note that  $\nu$  is well defined since  $h_x \cdot \nu_y = h_y \cdot \nu_x$ .

Since  $\rho \neq 0$ , we have

$$\bigcup_{x \in \mathcal{A}} K_x = K$$

and consequently  $\nu$  is defined on  $K$ .

Note that for all  $x, y \in \mathcal{A}$ ,

$$4xy = \sum_{\tau \in \{\pm 1, \pm i\}} \tau(x + \tau y^*)(x + \tau y^*)^*. \quad (4.3.2)$$

Since  $g$  is linear we will find the integral representation for  $g(xx^*)$  and use (4.3.2) to show the integral representation for  $g(xy)$ .

For every  $t \in \mathcal{A}$ ,  $\theta(t) = 0$ , thus we have

$$\begin{aligned} g(txx^*) &= \int_{K \cup \{\theta\}} \rho(t) d\nu_x(\rho) \\ &= \int_K \rho(t) d\nu_x(\rho) \\ &= \int_K \rho(t) g_x(\rho) d\nu(\rho) \\ &= \int_K \rho(t) |\rho(x)|^2 d\nu(\rho) \\ &= \int_K \rho(txx^*) d\nu(\rho). \end{aligned}$$

Thus for every  $t, x, y \in \mathcal{A}$  and every  $\tau \in \{\pm 1, \pm i\}$  we have

$$g(t(x + \tau y^*)(x + \tau y^*)^*) = \int_K \rho(t(x + \tau y^*)(x + \tau y^*)^*) d\nu(\rho).$$

Thus from equation (4.3.2),

$$g(txy) = \sum_{\tau \in \{\pm 1, \pm i\}} \frac{\tau}{4} g(t(x + \tau y^*)(x^* + \bar{\tau}y)) \quad (4.3.3)$$

we get

$$\begin{aligned}
g(txy) &= g\left(t \sum_{\tau \in \{\pm 1, \pm i\}} \frac{\tau}{4} (x + \tau y^*)(x^* + \bar{\tau}y)\right) \\
&= \sum_{\tau \in \{\pm 1, \pm i\}} \frac{\tau}{4} g(t(x + \tau y^*)(x + \tau y^*)^*) \\
&= \sum_{\tau \in \{\pm 1, \pm i\}} \frac{\tau}{4} \int_K \rho(t(x + \tau y^*)(x + \tau y^*)^*) d\nu(\rho) \\
&= \int_K \rho\left(t \sum_{\tau \in \{\pm 1, \pm i\}} \frac{\tau}{4} (x + \tau y^*)(x + \tau y^*)^*\right) d\nu(\rho) \\
&= \int_K \rho(txy) d\nu(\rho).
\end{aligned}$$

That is for every  $x, y, t \in \mathcal{A}$

$$g(txy) = \int_K \rho(txy) d\nu(\rho). \quad (4.3.4)$$

Now we want to show that, for all  $x \in \mathcal{A}$ ,  $\nu_x(\{\theta\}) = 0$ . Let  $x \in \mathcal{A}$  and  $\epsilon > 0$ , then there exist  $y, z \in \mathcal{A}$  such that  $g((x - yz)(x^* - y^*z^*)) < \epsilon$ . By (4.3.4) and the integral representation for  $g_z$  the following holds,

$$\begin{aligned}
&g(xx^*) - g((x - yz)(x^* - y^*z^*)) \\
&= g(xx^* - xx^* + xy^*z^* + yzx^* - yzy^*z^*) \\
&= g(xy^*z^*) + g(yzx^*) - g(yy^*zz^*) \\
&= \int_K \rho(xy^*z^*) d\mu(\rho) + \int_K \rho(yzx^*) d\mu(\rho) - \int_{K \cup \{\theta\}} \rho(yy^*) d\mu_z(\rho) \\
&= \int_K \rho(xy^*z^*) d\mu(\rho) + \int_K \rho(yzx^*) d\mu(\rho) - \int_K \rho(yy^*) d\mu_z(\rho) \\
&= \int_K \rho(xy^*z^*) d\mu(\rho) + \int_K \rho(yzx^*) d\mu(\rho) - \int_K \rho(yy^*) |\rho(z)|^2 d\mu(\rho)
\end{aligned}$$

$$\begin{aligned}
&= \int_K \rho(xy^*z^*)d\mu(\rho) + \int_K \rho(yzx^*)d\mu(\rho) - \int_K \rho(yy^*zz^*)d\mu(\rho) \\
&= \int_K \rho(xy^*z^* + yzx^* - yy^*zz^*)d\mu(\rho) \\
&= \int_K \rho(xx^* - xx^* + xy^*z^* + yzx^* - yy^*zz^*)d\mu(\rho) \\
&= \int_K \rho(xx^*)d\mu(\rho) - \int_K \rho(xx^* + xy^*z^* + yzx^* - yy^*zz^*)d\mu(\rho) \\
&= \int_K \rho(xx^*)d\mu(\rho) - \int_K \rho((x - yz)(x^* - y^*z^*))d\mu(\rho).
\end{aligned}$$

Therefore,

$$g(xx^*) - \int_K \rho(xx^*)d\mu(\rho) = g((x - yz)(x^* - y^*z^*)) - \int_K \rho((x - yz)(x^* - y^*z^*))d\mu(\rho).$$

Now we return to  $\nu_x(\{\theta\})$ ,

$$\begin{aligned}
\nu_x(\{\theta\}) &= \int_{K \cup \{\theta\}} d\nu_x - \int_K d\nu_x \\
&= g(xx^*) - \int_K |\rho(x)|^2 d\nu \\
&= g((x - yz)(x^* - y^*z^*)) - \int_K |\rho(x - yz)|^2 d\nu(\rho) \\
&\leq g((x - yz)(x^* - y^*z^*)) \\
&< \epsilon.
\end{aligned}$$

Thus we take  $\epsilon \rightarrow 0$  and it results that

$$\nu_x(\{\theta\}) = 0.$$

This means that for every  $x \in \mathcal{A}$ ,

$$g(xx^*) = \int_{K \cup \{\theta\}} d\nu_x = \int_K d\nu_x = \int_K \rho(xx^*) d\nu(\rho).$$

Thus for  $x, y \in \mathcal{A}$ ,

$$\begin{aligned} g(xy) &= g\left(\sum_{\tau \in \{\pm 1, \pm i\}} \frac{\tau}{4}(x + \tau y^*)(x^* + \bar{\tau}y)\right) \\ &= \sum_{\tau \in \{\pm 1, \pm i\}} \frac{\tau}{4} g((x + \tau y^*)(x + \tau y^*)^*) \\ &= \sum_{\tau \in \{\pm 1, \pm i\}} \frac{\tau}{4} \int_K \rho((x + \tau y^*)(x + \tau y^*)^*) d\nu(\rho) \\ &= \int_K \rho\left(\sum_{\tau \in \{\pm 1, \pm i\}} \frac{\tau}{4}(x + \tau y^*)(x + \tau y^*)^*\right) d\nu(\rho) \\ &= \int_K \rho(xy) d\nu(\rho). \end{aligned}$$

Thus, the function  $g$  admits a Plancherel representation.

Now we assume that  $g$  admits a Plancherel representation. We will show that there is a family  $(C_x)_{x \in \mathcal{A}}$  of positive real numbers such that  $|g(yxx^*)| \leq C_x p(y)$ , for every  $x, y \in \mathcal{A}$ , and for every  $x \in \mathcal{A}$  and every  $\epsilon > 0$  there are  $y$  and  $z$  in  $\mathcal{A}$  such that

$$g((x - yz)(x^* - y^*z^*)) < \epsilon.$$

First we have

$$\begin{aligned} g(xx^*) &= \int_K \rho(xx^*) d\mu(\rho) \\ &= \int_K |\rho(x)|^2 d\mu(\rho) \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned}
|g(yxx^*)| &= \left| \int_K \rho(y) |\rho(x)|^2 d\mu(\rho) \right| \\
&\leq \left( \int_K |\rho(x)|^2 d\mu(\rho) \right) p(y) \\
&= C_x p(y).
\end{aligned}$$

Next we have to show that for every  $x \in \mathcal{A}$  and every  $\epsilon > 0$  there are  $y$  and  $z$  in  $\mathcal{A}$  such that

$$g((x - yz)(x^* - y^*z^*)) = \int_K |\rho(x) - \rho(y)\rho(z)|^2 d\mu(\rho) \leq \epsilon.$$

Fix  $x \in \mathcal{A}$  and  $\epsilon > 0$ . Since continuous functions with compact support are dense in  $L^2(K)$ , there exists a continuous function with compact support  $\varphi$  such that

$$\int_K |\rho(x) - \varphi(\rho)|^2 d\mu(\rho) \leq \frac{\epsilon}{8}.$$

Let  $r \in \text{supp } \varphi$ , then there exist  $s \in \mathcal{A}$  such that  $r(s) \neq 0$ . Thus,  $r(ss^*) = |r(s)|^2 > 0$ . There exists a neighborhood,  $U_r$ , of  $r$  such that  $|t(s)|^2 > 0$  for every  $t$  in  $U_r$ . The set  $\{U_r\}_{r \in \text{supp } \varphi}$  form a cover of  $\text{supp } \varphi$ , thus there exists a finite set of  $r_k$ ,  $1 \leq k \leq n$ , whose neighborhoods cover  $\text{supp } \varphi$ . Let  $s_1, s_2, \dots, s_n$  be the corresponding  $s_k$ 's. Then  $r(s_1s_1^* + \dots + s_ns_n^*) > 0$  for every  $r \in \text{supp } \varphi$ .

By the Stone-Weierstrass theorem,  $\Gamma(\mathcal{A})$  is dense in  $\mathcal{C}_0(\hat{\mathcal{A}})$ . Let  $y = s_1s_1^* + \dots + s_ns_n^*$ . Since the function  $\frac{\varphi}{y}$  is continuous on  $K$  and has compact support, there exists  $z \in \mathcal{A}$  such that

$$\sup_{\rho \in K} \left| \frac{\varphi(\rho)}{\rho(y)} - \hat{z}(\rho) \right| < \sqrt{\frac{\epsilon}{8 \int_K |\rho(y)|^2 d\mu(\rho)}}.$$

Hence,

$$\begin{aligned}
\int_K |\varphi(\rho) - \rho(y)\rho(z)|^2 d\mu(\rho) &= \int_K |\rho(y)|^2 \left| \frac{\varphi(\rho)}{\rho(y)} - \rho(z) \right|^2 d\mu(\rho) \\
&\leq \frac{\epsilon}{8 \int_K |\rho(y)|^2 d\mu(\rho)} \int_K |\rho(y)|^2 d\mu(\rho) \\
&\leq \frac{\epsilon}{8}
\end{aligned}$$

and consequently

$$\begin{aligned}
\int_K |\rho(x) - \rho(y)\rho(z)|^2 d\mu(\rho) &= \int_K |\rho(x) - \varphi(\rho) + \varphi(\rho) - \rho(y)\rho(z)|^2 d\mu(\rho) \\
&\leq 4 \left( \int_K |\rho(x) - \varphi(\rho)|^2 d\mu(\rho) + \int_K |\varphi(\rho) - \rho(y)\rho(z)|^2 d\mu(\rho) \right) \\
&\leq 4 \left( \frac{\epsilon}{8} + \frac{\epsilon}{8} \right) \\
&= \epsilon.
\end{aligned}$$

□

## 4.4 Back to Bochner

We will now discuss some situations where a function that admits a Plancherel representation also admits a Bochner representation.

**Theorem 4.4.1.** *Let  $\mathcal{A}$  be a commutative algebra with involution  $*$  and without unity. Let  $g : \mathcal{A} \rightarrow \mathbb{C}$  be a positive linear function and  $p$  a multiplicative seminorm on  $\mathcal{A}$ . If there is a positive real number  $C$  such that  $|g(x)| \leq Cp(x)$ , for every  $x \in \mathcal{A}$ , and for every  $x \in \mathcal{A}$  and*

every  $\epsilon > 0$  there exists elements  $y$  and  $z$  in  $\mathcal{A}$  such that

$$p(x - yz) < \epsilon, \quad (4.4.1)$$

then  $g$  has a Plancherel-type integral representation as in Section 4.3.

If the measure  $\mu$  from Plancherel representation is finite, then  $g$  admits a Bochner-type integral representation.

*Proof.* We will use Theorem 4.3.1 to show that  $g$  has a Plancherel representation. Consider, for  $x, y \in \mathcal{A}$ ,

$$|g(yxx^*)| \leq Cp(yxx^*) \leq C|p(x)|^2p(y) = C_xp(y).$$

Fix  $x \in \mathcal{A}$  and  $\epsilon > 0$ . There exist  $y, z \in \mathcal{A}$  such that  $p(x - yz) < \sqrt{\frac{1}{C}\epsilon}$ . Therefore,

$$g((x - yz)(x^* - y^*z^*)) \leq Cp(x - yz)^2 < C \left( \sqrt{\frac{1}{C}\epsilon} \right)^2 = \epsilon.$$

Thus by Theorem 4.3.1,  $g$  has a Plancherel representation.

Next we will show that  $g$  admits a Bochner representation if  $\mu$  is finite. Suppose that  $\mu$  is finite. Let  $x \in \mathcal{A}$  and  $\epsilon > 0$ . We will show that  $|g(x) - \int_K \rho(x) d\mu(\rho)| < \epsilon$ . There exists  $y, z \in \mathcal{A}$  such that  $p(x - yz) < \epsilon$ . Consider

$$\begin{aligned} g(x) - g(x - yz) &= g(x - x + yz) \\ &= g(yz) \\ &= \int_K \rho(yz) d\mu(\rho) \\ &= \int_K \rho(x - x + yz) d\mu(\rho) \end{aligned}$$



$$\begin{aligned}
&= \int_K \rho(x) - \rho(x - yz) d\mu(\rho) \\
&= \int_K \rho(x) d\mu(\rho) - \int_K \rho(x - yz) d\mu(\rho).
\end{aligned}$$

Therefore

$$g(x) - \int_K \rho(x) d\mu(\rho) = g(x - yz) - \int_K \rho(x - yz) d\mu(\rho).$$

So we have the following,

$$\begin{aligned}
\left| g(x) - \int_K \rho(x) d\mu(\rho) \right| &= \left| g(x - yz) - \int_K \rho(x - yz) d\mu(\rho) \right| \\
&\leq |g(x - yz)| + \int_K |\rho(x - yz)| d\mu(\rho) \\
&\leq Cp(x - yz) + p(x - yz) \int_K d\mu(\rho) \\
&= C\mu(K)p(x - yz) \\
&< C\mu(K)\epsilon.
\end{aligned}$$

Thus we take the limit as  $\epsilon \rightarrow 0$  and we get that  $g$  has a Bochner representation.  $\square$

The next theorem is another theorem in which we get the Bochner representation from the Plancherel representation. As you will see, in this case there are more assumptions on  $\mathcal{A}$  that will lead to the boundedness of the measure  $\mu$ .

**Theorem 4.4.2.** *Let  $\mathcal{A}$  be a commutative algebra with involution  $*$  and without unity and  $p$  a multiplicative seminorm on  $\mathcal{A}$ . Let  $g : \mathcal{A} \rightarrow \mathbb{C}$  be a positive linear function such that the following hold,*

1. *There exists  $C > 0$  such that for every  $x \in \mathcal{A}$ ,  $|g(x)| \leq Cp(x)$*

2. For every  $\epsilon > 0$  and  $x \in \mathcal{A}$  there exists  $y, z \in \mathcal{A}$  such that

$$p(x - yz) < \epsilon.$$

If there is a sequence  $(e_n)_{n \in \mathbb{N}}$  in  $\mathcal{A}$  such that

1.  $p(e_n) \leq 1, n \in \mathbb{N}$

2.  $\lim_{n \rightarrow \infty} \rho(e_n) = 1, \rho \in K,$

then  $g$  admits a Bochner-type representation and we have  $g(x) = \lim_{n \rightarrow \infty} g(xe_n)$  for every  $x \in \mathcal{A}$ .

*Proof.* By Theorem 4.4.1  $g$  admits a Plancherel representation. From the Plancherel representation, we have

$$g(e_n e_n^*) = \int_K \rho(e_n e_n^*) d\mu(\rho) = \int_K |\rho(e_n)|^2 d\mu(\rho).$$

Using that  $g(e_n e_n^*) \leq Cp(e_n e_n^*) \leq C$ ,  $\liminf_{n \rightarrow \infty} |\rho(e_j)|^2 = 1$ , and Fatou's Lemma we obtain

$$\begin{aligned} C &\geq \liminf_{n \rightarrow \infty} \int_K |\rho(e_j)|^2 d\mu(\rho) \\ &\geq \int_K \liminf_{n \rightarrow \infty} |\rho(e_j)|^2 d\mu(\rho) \\ &= \mu(K). \end{aligned}$$

The measure  $\mu$  is finite and consequently, according to Theorem 4.4.1, the function  $g$  admits a Bochner-type representation. Now the fact that  $g(x) = \lim_{n \rightarrow \infty} g(xe_n)$  is a consequence of dominated convergence theorem because the function  $\hat{x}$  is  $\mu$ -integrable for every  $x \in \mathcal{A}$ .  $\square$

# CHAPTER 5

## PSEUDOQUOTIENTS ON COMMUTATIVE BANACH ALGEBRAS

The following results are taken from the paper [8].

### 5.1 Introduction

In this section we recall the construction of pseudoquotients and its basic properties. The construction of pseudoquotients was introduced in [21] under the name of “generalized quotients”. The motivation for the idea, early developments, and later modifications, are discussed in [22]. The construction of pseudoquotients has desirable properties. For instance, it preserves the algebraic structure of  $X$  and has good topological properties. There is growing evidence that pseudoquotients can be a useful tool (see, for example, [5], [6], or [7]).

Let  $X$  be a nonempty set and let  $S$  be a commutative semigroup acting on  $X$  injectively. The relation

$$(x, \varphi) \sim (y, \psi) \quad \text{if} \quad \psi x = \varphi y$$

is an equivalence in  $X \times S$ . We define  $\mathcal{B}(X, S) = (X \times S)/\sim$ . Elements of  $\mathcal{B}(X, S)$  are called *pseudoquotients*. The equivalence class of  $(x, \varphi)$  will be denoted by  $\frac{x}{\varphi}$ . Thus

$$\frac{x}{\varphi} = \frac{y}{\psi} \text{ means } \psi x = \varphi y.$$

Elements of  $X$  can be identified with elements of  $\mathcal{B}(X, S)$  via the embedding  $\iota : X \rightarrow \mathcal{B}(X, S)$  defined by

$$\iota(x) = \frac{\varphi x}{\varphi},$$

where  $\varphi$  is an arbitrary element of  $S$ . The action of  $S$  can be extended to  $\mathcal{B}(X, S)$  via

$$\varphi \frac{x}{\psi} = \frac{\varphi x}{\psi}.$$

If  $\varphi \frac{x}{\psi} = \iota(y)$ , for some  $y \in X$ , we simply write  $\varphi \frac{x}{\psi} \in X$  and  $\varphi \frac{x}{\psi} = y$ . For instance, we have  $\varphi \frac{x}{\varphi} = x$ .

In the case  $X$  is a topological space or a convergence space and  $S$  is a commutative semigroup of continuous injections acting on  $X$ , then we can define a convergence in  $\mathcal{B}(X, S)$  as follows: If, for a sequence  $F_n \in \mathcal{B}(X, S)$ , there exist  $\varphi \in S$  and  $F \in \mathcal{B}(X, S)$  such that  $\varphi F_n, \varphi F \in X$ , for all  $n \in \mathbb{N}$ , and  $\varphi F_n \rightarrow \varphi F$  in  $X$ , then we write  $F_n \xrightarrow{I} F$  in  $\mathcal{B}(X, S)$ . In other words,  $F_n \xrightarrow{I} F$  in  $\mathcal{B}(X, S)$  if

$$F_n = \frac{x_n}{\varphi}, F = \frac{x}{\varphi}, \text{ and } x_n \rightarrow x \text{ in } X,$$

for some  $x_n, x \in X$  and  $\varphi \in S$ .

This convergence is sometimes referred to as *type I* convergence. It is quite natural, but it need not be topological. For this reason we prefer to use the convergence defined as

follows:  $F_n \rightarrow F$  in  $\mathcal{B}(X, S)$  if every subsequence  $(F_{p_n})$  of  $(F_n)$  has a subsequence  $(F_{q_n})$  such that  $F_{q_n} \xrightarrow{I} F$ .

It is easy to show that the embedding  $\iota : X \rightarrow \mathcal{B}(X, S)$ , as well as the extension of any  $\varphi \in S$  to a map  $\varphi : \mathcal{B}(X, S) \rightarrow \mathcal{B}(X, S)$  defined above, are continuous.

The set of all positive linear functionals on an algebra  $\mathcal{A}$  is denoted by  $\mathcal{P}(\mathcal{A})$ . The following theorem (attributed to Maltese in [15]) describes  $\mathcal{P}(\mathcal{A})$  in terms of measures on  $\hat{\mathcal{A}}$ .

We say  $\mathcal{A}$  has a *symmetric involution*, if

$$\widehat{x^*} = \overline{\widehat{x}}, \text{ for all } x \in \mathcal{A}.$$

**Theorem 5.1.1.** *Let  $\mathcal{A}$  be a commutative Banach algebra with a bounded approximate identity and an isometric and symmetric involution. Let  $f$  be a linear functional on  $\mathcal{A}$ . Then  $f \in \mathcal{P}(\mathcal{A})$  if and only if*

$$f(x) = \int_{\hat{\mathcal{A}}} \widehat{x}(\xi) d\mu_f(\xi),$$

for all  $x \in \mathcal{A}$ , with respect to a unique positive Radon measure on  $\hat{\mathcal{A}}$  of total variation  $\|f\|$ .

Let  $\mathcal{F} : \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{M}_+^b(\hat{\mathcal{A}})$  be the map defined by Maltese's theorem, that is,  $\mathcal{F}(f) = \mu_f$ . In terms of the introduced notation, Theorem 5.1.1 states that  $\mathcal{F}$  is an isometry between  $\mathcal{P}(\mathcal{A})$  and  $\mathcal{M}_+^b(\hat{\mathcal{A}})$ . In this chapter we give conditions under which  $\mathcal{P}(\mathcal{A})$  can be extended to a space of pseudoquotients  $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$  such that  $\mathcal{F}$  can be extended to a bijection between  $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$  and  $\mathcal{M}_+(\hat{\mathcal{A}})$ .

In Section 5.2 we formulate and prove the main result of this chapter. In Section 5.3 we discuss some examples. We also show that the result in [6] is a special case of the construction presented here.

## 5.2 An extension of Maltese's theorem

In this section we will assume  $\mathcal{A}$  to be a nonunital commutative Banach algebra with bounded approximate identities and an isometric and symmetric involution. In addition, we assume that  $\mathcal{A}$  satisfies the following condition:

- $\Sigma$  There exists a sequence  $a_1, a_2, \dots \in \mathcal{A}$  such that  $\hat{a}_1, \hat{a}_2, \dots \in \mathcal{K}(\hat{\mathcal{A}})$  and for every  $\xi \in \hat{\mathcal{A}}$  there is an  $n$  such that  $\hat{a}_n(\xi) \neq 0$ .

The following are some examples of spaces where  $\Sigma$  is satisfied.

**Example 5.2.1** (Normal algebras). Let  $\mathcal{A}$  be a commutative Banach algebra. We say that  $\mathcal{A}$  is normal [17], if for every compact  $K \subset \hat{\mathcal{A}}$  and closed  $E \subset \hat{\mathcal{A}}$  such that  $K \cap E = \emptyset$ , there exists  $x \in \mathcal{A}$  such that

$$\hat{x}(\xi) = 1 \text{ for } \xi \in K \quad \text{and} \quad \hat{x}(\xi) = 0 \text{ for } \xi \in E.$$

If  $\mathcal{A}$  is a normal commutative Banach algebra and  $\hat{\mathcal{A}}$  is  $\sigma$ -compact, then  $\mathcal{A}$  satisfies condition  $\Sigma$ . Indeed, if  $\hat{\mathcal{A}}$  is  $\sigma$ -compact, there are compact sets  $K_n \subset \hat{\mathcal{A}}$  such that  $\hat{\mathcal{A}} =$

$\cup_{n=0}^{\infty} K_n$  and  $K_n \subset K_{n+1}^{\circ}$  for all  $n \in \mathbb{N}$ , where  $K_{n+1}^{\circ}$  is the interior of  $K_{n+1}$ . Since  $\mathcal{A}$  is regular, for every  $n \in \mathbb{N}$  there exists  $b_n \in \mathcal{A}$  such that

$$\hat{b}_n(\xi) = \begin{cases} 1 & \text{if } \xi \in K_n \\ 0 & \text{if } \xi \notin K_{n+1}^{\circ} \end{cases}.$$

Let  $a_n = b_n b_n^*$ . Then  $\hat{a}_n = |\hat{b}_n|^2 \geq 0$  and  $K_n \subset \text{supp } \hat{a}_n \subset K_{n+2}$ . Clearly, for every  $\xi \in \hat{\mathcal{A}}$ , there exists  $n$  such that  $\hat{a}_n > 0$ .

Note that a regular commutative Banach algebra is normal, [17].

**Example 5.2.2** (Algebras with  $\sigma$ -compact-open structure spaces). For our next example we use Shilov's idempotent theorem [23].

**Theorem 5.2.3** (Shilov). *Let  $\mathcal{A}$  be a commutative Banach algebra. If  $K$  is a compact and open subset of  $\hat{\mathcal{A}}$ , then there is a unique idempotent  $a \in \mathcal{A}$  such that  $\hat{a}$  is the characteristic function of  $K$ .*

Let  $\mathcal{A}$  be a commutative Banach algebra such that  $\hat{\mathcal{A}}$  is  $\sigma$ -compact-open, that is,  $\hat{\mathcal{A}} = \cup_{n=0}^{\infty} K_n$  where  $K_n$  are disjoint compact and open sets in the Gelfand topology in  $\hat{\mathcal{A}}$ . Since, by Shilov's idempotent theorem, for every  $n \in \mathbb{N}$  there exist a unique idempotent  $a_n \in \mathcal{A}$  such that  $\text{supp } \hat{a}_n = K_n$ ,  $\mathcal{A}$  satisfies condition  $\Sigma$ .

**Lemma 5.2.4.** *If  $\mathcal{A}$  satisfies  $\Sigma$ , the sequence of  $a_1, a_2, \dots \in \mathcal{A}$  can be chosen such that  $\hat{a}_n \geq 0$ .*

*Proof.* Suppose  $\mathcal{A}$  satisfies  $\Sigma$ . Then there exists a sequence  $a_1, a_2, \dots \in \mathcal{A}$  such that  $\hat{a}_1, \hat{a}_2, \dots \in \mathcal{K}(\hat{\mathcal{A}})$  and for every  $\xi \in \hat{\mathcal{A}}$  there is an  $n$  such that  $\hat{a}_n(\xi) \neq 0$ . Note that

for  $\xi \in \hat{\mathcal{A}}$  and  $n \in \mathbb{N}$ ,

$$\widehat{a_n a_n^*}(\xi) = \xi(a_n a_n^*) = \xi(a_n) \xi(a_n^*) = \hat{a}_n(\xi) \widehat{a_n^*}(\xi).$$

Since the involution on  $\mathcal{A}$  is symmetric, i.e.  $\hat{a}_n^* = \bar{\hat{a}}_n$ , we have

$$\widehat{a_n a_n^*}(\xi) = \hat{a}_n(\xi) \bar{\hat{a}}_n(\xi) = |\hat{a}_n(\xi)|^2 \geq 0.$$

We will show that  $a_1 a_1^*, a_2 a_2^*, \dots$  is a sequence that satisfies  $a_1 a_1^*, a_2 a_2^*, \dots \in \mathcal{K}(\hat{\mathcal{A}})$  and for every  $\xi \in \hat{\mathcal{A}}$  there is an  $n$  such that  $\widehat{a_n a_n^*}(\xi) \neq 0$ . Since  $\widehat{a_n a_n^*} = \hat{a}_n \widehat{a_n^*}$ ,  $\text{supp } \widehat{a_n a_n^*} \subset \text{supp } a_n$ . Thus  $\widehat{a_n a_n^*} \in \mathcal{K}(\hat{\mathcal{A}})$ .

Let  $\xi \in \hat{\mathcal{A}}$ . Then there exists  $n \in \mathbb{N}$  such that  $\hat{a}_n(\xi) \neq 0$ . Therefore,  $\widehat{a_n a_n^*}(\xi) = |\hat{a}_n(\xi)|^2 \neq 0$ .  $\square$

For  $a \in \mathcal{A}$ , by  $\Lambda_a$  we denote the operation on linear functionals on  $\mathcal{A}$  defined by  $(\Lambda_a f)(x) = f(ax)$ . Let

$$\mathcal{S} = \left\{ \Lambda_a : \hat{a} > 0 \text{ on } \hat{\mathcal{A}} \right\}.$$

**Lemma 5.2.5.** *If  $\mathcal{A}$  satisfies  $\Sigma$ , then  $\mathcal{S}$  is a nonempty commutative semigroup of injective maps acting on  $\mathcal{P}(\mathcal{A})$ .*

*Proof.* Since  $\mathcal{A}$  satisfies  $\Sigma$ , there exists  $(a_n) \in \mathcal{A}$  such that for every  $\xi \in \hat{\mathcal{A}}$  there exists an  $n$  such that  $\hat{a}_n(\xi) > 0$ . By Lemma 5.2.4 we may assume that  $\hat{a}_n \geq 0$ . If we choose  $\lambda_n > 0$  such that  $\sum_{n=1}^{\infty} \|\lambda_n a_n\| < \infty$ . Since  $\mathcal{A}$  is complete, there exists  $a \in \mathcal{A}$  such that  $a = \sum_{n=1}^{\infty} \lambda_n a_n$ . Therefore  $\Lambda_a \in \mathcal{S}$ .



Clearly,  $\mathcal{S}$  is a commutative semigroup because  $\mathcal{A}$  is commutative. Let  $f \in \mathcal{P}(\mathcal{A})$  and  $\Lambda_a \in \mathcal{S}$ . By Maltese's theorem [15],  $f(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu(\xi)$  for some  $\mu \in \mathcal{M}_+^b(\hat{\mathcal{A}})$ . Thus

$$(\Lambda_a f)(x) = f(ax) = \int_{\hat{\mathcal{A}}} \widehat{ax}(\xi) d\mu(\xi) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) \hat{a}(\xi) d\mu(\xi).$$

Note that  $\hat{a}$  is a positive bounded function on  $\hat{\mathcal{A}}$ . Since  $\hat{a}(\xi) > 0$  for all  $\xi \in \hat{\mathcal{A}}$ ,  $\tilde{\mu} = \hat{a}\mu \in \mathcal{M}_+^b(\hat{\mathcal{A}})$  and  $\Lambda_a f(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\tilde{\mu}(\xi)$ . By Maltese's theorem [15],  $\Lambda_a f \in \mathcal{P}(\mathcal{A})$ .

If  $\Lambda_a f = 0$ , then

$$0 = f(ax) = \int_{\hat{\mathcal{A}}} \widehat{ax}(\xi) d\mu(\xi) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) \hat{a}(\xi) d\mu(\xi),$$

for all  $x$  in  $\mathcal{A}$ . By the Stone-Weierstrass theorem,  $\Gamma(\mathcal{A})$  is dense in  $C_0(\hat{\mathcal{A}})$ . Therefore  $\hat{a}\mu = 0$  which implies  $\mu = 0$ , because  $\hat{a} > 0$ . Thus

$$f(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu(\xi) = 0.$$

Hence  $\Lambda_a$  is injective. □

The map  $\mathcal{F} : \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{M}_+^b(\hat{\mathcal{A}})$  defined by Maltese's theorem, can be extended to a map  $\mathcal{F} : \mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S}) \rightarrow \mathcal{M}_+(\hat{\mathcal{A}})$  in the natural way:

$$\mathcal{F} \left( \frac{f}{\Lambda_a} \right) = \frac{\mathcal{F}(f)}{\hat{a}} = \frac{1}{\hat{a}} \mu_f. \quad (5.2.1)$$

It is clear that  $\mathcal{F}$  is well-defined.

**Theorem 5.2.6.** *Let  $\mathcal{A}$  be a nonunital commutative Banach algebra with a bounded approximate identity and an isometric and symmetric involution. If  $\mathcal{A}$  satisfies  $\Sigma$ , then the extended  $\mathcal{F}$  defined by (5.2.1) is an bijection from  $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$  to  $\mathcal{M}_+(\hat{\mathcal{A}})$ .*

*Proof.* First we will show that  $\mathcal{F}$  is injective. Suppose that  $\mathcal{F}\left(\frac{f}{\Lambda_a}\right) = 0$ , that is  $\frac{1}{\hat{a}}\mu_f = 0$ , which implies  $\mu_f = 0$ . Therefore  $f(x) = 0$  because  $f(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu_f(\xi)$ . So,  $\frac{f}{\Lambda_a} = 0$  and  $\mathcal{F}$  is injective.

Next we will show that  $\mathcal{F}$  is surjective. Let  $\mu \in \mathcal{M}_+(\hat{\mathcal{A}})$ . There are  $a_n \in \mathcal{A}$  such that  $\hat{a}_n \geq 0$ ,  $\text{supp } \hat{a}_n$  is compact, and such that for every  $\xi \in \hat{\mathcal{A}}$  there exists an  $n$  such that  $\hat{a}_n(\xi) > 0$ . Then  $\hat{a}_n\mu$  is a finite positive Radon measure on  $\hat{\mathcal{A}}$  for every  $n \in \mathbb{N}$ . There exist positive numbers  $\lambda_1, \lambda_2, \dots$  such that  $\sum_{n=1}^{\infty} \lambda_n \hat{a}_n\mu$  defines a finite positive Radon measure on  $\hat{\mathcal{A}}$  and  $\sum_{n=1}^{\infty} \|\lambda_n a_n\| < \infty$ . By Maltese's theorem there exists  $f \in \mathcal{P}(\mathcal{A})$  such that

$$\mu_f = \sum_{n=1}^{\infty} \lambda_n \hat{a}_n\mu.$$

Since  $\mathcal{A}$  is complete there exists  $a \in \mathcal{A}$  such that  $a = \sum_{n=1}^{\infty} \lambda_n a_n$ . Then  $\Lambda_a \in \mathcal{S}$  and  $\sum_{n=1}^{\infty} \lambda_n \hat{a}_n\mu = \hat{a}\mu$ . Thus

$$\mathcal{F}\left(\frac{f}{\Lambda_a}\right) = \frac{\mu_f}{\hat{a}} = \frac{\hat{a}\mu}{\hat{a}} = \mu.$$

□

**Theorem 5.2.7.** *The map  $\mathcal{F} : \mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S}) \rightarrow \mathcal{M}_+(\hat{\mathcal{A}})$  is a sequential homeomorphism.*

*Proof.* If  $F_n \xrightarrow{I} F$  in  $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$ , then  $F_n = \frac{f_n}{\Lambda_a}$ ,  $F = \frac{f}{\Lambda_a}$ , and  $f_n \rightarrow f$  in  $\mathcal{P}(\mathcal{A})$  for some  $f_n, f \in \mathcal{P}(\mathcal{A})$ , where  $f_n \rightarrow f$  means  $f_n(x) \rightarrow f(x)$  for all  $x \in \mathcal{A}$ . Consequently,

$$\int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu_{f_n}(\xi) \rightarrow \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu_f(\xi)$$

for all  $x \in \mathcal{A}$ . Since the involution in  $\mathcal{A}$  is symmetric and  $\Gamma(\mathcal{A}) = \{\hat{x} : x \in \mathcal{A}\}$  strongly separates points in  $\hat{\mathcal{A}}$  (see, for example, Theorem 2.2.7 in [16]). Thus by the Stone-Weirstrass

theorem  $\Gamma(\mathcal{A})$  is dense in  $\mathcal{C}_0(\hat{\mathcal{A}})$ . We obtain

$$\int_{\hat{\mathcal{A}}} \varphi(\xi) d\mu_{f_n}(\xi) \rightarrow \int_{\hat{\mathcal{A}}} \varphi(\xi) d\mu_f(\xi)$$

for all  $\varphi \in \mathcal{K}(\hat{\mathcal{A}})$ . Therefore,

$$\int_{\hat{\mathcal{A}}} \varphi(\xi) \frac{d\mu_{f_n}(\xi)}{\hat{a}(\xi)} \rightarrow \int_{\hat{\mathcal{A}}} \varphi(\xi) \frac{d\mu_f(\xi)}{\hat{a}(\xi)}$$

for all  $\varphi \in \mathcal{K}(\hat{\mathcal{A}})$ , which means that  $\mathcal{F}(F_n) \rightarrow \mathcal{F}(F)$  in  $\mathcal{M}_+(\hat{\mathcal{A}})$ .

Now assume  $\mu_n, \mu \in \mathcal{M}_+(\hat{\mathcal{A}})$  and

$$\int_{\hat{\mathcal{A}}} \varphi(\xi) d\mu_n(\xi) \rightarrow \int_{\hat{\mathcal{A}}} \varphi(\xi) d\mu(\xi)$$

for all  $\varphi \in \mathcal{K}(\hat{\mathcal{A}})$ . There exist  $\lambda_k > 0$ ,  $k \in \mathbb{N}$ , such that  $\sum_{k=1}^{\infty} \lambda_k \hat{a}_k \mu_n$  is a finite measure for

all  $n \in \mathbb{N}$  and  $\Lambda_a \in \mathcal{S}$ , where  $a = \sum_{k=1}^{\infty} \lambda_k a_k$ . Let

$$f_n = \mathcal{F}^{-1} \left( \sum_{k=1}^{\infty} \lambda_k \hat{a}_k \mu_n \right) = \mathcal{F}^{-1}(\hat{a} \mu_n)$$

and

$$f = \mathcal{F}^{-1} \left( \sum_{k=1}^{\infty} \lambda_k \hat{a}_k \mu \right) = \mathcal{F}^{-1}(\hat{a} \mu).$$

Then  $\mathcal{F}^{-1}(\mu_n) = \mathcal{F}^{-1} \left( \frac{\hat{a} \mu_n}{\hat{a}} \right) = \frac{f_n}{\Lambda_a}$  and  $\mathcal{F}^{-1}(\mu) = \mathcal{F}^{-1} \left( \frac{\hat{a} \mu}{\hat{a}} \right) = \frac{f}{\Lambda_a}$ . Moreover,

$$f_n(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) \hat{a}(\xi) d\mu_n(\xi) \rightarrow \int_{\hat{\mathcal{A}}} \hat{x}(\xi) \hat{a}(\xi) d\mu(\xi) = f(x)$$

for every  $x \in \mathcal{A}$ . Therefore  $\frac{f_n}{\Lambda_a} \xrightarrow{I} \frac{f}{\Lambda_a}$  in  $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$ . □

### 5.3 Examples

In this section we give some examples of spaces where the assumptions of Theorem 5.2.6 are satisfied.

#### Example 5.3.1. Locally compact groups

Let  $G$  be a locally compact abelian group. A continuous function  $f : G \rightarrow \mathbb{C}$  is called positive definite if

$$\sum_{k,l=1}^n c_k \bar{c}_l f(x_l^{-1} x_k) \geq 0$$

for all  $c_1, \dots, c_n \in \mathbb{C}$  and  $x_1, \dots, x_n \in G$  for any  $n \in \mathbb{N}$ . We denote the cone of positive definite functions on  $G$  by  $\mathcal{P}_+(G)$ . A character  $\alpha$  on  $G$  is a continuous homomorphism from  $G$  into the unit circle group  $\mathbb{T}$ . Let  $\widehat{G}$  denote the group of characters. By Bochner's theorem [13],  $f \in \mathcal{P}_+(G)$  if and only if there exists a unique bounded positive Radon measure  $\mu_f$  on  $\widehat{G}$  such that

$$f(x) = \int_{\widehat{G}} \hat{x} d\mu_f.$$

In [6] it was shown that, if  $\widehat{G}$  is  $\sigma$ -compact, then the map  $f \mapsto \mu_f$  defined by Bochner's theorem can be extended to a map from a space of pseudoquotients to all positive measures on  $\widehat{G}$ . That space of pseudoquotients was  $\mathcal{B}(\mathcal{P}_+(G), \mathcal{S})$  where

$$\mathcal{S} = \left\{ \varphi \in L^1(G) : \widehat{\varphi}(\xi) > 0 \text{ for all } \xi \in \widehat{G} \right\}.$$

We will show that this extension is a special case of the extension presented in this note.

Since the convolution algebra  $L^1(G)$  is regular, it satisfies  $\Sigma$ , as indicated in 5.2.1.

For  $\alpha \in \widehat{G}$  we define  $\varphi_\alpha : L^1(G) \rightarrow \mathbb{C}$  by

$$\varphi_\alpha(f) = \int_G f(x) \overline{\alpha(x)} dx,$$

where  $dx$  indicates the integral with respect to the Haar measure on  $G$ . The map  $\alpha \mapsto \varphi_\alpha$  is a bijection from  $\widehat{G}$  onto  $\widehat{L^1(G)}$  (see, for example, [16]). This allows us to identify  $\mathcal{M}_+(\widehat{G})$  and  $\mathcal{M}_+(\widehat{L^1(G)})$ . If  $f$  is a positive definite function on  $G$ , we define a positive functional on  $L^1(G)$  by

$$F(\psi) = \int_G f(x) \psi(x) dx$$

and a map from  $\mathcal{B}(\mathcal{P}_+(G), \mathcal{S})$  to  $\mathcal{B}(L^1(G), \mathcal{S})$  by  $\frac{f}{\psi} \mapsto \frac{F}{\Lambda_{\tilde{\psi}}}$ , where  $\tilde{\psi}(x) = \psi(x^{-1})$ . We will show that  $F$  is positive.

$$\begin{aligned} F(\psi * \psi^*) &= \int_G \psi * \psi^*(x) f(x) dx \\ &= \int_G \int_G \psi(xy^{-1}) \psi^*(y) f(x) dy dx \\ &= \int_G \int_G \psi(xy^{-1}) \overline{\psi(y^{-1})} f(x) dy dx \\ &= \int_G \int_G \psi(x) \overline{\psi(y^{-1})} f(xy) dy dx \\ &= \int_G \int_G \psi(x) \overline{\psi(y)} f(xy^{-1}) dy dx. \end{aligned}$$

Riemann sums for this integral are of the form  $\sum_{k,l=1}^n c_k \bar{c}_l f(x_l^{-1} x_k)$  which are nonnegative since  $f$  is positive definite. Therefore  $F(\psi * \psi^*) \geq 0$ . Thus for all  $\psi$  in  $L^1(G)$ ,

$$F(\psi) = \int_{\widehat{L^1(G)}} \varphi_\alpha(\psi) d\mu_F(\alpha)$$

$$\begin{aligned}
&= \int_{\widehat{L^1(G)}} \int_G \psi(x) \overline{\alpha(x)} dx d\mu_F(\alpha) \\
&= \int_G \int_{\hat{G}} \psi(x) \overline{\alpha(x)} d\mu_f(\alpha) dx.
\end{aligned}$$

Therefore for all  $\psi$  in  $L^1(G)$ ,

$$\int_G \psi(x) f(x) dx = \int_G \int_{\hat{G}} \psi(x) \overline{\alpha(x)} d\mu_f(\alpha) dx$$

which implies,

$$f(x) = \int_{\hat{G}} \overline{\alpha(x)} d\mu_f(\alpha).$$

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