DYNAMICAL INVARIANTS AND THE FLUID IMPULSE IN PLASMA MODELS

by

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ABSTRACT

Much progress has been made in understanding of plasmas through the use of the MHD equations and newer models such as Hall MHD and electron MHD. As with most equations of fluid behavior, these equations are nonlinear, and no general solutions can be found. The use of invariant structures allows limited predictions of fluid behavior without requiring a full solution of the underlying equations. The use of gauge transformation can allow the creation of new invariants, while differential geometry offers useful tools for constructing additional invariants from those that are already known. Using these techniques, new geometric, integral and topological invariants are constructed for Hall and electron MHD models. Both compressible and incompressible models are considered, where applicable. An application of topological invariants to magnetic reconnection is provided. Finally, a particular geometric invariant, which can be interpreted as the fluid impulse density, is studied in greater detail, its nature and invariance in plasma models is demonstrated, and its behavior is predicted in particular geometries under different models.

I would like to dedicate this dissertation to my parents. To my mother, without whose care and nurturing I would not have made it this far, and to my father, one of the greatest scientists I have ever known. From a young age, both filled me with a desire to understand the world.

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1 INTRODUCTION

It has been said that most of the ordinary matter in the universe is in the plasma state. On Earth, plasmas are rather rare. They can be found in lightning bolts, specialized electronics, the interiors of fusion reactors, high energy experiments and other exotic applications. While uncommon on Earth, most of the volume of a star is in the plasma state, as is much of the interstellar medium. **[Gurnett, Bhattacharjee]** The understanding of plasma behavior is key to an understanding of fusion reactions, stellar astrophysics and the solar wind, which, in turn, is a major contributor to space weather. Additionally, as technology advances, plasmas appear ever more frequently in new devices and applications.

A plasma is a charged, quasi-neutral fluid. In a plasma many of the particles have an electric charge. Having an electric charge, they generate, and react to, electric and magnetic fields as described by Maxwell's equations. While not all particles in a plasma must be charged, it must be a sufficient proportion such that electromagnetic interactions have a significant impact on the fluid behavior. Though the particles are charged, the plasma is quasi-neutral; the electric charge of each charged particle is shielded by a shell of particles of opposite charge, so that electrical forces are not felt beyond a distance known as the Debye length. Thus, in a plasma, there are no long range electrical interactions, only magnetic. Finally, the plasma must be a fluid. A fluid is a substance that responds to a force applied, at a point, by a local acceleration proportional to that force, and obeys a continuum model. A continuum model requires that the substance can be viewed as a continuous entity, and can be entirely described by properties which vary continuously. This continuity allows the description of the plasma in terms of differential equations. Since the particles that make up a fluid are discrete objects, in order to view the substance as continuous, we must describe it at a scale which is sufficiently large to ignore this discontinuity. On the other hand, in order for the properties of the fluid to be viewed as continuous, there must be a means for each particle to interact with nearby particles in a way that can diffuse this property to its neighbors. The continuity of properties also restricts the scale. The size scale must be sufficiently large that the properties are well defined, but sufficiently small that the rates of change of these properties in space and time are gradual. It is at this scale that the plasma models are valid.

Since plasmas are fluids, the equations governing plasmas are an extension of the hydrodynamic equations governing uncharged fluids. Thus, this research also includes extensive study of the behavior of hydrodynamic systems. However, a plasma must also satisfy Maxwell's equations, and these are used to modify the hydrodynamic equations in order to construct plasma models. It is well known that, due to nonlinearity, the hydrodynamic equations cannot be solved in general. Inclusion of Maxwell's equations makes solution even more difficult. Consequently, a variety of assumptions are made to reduce the problem to a finite set of partial differential equations.

Different assumptions result in different plasma models. The magnetohydrodynamic model (MHD) assumes, among other things, that fluid velocities are on the order of thermal velocities **[Fitzpatrick]**, **[Chen].** Hall MHD does not discard the Hall

Effect term, which is neglected in MHD. Hall MHD applies for length scales and time scales which are small compared to ion inertial length and cyclotron period, respectively. **[Huba]** Electron MHD assumes that accelerations are slowly varying, but time scales are so short that ions, having much more mass than electrons, have too little time to substantially react to applied forces, and are approximated to be stationary. Only electron motion matters in this case. **[Kingsep] [Jain]**

Despite the assumptions made in constructing various plasma models, the resulting equations are still nonlinear and, generally, unsolvable. Additionally, different techniques are frequently required for treating compressible versus incompressible fluids. To overcome this complexity, a variety of techniques have been developed to extract information from various models under restricted conditions, or to extract limited information under general conditions. One group of such techniques is the finding of invariants. Invariants are physical quantities or mathematical objects that maintain particular properties despite the effects of fluid motion. Once an invariant is identified, its future properties are known without needing to solve the fluid equations.

A variety of different types of invariants exist, and some have been found, for particular models, by a number of authors. Local, dynamical invariants have a geometrical nature which is preserved despite fluid flow. For example, the vorticity behaves as a frozen-in field for the incompressible hydrodynamic system. Topological invariants, like Moffatt's vortex ring knottedness **[Moffatt]** maintain topological properties under fluid motion. Finally, integral invariants maintain the numerical result of an integral, such as the total fluid impulse for an incompressible hydrodynamic fluid, as shown by **[Batchelor]**, or the famous Kelvin's Circulation Theorem. In addition to the finding of individual invariants, techniques have been demonstrated by [**Tur & Yanovsky**] and [**Hollmann**] for generating new invariants from ones that are known.

One, specific, geometric invariant has particular physical interest. It can be related to the fluid impulse density, as defined for the incompressible hydrodynamic case by **[Batchelor]**, The fluid impulse density is the local total momentum change required to generate a particular fluid motion from rest. It was shown, by **[Kuzmin],** to behave like a material surface element in the incompressible hydrodynamic case.

This research will focus on constructing and describing invariants for various models, with special focus on the newer plasma models, Hall MHD and electron MHD. We will also look at the formulation and behavior of the fluid impulse and fluid impulse density in compressible and incompressible hydrodynamic and plasma models.

1.1 Coordinate Systems and the Material Derivative

We begin with a description of the mathematical tools required for working with fluid models. There are two coordinate systems that are commonly used to specify a scalar or vector field over a fluid: the Eulerian system addresses locations within the fluid via the static laboratory coordinate system, while the Lagrangian system identifies material elements of the fluid which are carried along by the flow, and attaches physical properties to the mobile fluid element rather than a fixed location. Through the use of the material derivative, which we will now derive, we can describe changes in a Lagrangian field in terms of Eulerian coordinates, thus converting between the two systems.

Consider a scalar field, $\varphi(\vec{x},t)$, defined over a fluid with velocity field $\vec{v}(\vec{x},t)$. We want to compute how the field changes with respect to a particular fluid element that is being carried by the flow, given that we know how the field is changing in the lab frame. To find how the field on a fluid element has changed from time t to time $t+\Delta t$, we must also take into account the fact that the fluid element has moved from position $\vec{x}$ to position $\vec{x}+\Delta\vec{x} = \vec{x}+\vec{v}\Delta t$. Thus, we define a new derivative,

$$\begin{aligned}\frac{D\varphi}{Dt} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}[\varphi(\vec{x}+\Delta\vec{x},t+\Delta t)-\varphi(\vec{x},t)]\\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}[\varphi(\vec{x}+\Delta\vec{x},t+\Delta t)-\varphi(\vec{x},t+\Delta t)+\varphi(\vec{x},t+\Delta t)-\varphi(\vec{x},t)] \\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}[\varphi(\vec{x}+\Delta\vec{x},t+\Delta t)-\varphi(\vec{x},t+\Delta t)]+\lim_{\Delta t\to 0}\frac{1}{\Delta t}[\varphi(\vec{x},t+\Delta t)-\varphi(\vec{x},t)]\end{aligned} \quad . \tag{1.1}$$

The second limit is merely the partial derivative of $\varphi(\vec{x},t)$ with respect to t. The first part gives the change in $\varphi(\vec{x},t)$ for changing $\vec{x}$ with respect to changing t. However, since the change in $\vec{x}$ is given in terms of the change in t, we can employ the chain rule to give

$$\begin{aligned}&\lim_{\Delta t\to 0}\frac{1}{\Delta t}[\varphi(\vec{x}+\Delta\vec{x},t+\Delta t)-\varphi(\vec{x},t+\Delta t)]\\&=\frac{\partial\varphi(\vec{x},t)}{\partial\vec{x}}\cdot\frac{d\vec{x}}{dt}=\frac{\partial\varphi(\vec{x},t)}{\partial\vec{x}}\cdot\frac{\vec{v}dt}{dt}=\frac{\partial\varphi(\vec{x},t)}{\partial\vec{x}}\cdot\vec{v}=(\vec{v}\cdot\nabla)\varphi(\vec{x},t)\end{aligned}. \qquad (1.2)$$

Thus, the material derivative is given by

$$\frac{D\varphi}{Dt}=\frac{\partial\varphi}{\partial t}+(\vec{v}\cdot\nabla)\varphi\,, \qquad (1.3)$$

where the first term captures the change in the field at a particular fixed point in the lab frame, while the second term, $(\vec{v}\cdot\nabla)\varphi$, is the convective derivative, and captures the difference in the field at its current position compared to its value a small distance further along in the flow. Though this derivation was specific to a scalar field, the material derivative can also be applied to a vector field, with similar derivation and identical results.

1.2 The Equations of Hydrodynamics and Plasma Physics

The models we will be using to study plasma behavior are the equations of magnetohydrodynamics (MHD), Hall magnetohydrodynamics (Hall MHD), and electron magnetohydrodynamics (eMHD). Additionally, we will be using the equations of hydrodynamics which exhibit many of the features of the MHD models but are easier to analyze. In preparation, we will want to have an understanding of the various equations involved, and an understanding of the assumptions made during their derivations.

The simplest set of equations that we will be considering will be those of fluid mechanics. While these equations do not capture the physics of a plasma, they are similar to the plasma equations. This allows us to examine mathematical tools in a simpler arena, while still maintaining their relevancy to the more complex plasma equations.

The general form of the fluid equations is

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \\ &\rho \frac{D\vec{v}}{Dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{v} \end{aligned} , \tag{1.4}$$

The first equation is the continuity equation for mass, a statement of mass conservation, and codifies the equivalence of the change in density over time with the rate of flow of mass into a region. In the case that the fluid is incompressible, this equation is replaced by

$$\nabla \cdot \vec{v} = 0 . \tag{1.5}$$

The second equation is a mathematical statement of Newton's Second Law. The left hand side is mass density times acceleration, while the right lists the forces acting on the fluid: those due to external body forces, $\vec{F}$, fluid pressure, p , and viscosity, μ .

The equations of MHD are similar. Ignoring viscosity and external forces gives us the equations of ideal MHD:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \\
&\rho \frac{D\vec{v}}{Dt} = -\nabla p + (\nabla \times \vec{B}) \times \vec{B} \\
&\frac{3}{2}\frac{Dp}{Dt} + \frac{5}{2} p \nabla \cdot \vec{v} = 0 \\
&\frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right) = \left(\frac{\vec{B}}{\rho} \cdot \nabla\right)\vec{v}
\end{aligned} \quad . \qquad (1.6)$$

The first equation is identical to the hydrodynamic case. The second equation includes a term for the force on the fluid due to the magnetic field, $\vec{B}$. The third equation is a statement of energy conservation, with the change in momentum in the Lagrangian frame balanced by work done by the expansion of the fluid in opposition to pressure. The fourth describes the evolution of the magnetic field. For a detailed derivation of the MHD equations, see the **Appendix**.

Derivation of the MHD equations involves the use of Maxwell's equations which include equations for the electric field. The approach to eliminating the electric field in terms of the magnetic distinguishes MHD from Hall MHD (see **Appendix**). Ideal MHD assumes perfect conductivity. This implies

$$\vec{E} + \vec{v} \times \vec{B} = \vec{0} \qquad (1.7)$$

which is the same as the statement of Ohm's Law in a perfect conductor. Hall MHD includes a term corresponding to the Hall Effect, replacing the above equation with

$$\vec{E} + \vec{v} \times \vec{B} = \frac{\vec{J}}{ne} \times \vec{B}. \tag{1.8}$$

Here, $\vec{J}$ is the plasma current, $\vec{v}$ is the center of mass velocity, which is equal to the ion velocity, n is the particle number density and e is the electric charge. This leads to a Hall current term in the magnetic portion of the equation of motion, the second equation in (1.6), giving

$$\begin{gathered} \rho \frac{D\vec{v}}{Dt} = -\nabla p + \frac{1}{c} \vec{J} \times \vec{B} \\ p = p_e + p_i \end{gathered}. \tag{1.9}$$

This replacement of the electric field also alters the evolution equation for the magnetic field. The details of a useful form for that equation will be reserved until **section 2.3**.

In electron MHD, the ions are assumed to be stationary. All motion is due to electrons. All activity in this model is expressed in terms of the generalized magnetic field,

$$\vec{B}_e = \vec{B} - d_e{}^2 \nabla^2 \vec{B}, \tag{1.10}$$

where $d_e = \frac{c}{2e}\sqrt{\frac{m_e}{\pi n_e}}$ is the electron skin depth. The electron velocity is entirely dependent upon this field. The generalized magnetic field obeys

$$\frac{\partial \vec{B}_e}{\partial t} = \nabla \times (\vec{\mathrm{v}}_\mathrm{e} \times \vec{B}_e) \qquad (1.11)$$

and, in turn, gives the velocity field for electrons by

$$\vec{\mathrm{v}}_\mathrm{e} = -\frac{c}{4\pi ne}(\nabla \times \vec{B})\ . \qquad (1.12)$$

1.3 Gauge Transformations

Gauge transformations are powerful techniques which have seen much use in quantum field theory. The motivation behind a gauge transformation is an observation that equations governing the behavior of a system maybe be invariant under a particular transformation. For instance, we observe that probabilities determined from quantum mechanical wave functions are only dependent upon differences in phase, not upon the phase, itself. Since these probabilities determine the behavior of a system, we see that simultaneously shifting the phase of all wave functions, everywhere, will cause no change in any measurement. The simultaneous, universal phase shift is a global gauge transformation. A phase-shift that is spatially dependent, a local gauge transformation, is required to generate an observable effect. When we notice that passage through a magnetic field generates exactly this sort of phase shift, we can invert this relationship and explain the existence of the electromagnetic field as a consequence of local changes in phase. That is to say, the "difference in physics" due to a local gauge transformation in wave function phase manifests as the electromagnetic field. We then notice that Maxwell's equations have a corresponding uncertainty and only determine the electric and magnetic fields only up to an arbitrary time derivative of a scalar field and gradient of a potential, respectively.

$$\begin{aligned} \vec{B} &= \nabla \times (\vec{A} + \nabla \varphi) \\ E &= -\nabla\left(f - \frac{\partial \varphi}{\partial t}\right) + \frac{\partial \vec{A}}{\partial t} \end{aligned} \tag{1.13}$$

The arbitrary function is the "gauge field" for the electromagnetic field and is the mechanism through which the local gauge transformation acts. Other gauge transformations can be used to generate other fundamental forces. **[t'Hooft]**, **[Mills]**

The gauge transformation that we will consider is more approximate, but analogous. Consider a transformation of the velocity field given by

$$\vec{q} = \vec{v} + \nabla \varphi \tag{1.14}$$

[Kuzmin]. Here, the scalar field, φ, is the gauge. Many properties of a fluid system depend, not upon the velocity field itself, but the vorticity $\vec{\omega} = \nabla \times \vec{v}$. We notice that the flow by this new quantity has the same vorticity as the original velocity field. Insofar as a system is described by its vorticity, we see that the chosen transformation will not alter the system behavior. It is in this sense that this transformation is a gauge transformation.

As in quantum field theory, we can define an analog to global and local transformations for this gauge structure. Substitution of the transformation into the equation of motion for a fluid model yields

$$\frac{D\vec{q}}{Dt} + \frac{D\varphi}{Dt} = -\frac{\nabla p}{\rho} + \vec{F}(\vec{x}, t)$$

Where $\vec{F}(\vec{x}, t)$ varies depending upon the model. We see that, if the gradient of the gauge field is strictly a time independent function of Lagrangian variables only, i.e. $\frac{D\nabla\varphi}{Dt} = 0$, the velocity field is unaltered by the transformation, much as a quantum theoretic global gauge transformation. On the other hand, if φ is allowed to vary more generally, as will be the case for the gauges we will consider, we see that the transformed field does have a new behavior. This is a local gauge transformation and implies the existence of a gauge field, which, in our case carries momentum, and modifies the flow to account for the difference in motion.

In our attempts to find invariant structures, we will find that certain equations, particularly the equation of motion, do not have the right form to yield proper invariants.

Fortunately, we can use a gauge transformation to transform quantities that do not evolve in the proper ways into ones that do. The new quantity should be chosen such that it obeys many of the physically important equations of the original quantity, but has an evolution equation that is suitable for a particular application, such as the finding of invariants. There are many choices of gauge possible and each, upon substitution into the fluid equations, yields a new system of equations which may have more useful properties than the original fluid equations. For the barotropic hydrodynamic case, the choice of gauge given by

$$\frac{D\varphi}{Dt} = P - \frac{1}{2}|\vec{v}|^2 \tag{1.15}$$

is called the geometric gauge, because, in this gauge, for the hydrodynamic case, the field $\vec{q}$ becomes a geometrical invariant which evolves as the dual of a material surface element (equations (1.33) and (2.14))[**Kuzmin]**. It is this gauge in which we will be most interested, and a more detailed discussion for the hydrodynamic case is presented in **section 2.1**. With some modification, geometric gauges can be constructed for the plasma models, though these gauges and their physical implications have not been well studied. These are considered in **sections 2.2** through **2.5**. Important considerations and techniques for constructing new invariants via gauge transformations are developed in **sections 2.6** through **2.9**. The applications of other gauges to plasma models are considered in **section 2.10**.

1.4 The Fluid Impulse and Fluid Impulse Density

The fluid impulse, $\vec{I}$, is the total amount of momentum change required to generate a given fluid motion from rest. That is to say,

$$\frac{\partial \vec{I}}{\partial t} = \iiint_V \rho \vec{F} dV . \tag{1.16}$$

Here, the left side gives the change in the fluid impulse in Lagrangian coordinates, while the right gives the fluid density, ρ , times the external force per unit density, $\vec{F}$, integrated over the entirety of the fluid. That is to say, the change in the fluid impulse is the sum of all external, nonconservative forces. This is modified from **[Saffman]** who gives the fluid impulse per unit density for an incompressible, hydrodynamic fluid of uniform density. Care must be taken when defining the fluid impulse in this way, since the integral over an infinite fluid is, in general, not absolutely convergent.

The fluid impulse density, $\vec{p}$, is a vector field which gives the total momentum change at each location required to, from rest, generate a particular fluid motion at that location. It was shown, by **[Buttke (1993a), (1993b)]** and **[Russo & Smereka]**, that, for the incompressible, hydrodynamic case, $\vec{q}$, as defined in (1.14) with the geometric gauge (1.15) can be interpreted as the fluid impulse density per unit mass.

The fluid impulse and fluid impulse density per unit mass density has been examined by **[Shivamoggi (2011), (2007), (2009)]**. The definition of, and behavior of the fluid impulse and fluid impulse density for plasma models has not been well studied. **Chapter 3** discusses the fluid impulse density and its interpretation as the gauge transformed velocity field, $\vec{q}$, in the known incompressible hydrodynamic case, and then extends the definition to incompressible Hall MHD and eMHD and proves their invariance in these models. In **Chapter 6**, we make specific predictions for the behavior of the fluid impulse in cylindrical geometry for different initial conditions and for different models.

1.5 Varieties of Geometrical Dynamical Invariants

We now introduce the idea of dynamical invariants to extract geometrical properties from the fluid equations. A dynamical invariant is a structure that persists even as it is moved by the flow.

The material derivative, by capturing the change in a field due to the flow, gives us our first, and most familiar, dynamical invariant. A scalar field, I, having

$$\frac{DI}{Dt} = 0 \qquad (1.17)$$

is a Lagrangian invariant. The entirety of the change in this a scalar field in the lab frame is that which is due to its advection by the flow.

Consider, instead, a small vector in the fluid. We can compute the change in the vector in the lab frame due to its transport by the flow. That is to say, we first chose a vector at a location in the lab frame at an initial time and identify the material elements at its endpoints. These material elements will be transported by the flow. A short time, Δt, later, we compute the vector difference between the new locations of these fluid elements. We then subtract the new vector from the original vector to find the change in the vector due to the flow. Once we see how a vector field is changed by fluid motion, we will be able to decompose the time evolution of an vector field into a component due to the action of the flow and a component relative to the flow.

For a vector field, $\vec{J}(\vec{x},t)$, $\frac{D\vec{J}}{Dt}$ represents the differential change in the vector field due to the motion of the flow. This differential change can be found by the limit

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}[\vec{J}(\vec{x}+\Delta\vec{x}, t+\Delta t) - \vec{J}(\vec{x},t)]. \tag{1.18}$$

If this vector represents a material line element then the only change in the vector $\vec{J}(\vec{x},t)$ is due to transport by the fluid. In this case, the above expression should be equal to the change in the difference between the positions of the material elements associated with the endpoints of the vector. Therefore, we have

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}[(\vec{x}+\vec{J}+\vec{v}(\vec{x}+\vec{J})\Delta t)-(\vec{x}+\vec{v}(\vec{x})\Delta t)]-[(\vec{x}+\vec{J})-\vec{x}]$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\vec{v}(\vec{x}+\vec{J})\Delta t-\vec{v}(\vec{x})\Delta t] \quad . \tag{1.19}$$

Note that, in the above, $\vec{v}(\vec{x})$ refers to the value of the velocity field at $\vec{x}$, not multiplication. If we assume $\vec{J}(\vec{x},t)$ to be a differential line element, we can expand around $\vec{x}$ giving

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}[\vec{v}(\vec{x})\Delta t+\nabla\vec{v}\cdot\vec{J}\Delta t+\frac{1}{2}\nabla_j\nabla_i\vec{v}\vec{J}_i\vec{J}_j\Delta t+...-\vec{v}(\vec{x})\Delta t]$$
$$=\nabla\vec{v}\cdot\vec{J}+\nabla_j\nabla_i\vec{v}\vec{J}_i\vec{J}_j+... \quad . \tag{1.20}$$

Then, taking the leading term gives

$$\nabla\vec{v}\cdot\vec{J} \; . \tag{1.21}$$

Thus, for a material line element

$$\frac{D\vec{J}}{Dt}=\nabla\vec{v}\cdot\vec{J} \tag{1.22}$$

[Batchelor] or

$$\frac{D\vec{J}}{Dt} = (\vec{J} \cdot \nabla)\vec{v} . \tag{1.23}$$

Any vector field which obeys this same equation will behave exactly as a material line element advected by the flow.

A third type of dynamical invariant is provided by continuity equations. Continuity equations correspond to conserved quantities. For example, in fluid mechanics, we have the conservation of mass stated as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 . \tag{1.24}$$

The fourth type of equation is related to surfaces advected by the flow. We consider, not the surfaces themselves, but the duals to such surfaces via the volume form, ρ. Duals and volume forms are discussed in **section 1.6**, but, for our purposes here, we note that the relevant quantity, $\vec{S}(\vec{x},t)$, can be found by taking the vector normal to a material surface and multiplying it by ρ. One way to find the equation describing the behavior of $\vec{S}(\vec{x},t)$ is, again, to consider the conservation of mass. For a line element, $\vec{J}$, which is advected by the flow, $\vec{J} \cdot \vec{S}$ is a incompressible volume element that is advected by the flow, and $\rho \vec{J} \cdot \vec{S}$ is its mass, which should be conserved. Thus, we have

$$\frac{\partial(\rho\vec{J}\cdot\vec{S})}{\partial t}+\nabla\cdot\left(\rho\vec{v}(\vec{J}\cdot\vec{S})\right)=0. \tag{1.25}$$

Converting to tensorial notation for convenience gives

$$\frac{\partial(\rho J_i S_i)}{\partial t}+\frac{\partial}{\partial x_i}(\rho v J_j S_j)_i=0, \tag{1.26}$$

which gives

$$\left(\frac{\partial\rho}{\partial t}J_i S_i+\rho\frac{\partial J_i}{\partial t}S_i+\rho J_i\frac{\partial S_i}{\partial t}\right)+\left(\frac{\partial(\rho v)_i}{\partial x_i}J_j S_j+(\rho v)_i\frac{\partial J_j}{\partial x_i}S_j+(\rho v)_i J_j\frac{\partial S_j}{\partial x_i}\right)=0. \tag{1.27}$$

By mass conservation, $\left(\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)_i}{\partial x_i}\right)J_j S_j=0$, giving

$$\left(\rho\frac{\partial J_i}{\partial t}S_i+\rho J_i\frac{\partial S_i}{\partial t}\right)+\left((\rho v)_i\frac{\partial J_j}{\partial x_i}S_j+(\rho v)_i J_j\frac{\partial S_j}{\partial x_i}\right)=0. \tag{1.28}$$

Changing indices results in

$$\left(\rho\frac{\partial J_i}{\partial t}S_i+\rho J_i\frac{\partial S_i}{\partial t}\right)+\left((\rho v)_j\frac{\partial J_i}{\partial x_j}S_i+(\rho v)_j J_i\frac{\partial S_i}{\partial x_j}\right)$$
$$=\rho\left(\frac{\partial J_i}{\partial t}+v_j\frac{\partial J_i}{\partial x_j}\right)S_i+\rho J_i\frac{\partial S_i}{\partial t}+(\rho v)_j J_i\frac{\partial S_i}{\partial x_j}=0 \quad . \tag{1.29}$$

Then, since $\vec{J}$ is a material line element, we have

$$\rho\left(\frac{\partial v_i}{\partial x_j}J_j\right)S_i+\rho J_i\frac{\partial S_i}{\partial t}+(\rho v)_j J_i\frac{\partial S_i}{\partial x_j}=0\,. \tag{1.30}$$

Changing indices again gives

$$\rho\left(\frac{\partial v_j}{\partial x_i}S_j+\frac{\partial S_i}{\partial t}+v_j\frac{\partial S_i}{\partial x_j}\right)J_i=0\,. \tag{1.31}$$

Finally, ρ can be canceled from both sides, and since the above must be true for any $\vec{J}$, we must have

$$\frac{\partial S_i}{\partial t}+v_j\frac{\partial S_i}{\partial x_j}=-\frac{\partial v_j}{\partial x_i}S_j\,, \tag{1.32}$$

or

$$\frac{D\vec{S}}{Dt} = -(\nabla v)^T \cdot S \,. \tag{1.33}$$

This, then, is the equation obeyed by vectors that behave as if they were dual to material surface elements. **[Batchelor]**

1.6 Connection to Differential Geometry

Though we can use the four equations derived above to identify Lagrangian invariants (1.17), conserved quantities (1.24), and vector fields that behave either like material line elements (1.22) or duals to material surface elements (1.33), we are still left with the problem of finding such dynamical invariants. The process of finding new dynamical invariants is simplified dramatically by using the language of exterior calculus as shown by **[Tur & Yanovsky]**. The equations for the four dynamical invariants, though very different in coordinate form are unified into one equation in differential geometry. However, to use the power of differential geometry, we must see how to convert the partial differential equations in the previous section into the proper language.

Differential k-forms, $\tilde{\omega}_k$, are fully anti-symmetric tensor products of elements of the co-tangent bundle. **[Schutz]** That is to say, if we consider a given point on a manifold, we can find the set of all tangent vectors to the manifold at that point. The set of all such vectors is the tangent space at that point. The vector space dual of a tangent vector at a point is a one-form at that point. Dual, meaning, here, that when the one-form is combined with the vector, the result is a scalar. The collection of all one-forms at the point give the dual of the tangent space, called the co-tangent space. If we now construct a co-tangent space at every point of the manifold, this collection of spaces is the co-tangent bundle, and a one-form, $\tilde{\omega}_1$, is a collection of elements, one from each co-tangent space, chosen so that they vary continuously. These one-forms can then be combined together by using the tensor product, $\otimes$, to form tensors of higher rank. The wedge product, $\wedge$, is a fully anti-symmetric tensor product. That is to say, it is a tensor product in which an exchange of any two adjacent factors negates the value of the product. A wedge product of k one-forms is a differential k-form, $\tilde{\omega}_k$ or just $\tilde{\omega}$. Wedge products of tangent vectors can also be constructed, yielding k-vector fields.

The Lie derivative, $\mathcal{L}_{\vec{v}}$, along a vector field, $\vec{v}$, of a k-form or k-vector field, describes how the k-form or k-vector field changes as it is dragged along the vector field, $\vec{v}$. If we take $\vec{v}$ to be the velocity field of a fluid, then, we may expect, intuitively, that the Lie derivative is related to the convective derivative, as defined in **section 1.1**, for the flow. Let us consider the coordinate representation of the operator $\partial_t + \mathcal{L}_{\vec{v}}$ on various k-forms to determine what this relationship might be.

The action of the Lie derivative on a form, $\tilde{\omega}$, can be expressed in terms of differentials as

$$\mathcal{L}_{\vec{v}}\tilde{\omega} = d[\tilde{\omega}(\vec{v})] + (d\tilde{\omega})(\vec{v}), \tag{1.34}$$

where $\tilde{\omega}(\vec{v})$ is the contraction of the form with the vector. Here, the operator, d, is the differential. The differential takes a k-form and returns a (k+1)-form. In coordinates, the differential operates on a form $\tilde{\omega} = a_{ij\ldots} dx^i \wedge dx^j \wedge \ldots$ by

$$d[a_{ij\ldots} dx^i \wedge dx^j \wedge \ldots] = \frac{\partial a_{ij\ldots}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j \wedge \ldots \tag{1.35}$$

In $\mathbb{R}^3$ we have 4 ranks of k-forms to choose from. A 0-form is merely a smooth function, $\tilde{\omega}_0 = I$. Applying the operator $\partial_t + \mathcal{L}_{\vec{v}}$ gives

$$\begin{aligned} &\partial_t I + \mathcal{L}_{\vec{v}} I \\ &= \partial_t I + d[I(\vec{v})] + (dI)(\vec{v}) \\ &= \partial_t I + d[I(\vec{v})] + (\partial_{x^i} I) v^i \end{aligned} \tag{1.36}$$

Since the contraction of a scalar with a vector is, by definition, zero we have

$$\partial_t I + \mathcal{L}_{\vec{v}} I = \partial_t I + (\partial_{x^i} I) v^i \,, \tag{1.37}$$

which has a coordinate representation of

$$\partial_t I + \mathcal{L}_{\vec{v}} I = \frac{\partial I}{\partial t} + v \cdot \nabla I \\ = \frac{DI}{Dt} . \tag{1.38}$$

If we set this equal to zero, we have exactly the time evolution equation of a Lagrangian invariant, (1.17).

To continue, we must define the dual operation. The dual is defined on k-vectors as a contraction with a particular n-form called the volume form, where n is the dimension of the manifold: 3, for our purposes. The volume form for the fluid systems will come from the mass density,

$$\tilde{\omega}_{vol} = \frac{1}{3!} \rho_{ijk} dx^i \wedge dx^j \wedge dx^k , \tag{1.39}$$

where

$$\rho_{ijk} = \varepsilon_{ijk} \rho . \tag{1.40}$$

The dual of a k-vector is a (n-k)-form. We can, similarly, define the dual on k-forms by a contraction with a 3-vector, the inverse of the volume form, such that successive applications of the dual give the identity. This implies that the dual of a k-form is a (n-k)-vector.

For incompressible systems, the mass density can often be taken to be constant. In these cases, the volume form is just the trivial unit 3-form. It is the compressible cases that are more interesting. When converted to coordinate form, conversion of forms and vectors into their duals will result in multiplication by or division by ρ.

Next, with that in place, consider a scalar field, $f(\vec{x},t)$. Suppose that this scalar field is the coefficient of a 3-form

$$\tilde{\omega}_3 = \frac{1}{3!} f_{ijk} dx^i \wedge dx^j \wedge dx^k \tag{1.41}$$

with $f_{ijk} = \varepsilon_{ijk} f$. Suppose that this 3-form satisfies $\partial_t \tilde{\omega}_3 + \mathcal{L}_{\vec{v}} \tilde{\omega}_3 = 0$. Applying this operator and converting to component form gives

$$\begin{aligned}
&\partial_t \tilde{\omega}_3 + \mathcal{L}_{\vec{v}} \tilde{\omega}_3 \\
&= \partial_t \tilde{\omega}_3 + d[\tilde{\omega}_3(\vec{v})] + (d\tilde{\omega}_3)(\vec{v}) \\
&= \partial_t \frac{1}{3!} f_{ijk} dx^i \wedge dx^j \wedge dx^k + d[\frac{1}{3!} f_{ijk} v_i dx^j \wedge dx^k - \frac{1}{3!} f_{ijk} v_j dx^i \wedge dx^k \; . \\
&+ \frac{1}{3!} f_{ijk} v_k dx^i \wedge dx^j] + (d \frac{1}{3!} f_{ijk} dx^i \wedge dx^j \wedge dx^k)(\vec{v}) = 0
\end{aligned} \tag{1.42}$$

By a renaming indices and utilizing the antisymmetry of ρ_{ijk}, we have

$$-\frac{1}{3!} f_{ijk} v_j dx^i \wedge dx^k = -\frac{1}{3!} f_{jik} v_i dx^j \wedge dx^k = \frac{1}{3!} f_{ijk} v_i dx^j \wedge dx^k$$
$$\frac{1}{3!} f_{ijk} v_k dx^i \wedge dx^j = \frac{1}{3!} f_{jki} v_i dx^j \wedge dx^k = \frac{1}{3!} f_{ijk} v_i dx^j \wedge dx^k \tag{1.43}$$

giving

$$\partial_t \tilde{\omega}_3 + \mathcal{L}_{\vec{v}} \tilde{\omega}_3 = \partial_t \frac{1}{3!} f_{ijk} dx^i \wedge dx^j \wedge dx^k + d[\frac{1}{2!} f_{ijk} v_i dx^j \wedge dx^k] + (d \frac{1}{3!} f_{ijk} dx^i \wedge dx^j \wedge dx^k)(\vec{v}) = 0$$
$$. \tag{1.44}$$

Since there are no 4-forms in $\mathbb{R}^3$, we have $(d \frac{1}{3!} f_{ijk} dx^i \wedge dx^j \wedge dx^k) = 0$. This gives

$$\partial_t \frac{1}{3!} f_{ijk} dx^i \wedge dx^j \wedge dx^k + \frac{1}{2!} \frac{\partial(f_{ijk} v_i)}{\partial x^l} dx^l \wedge dx^j \wedge dx^k = 0 . \tag{1.45}$$

Expanding the second term and accounting for antisymmetry gives

$$\frac{1}{2!} \frac{\partial(f_{ijk} v_i)}{\partial x^l} dx^l \wedge dx^j \wedge dx^k = \frac{1}{2!} \frac{\partial(f_{123} v_1)}{\partial x^1} dx^1 \wedge dx^2 \wedge dx^3 + \frac{1}{2!} \frac{\partial(f_{132} v_1)}{\partial x^1} dx^1 \wedge dx^3 \wedge dx^2$$
$$+ \frac{1}{2!} \frac{\partial(f_{231} v_2)}{\partial x^2} dx^2 \wedge dx^3 \wedge dx^1 + \frac{1}{2!} \frac{\partial(f_{213} v_2)}{\partial x^2} dx^2 \wedge dx^1 \wedge dx^3$$
$$+ \frac{1}{2!} \frac{\partial(f_{312} v_3)}{\partial x^3} dx^3 \wedge dx^1 \wedge dx^2 + \frac{1}{2!} \frac{\partial(f_{321} v_3)}{\partial x^3} dx^3 \wedge dx^2 \wedge dx^1 \qquad .(1.46)$$
$$= \frac{\partial(f_{123} v_1)}{\partial x^1} dx^1 \wedge dx^2 \wedge dx^3 + \frac{\partial(f_{231} v_2)}{\partial x^2} dx^2 \wedge dx^3 \wedge dx^1 + \frac{\partial(f_{312} v_3)}{\partial x^3} dx^3 \wedge dx^1 \wedge dx^2$$

Now, permuting the wedge products and using the antisymmetry of f_{ijk} gives

$$\begin{aligned}\frac{1}{2!}\frac{\partial(f_{ijk}v_i)}{\partial x^l}dx^l\wedge dx^j\wedge dx^k &= \frac{\partial(f_{123}v_1)}{\partial x^1}+\frac{\partial(f_{123}v_2)}{\partial x^2}+\frac{\partial(f_{123}v_3)}{\partial x^3}dx^1\wedge dx^2\wedge dx^3 \\ &= \frac{\partial(f_{123}v_m)}{\partial x^m}dx^1\wedge dx^2\wedge dx^3\end{aligned} \quad . \tag{1.47}$$

The above equation is in index sorted form. Since all of its coefficients are independent of index up to antisymmetry, we can convert back to unordered form by simply taking

$$\frac{\partial(f_{123}v_m)}{\partial x^m}dx^1\wedge dx^2\wedge dx^3 = \frac{1}{3!}\frac{\partial(f_{ijk}v_m)}{\partial x^m}dx^i\wedge dx^j\wedge dx^k\ . \tag{1.48}$$

Substituting this into (1.45) gives

$$\begin{aligned}&\partial_t\frac{1}{3!}f_{ijk}dx^i\wedge dx^j\wedge dx^k+\frac{1}{3!}\frac{\partial(f_{ijk}v_m)}{\partial x^m}dx^i\wedge dx^j\wedge dx^k = \\ &\frac{1}{3!}\varepsilon_{ijk}\partial_t f dx^i\wedge dx^j\wedge dx^k+\frac{1}{3!}\varepsilon_{ijk}\frac{\partial(fv_m)}{\partial x^m}dx^i\wedge dx^j\wedge dx^k\ . \\ &\frac{1}{3!}\varepsilon_{ijk}\left(\partial_t f+\frac{\partial(fv_m)}{\partial x^m}\right)dx^i\wedge dx^j\wedge dx^k = 0\end{aligned} \tag{1.49}$$

Thus, f must satisfy the continuity equation

$$\frac{\partial f}{\partial t} + \nabla \cdot (f\vec{v}) = 0. \tag{1.50}$$

We would now like to reexamine the behavior of surface elements. A surface element is a 2-vector. We can transform the 2-vector into a one-vector, the vector normal to the surface element, by use of the Hodge dual. Instead, we will consider the volume form dual, which will yield a one-form, $\tilde{\omega}_1 = S_i dx^i$. Applying the operator $\partial_t \tilde{\omega} + \mathcal{L}_{\vec{v}} \tilde{\omega} = 0$ to the one-form gives

$$\begin{aligned}
&\partial_t \tilde{\omega}_1 + \mathcal{L}_{\vec{v}} \tilde{\omega}_1 \\
&= \partial_t S_i dx^i + \mathcal{L}_{\vec{v}} S_i dx^i \\
&= \partial_t S_i dx^i + d[S_i dx^i(\vec{v})] + (d[S_i dx^i])(\vec{v}) \\
&= \partial_t S_i dx^i + d[S_i v^i] + (d[S_i dx^i])(\vec{v}) = 0
\end{aligned}. \tag{1.51}$$

Proceeding by the product rule leads to

$$\partial_t S_i dx^i + d[S_i]v^i + S_i d[v^i] + (d[S_i dx^i])(\vec{v}) = 0. \tag{1.52}$$

Then, converting the differentials into coordinate form gives

$$\partial_t S_i dx^i + \frac{\partial S_i}{\partial x^j} v^i dx^j + S_i \frac{\partial v^i}{\partial x^j} dx^j + \left(\frac{\partial S_i}{\partial x^j} dx^j \wedge dx^i \right)(\vec{v}) = 0 \,. \tag{1.53}$$

Then, we have

$$\begin{aligned} &\partial_t S_i dx^i + \frac{\partial S_i}{\partial x^j} v^i dx^j + S_i \frac{\partial v^i}{\partial x^j} dx^j + \frac{\partial S_i}{\partial x^j} v^j dx^i - \frac{\partial S_i}{\partial x^j} v^i dx^j \\ &= \partial_t S_i dx^i + S_i \frac{\partial v^i}{\partial x^j} dx^j + \frac{\partial S_i}{\partial x^j} v^j dx^i \\ &= \frac{\partial \vec{S}}{\partial t} + (\nabla v)^T \cdot S + (\vec{v} \cdot \nabla)\vec{S} = 0 \end{aligned} \,. \tag{1.54}$$

This gives

$$\frac{D\vec{S}}{Dt} = -(\nabla v)^T \cdot S \,, \tag{1.55}$$

which is exactly the evolution equation for the dual of a material surface element, (1.33).

Finally, we would like to consider the coordinate representation of $\partial_t \tilde{\omega} + \mathcal{L}_{\vec{v}} \tilde{\omega} = 0$ applied to a 2-form. There are no 2-forms in traditional vector calculus. Instead, they arise as the duals of vectors. Consider the vector

$$\vec{J} = J^k dx_k \,. \tag{1.56}$$

We take its dual to create the 2-form

$$\tilde{\omega}_2 = \frac{\rho}{2!} J_{ij} dx^i \wedge dx^j , \tag{1.57}$$

where

$$J_{ij} = \varepsilon^{ij}{}_k J^k \tag{1.58}$$

and

$$\frac{\rho}{2!} dx^i \wedge dx^j = \varepsilon^{ij}{}_k \frac{\rho}{2!} dx^k = \varepsilon^{ijk} \frac{1}{2!} dx_k . \tag{1.59}$$

In the above expression, the first equality is the Hodge dual. The equality between the second and third quantities is the vector space dual. Their composition, i.e. the equality between the first and third quantities, is the dual. **[Schutz]** The factor of one half remains, because the indices are not summed, as they were in (1.57), and we must avoid overcounting duplicates. Applying $\partial_t \tilde{\omega} + \mathcal{L}_{\vec{v}} \tilde{\omega} = 0$ to the 2-form gives

$$
\begin{aligned}
&\partial_t \tilde{\omega}_2 + \mathcal{L}_{\vec{v}} \tilde{\omega}_2 \\
&= \partial_t \tilde{\omega}_2 + d[\tilde{\omega}_2(\vec{v})] + (d\tilde{\omega}_2)(\vec{v}) \\
&= \partial_t \frac{\rho}{2!} J_{ij} dx^i \wedge dx^j + d[\frac{\rho}{2!} J_{ij} dx^i \wedge dx^j (\vec{v})] + (d\frac{\rho}{2!} J_{ij} dx^i \wedge dx^j)(\vec{v}) \\
&= \partial_t \frac{\rho}{2!} J_{ij} dx^i \wedge dx^j + d[\frac{\rho}{2!} J_{ij} v_i dx^j - \frac{\rho}{2!} J_{ij} v_j dx^i] + \frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k} dx^k \wedge dx^i \wedge dx^j\right)(\vec{v}) \quad . \ (1.60) \\
&= \partial_t \frac{\rho}{2!} J_{ij} dx^i \wedge dx^j + \frac{1}{2!}\frac{\partial(\rho J_{ij} v_i)}{\partial x^k} dx^k \wedge dx^j - \frac{1}{2!}\frac{\partial(\rho J_{ij} v_j)}{\partial x^k} dx^k \wedge dx^i \\
&+ \frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k} v_k dx^i \wedge dx^j\right) - \frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k} v_i dx^k \wedge dx^j\right) + \frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k} v_j dx^k \wedge dx^i\right) = 0
\end{aligned}
$$

Expanding using product rule gives

$$
\begin{aligned}
&\frac{1}{2!} J_{ij} \partial_t \rho dx^i \wedge dx^j + \frac{1}{2!} \rho \partial_t J_{ij} dx^i \wedge dx^j + \frac{1}{2!}\frac{\partial(v_i)}{\partial x^k} \rho J_{ij} dx^k \wedge dx^j - \frac{1}{2!}\frac{\partial(v_j)}{\partial x^k} \rho J_{ij} dx^k \wedge dx^i \\
&+ \frac{1}{2!}\frac{\partial(\rho J_{ij})}{\partial x^k} v_i dx^k \wedge dx^j - \frac{1}{2!}\frac{\partial(\rho J_{ij})}{\partial x^k} v_j dx^k \wedge dx^i + \frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k} v_k dx^i \wedge dx^j\right) \qquad .(1.61) \\
&- \frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k} v_i dx^k \wedge dx^j\right) + \frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k} v_j dx^k \wedge dx^i\right) = 0
\end{aligned}
$$

Canceling terms gives

$$
\begin{aligned}
&\frac{1}{2!} J_{ij} \partial_t \rho dx^i \wedge dx^j + \frac{1}{2!} \rho \partial_t J_{ij} dx^i \wedge dx^j + \frac{1}{2!}\frac{\partial(v_i)}{\partial x^k} \rho J_{ij} dx^k \wedge dx^j - \frac{1}{2!}\frac{\partial(v_j)}{\partial x^k} \rho J_{ij} dx^k \wedge dx^i \\
&+ \frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k} v_k dx^i \wedge dx^j\right) = 0 \qquad .(1.62)
\end{aligned}
$$

Now, changing indices on the fourth term gives

$$\frac{1}{2!}J_{ij}\partial_t\rho dx^i\wedge dx^j+\frac{1}{2!}\rho\partial_t J_{ij}dx^i\wedge dx^j+\frac{1}{2!}\frac{\partial(v_i)}{\partial x^k}\rho J_{ij}dx^k\wedge dx^j-\frac{1}{2!}\frac{\partial(v_i)}{\partial x^k}\rho J_{ji}dx^k\wedge dx^j$$
$$+\frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k}v_k dx^i\wedge dx^j\right)=0 \quad ,(1.63)$$

and, by the antisymmetry of J_{ij} ,

$$\frac{1}{2!}J_{ij}\partial_t\rho dx^i\wedge dx^j+\frac{1}{2!}\rho\partial_t J_{ij}dx^i\wedge dx^j+\frac{\partial(v_i)}{\partial x^k}\rho J_{ij}dx^k\wedge dx^j+\frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k}v_k dx^i\wedge dx^j\right)=0 \quad .(1.64)$$

Expanding the third term gives

$$\frac{\partial(v_i)}{\partial x^k}\rho J_{ij}dx^k\wedge dx^j=\frac{\partial(v_3)}{\partial x^1}\rho J_{32}dx^1\wedge dx^2+\frac{\partial(v_2)}{\partial x^1}\rho J_{23}dx^1\wedge dx^3+\frac{\partial(v_3)}{\partial x^2}\rho J_{31}dx^2\wedge dx^1$$
$$+\frac{\partial(v_1)}{\partial x^2}\rho J_{13}dx^2\wedge dx^3+\frac{\partial(v_2)}{\partial x^3}\rho J_{21}dx^3\wedge dx^1+\frac{\partial(v_1)}{\partial x^3}\rho J_{12}dx^3\wedge dx^2 \quad .(1.65)$$

Now, we split each term in half

$$\frac{\partial(v_i)}{\partial x^k}\rho J_{ij}dx^k\wedge dx^j=\frac{1}{2}\frac{\partial(v_3)}{\partial x^1}\rho J_{32}dx^1\wedge dx^2+\frac{1}{2}\frac{\partial(v_3)}{\partial x^1}\rho J_{32}dx^1\wedge dx^2+\frac{1}{2}\frac{\partial(v_2)}{\partial x^1}\rho J_{23}dx^1\wedge dx^3$$
$$+\frac{1}{2}\frac{\partial(v_2)}{\partial x^1}\rho J_{23}dx^1\wedge dx^3+\frac{1}{2}\frac{\partial(v_3)}{\partial x^2}\rho J_{31}dx^2\wedge dx^1+\frac{1}{2}\frac{\partial(v_3)}{\partial x^2}\rho J_{31}dx^2\wedge dx^1$$
$$+\frac{1}{2}\frac{\partial(v_1)}{\partial x^2}\rho J_{13}dx^2\wedge dx^3+\frac{1}{2}\frac{\partial(v_1)}{\partial x^2}\rho J_{13}dx^2\wedge dx^3+\frac{1}{2}\frac{\partial(v_2)}{\partial x^3}\rho J_{21}dx^3\wedge dx^1$$
$$+\frac{1}{2}\frac{\partial(v_2)}{\partial x^3}\rho J_{21}dx^3\wedge dx^1+\frac{1}{2}\frac{\partial(v_1)}{\partial x^3}\rho J_{12}dx^3\wedge dx^2+\frac{1}{2}\frac{\partial(v_1)}{\partial x^3}\rho J_{12}dx^3\wedge dx^2 \quad ,(1.66)$$

anticommute half of the wedge products

$$\frac{1}{2}\frac{\partial(v_3)}{\partial x^1}\rho J_{32}dx^1\wedge dx^2-\frac{1}{2}\frac{\partial(v_3)}{\partial x^1}\rho J_{32}dx^2\wedge dx^1+\frac{1}{2}\frac{\partial(v_2)}{\partial x^1}\rho J_{23}dx^1\wedge dx^3$$
$$-\frac{1}{2}\frac{\partial(v_2)}{\partial x^1}\rho J_{23}dx^3\wedge dx^1+\frac{1}{2}\frac{\partial(v_3)}{\partial x^2}\rho J_{31}dx^2\wedge dx^1-\frac{1}{2}\frac{\partial(v_3)}{\partial x^2}\rho J_{31}dx^1\wedge dx^2$$
$$+\frac{1}{2}\frac{\partial(v_1)}{\partial x^2}\rho J_{13}dx^2\wedge dx^3-\frac{1}{2}\frac{\partial(v_1)}{\partial x^2}\rho J_{13}dx^3\wedge dx^2+\frac{1}{2}\frac{\partial(v_2)}{\partial x^3}\rho J_{21}dx^3\wedge dx^1 ,$$
$$-\frac{1}{2}\frac{\partial(v_2)}{\partial x^3}\rho J_{21}dx^1\wedge dx^3+\frac{1}{2}\frac{\partial(v_1)}{\partial x^3}\rho J_{12}dx^3\wedge dx^2-\frac{1}{2}\frac{\partial(v_1)}{\partial x^3}\rho J_{12}dx^2\wedge dx^3 \quad (1.67)$$

and regroup

$$\frac{1}{2}\rho\left(\frac{\partial(v_3)}{\partial x^1}J_{32}-\frac{\partial(v_3)}{\partial x^2}J_{31}\right)dx^1\wedge dx^2+\frac{1}{2}\rho\left(\frac{\partial(v_2)}{\partial x^1}J_{23}-\frac{\partial(v_2)}{\partial x^3}J_{21}\right)dx^1\wedge dx^3$$
$$+\frac{1}{2}\rho\left(\frac{\partial(v_3)}{\partial x^2}J_{31}-\frac{\partial(v_3)}{\partial x^1}J_{32}\right)dx^2\wedge dx^1+\frac{1}{2}\rho\left(\frac{\partial(v_1)}{\partial x^2}J_{13}-\frac{\partial(v_1)}{\partial x^3}J_{12}\right)dx^2\wedge dx^3 .$$
$$+\frac{1}{2}\rho\left(\frac{\partial(v_2)}{\partial x^3}J_{21}-\frac{\partial(v_2)}{\partial x^1}J_{23}\right)dx^3\wedge dx^1+\frac{1}{2}\rho\left(\frac{\partial(v_1)}{\partial x^3}J_{12}-\frac{\partial(v_1)}{\partial x^2}J_{13}\right)dx^3\wedge dx^2 \quad (1.68)$$

Employing the antisymmetry of J_{ij} and factoring out negatives gives

$$\begin{aligned}
&-\frac{1}{2}\rho\left(\frac{\partial(v_3)}{\partial x^1}J_{23}+\frac{\partial(v_3)}{\partial x^2}J_{31}\right)dx^1\wedge dx^2-\frac{1}{2}\rho\left(\frac{\partial(v_2)}{\partial x^1}J_{32}+\frac{\partial(v_2)}{\partial x^3}J_{21}\right)dx^1\wedge dx^3\\
&-\frac{1}{2}\rho\left(\frac{\partial(v_3)}{\partial x^2}J_{13}+\frac{\partial(v_3)}{\partial x^1}J_{32}\right)dx^2\wedge dx^1-\frac{1}{2}\rho\left(\frac{\partial(v_1)}{\partial x^2}J_{31}+\frac{\partial(v_1)}{\partial x^3}J_{12}\right)dx^2\wedge dx^3\ .\\
&-\frac{1}{2}\rho\left(\frac{\partial(v_2)}{\partial x^3}J_{12}+\frac{\partial(v_2)}{\partial x^1}J_{23}\right)dx^3\wedge dx^1-\frac{1}{2}\rho\left(\frac{\partial(v_1)}{\partial x^3}J_{21}+\frac{\partial(v_1)}{\partial x^2}J_{13}\right)dx^3\wedge dx^2
\end{aligned} \tag{1.69}$$

Adding and subtracting $\frac{1}{2}\frac{\partial(v_k)}{\partial x^k}\rho J_{ij}dx^i\wedge dx^j$ and regrouping the new terms yields

$$\begin{aligned}
&-\frac{1}{2}\rho\left(\frac{\partial(v_3)}{\partial x^1}J_{23}+\frac{\partial(v_3)}{\partial x^2}J_{31}+\frac{\partial(v_3)}{\partial x^3}J_{12}\right)dx^1\wedge dx^2-\frac{1}{2}\rho\left(\frac{\partial(v_2)}{\partial x^1}J_{32}+\frac{\partial(v_2)}{\partial x^3}J_{21}+\frac{\partial(v_2)}{\partial x^2}J_{13}\right)dx^1\wedge dx^3\\
&-\frac{1}{2}\rho\left(\frac{\partial(v_3)}{\partial x^2}J_{13}+\frac{\partial(v_3)}{\partial x^1}J_{32}+\frac{\partial(v_3)}{\partial x^3}J_{21}\right)dx^2\wedge dx^1-\frac{1}{2}\rho\left(\frac{\partial(v_1)}{\partial x^2}J_{31}+\frac{\partial(v_1)}{\partial x^3}J_{12}+\frac{\partial(v_1)}{\partial x^1}J_{23}\right)dx^2\wedge dx^3\\
&-\frac{1}{2}\rho\left(\frac{\partial(v_2)}{\partial x^3}J_{12}+\frac{\partial(v_2)}{\partial x^1}J_{23}+\frac{\partial(v_2)}{\partial x^2}J_{31}\right)dx^3\wedge dx^1-\frac{1}{2}\rho\left(\frac{\partial(v_1)}{\partial x^3}J_{21}+\frac{\partial(v_1)}{\partial x^2}J_{13}+\frac{\partial(v_1)}{\partial x^1}J_{32}\right)dx^3\wedge dx^2\\
&+\frac{1}{2}\frac{\partial(v_k)}{\partial x^k}\rho J_{ij}dx^i\wedge dx^j\\
&.
\end{aligned} \tag{1.70}$$

However, all of this can be written in index form as

$$\frac{\partial(v_i)}{\partial x^k}\rho J_{ij}dx^k\wedge dx^j=-\frac{1}{2}\varepsilon^l{}_{mn}\varepsilon^{ijk}\rho J_{ij}\frac{\partial(v_l)}{\partial x^k}dx^m\wedge dx^n+\frac{1}{2}\frac{\partial(v_k)}{\partial x^k}\rho J_{ij}dx^i\wedge dx^j\ . \tag{1.71}$$

Substituting this back into (1.64) gives

$$\frac{1}{2!}J_{ij}\partial_t\rho dx^i\wedge dx^j+\frac{1}{2!}\rho\partial_t J_{ij}dx^i\wedge dx^j-\frac{1}{2!}\varepsilon^l{}_{mn}\varepsilon^{ijk}\rho J_{ij}\frac{\partial(v_l)}{\partial x^k}dx^m\wedge dx^n$$
$$+\frac{1}{2!}\frac{\partial(v_k)}{\partial x^k}\rho J_{ij}dx^i\wedge dx^j+\frac{1}{2!}\left(\frac{\partial(\rho J_{ij})}{\partial x^k}v_k dx^i\wedge dx^j\right)=0 \qquad . \qquad (1.72)$$

Using the product rule on the last term gives

$$\frac{1}{2!}J_{ij}\partial_t\rho dx^i\wedge dx^j+\frac{1}{2!}\rho\partial_t J_{ij}dx^i\wedge dx^j-\frac{1}{2!}\varepsilon^l{}_{mn}\varepsilon^{ijk}\rho J_{ij}\frac{\partial(v_l)}{\partial x^k}dx^m\wedge dx^n$$
$$+\frac{1}{2!}\frac{\partial(v_k)}{\partial x^k}\rho J_{ij}dx^i\wedge dx^j+\frac{1}{2!}\frac{\partial(\rho)}{\partial x^k}J_{ij}v_k dx^i\wedge dx^j+\frac{1}{2!}\frac{\partial(J_{ij})}{\partial x^k}\rho v_k dx^i\wedge dx^j=0 \qquad . \qquad (1.73)$$

Regrouping and factoring then produces

$$\frac{1}{2!}J_{ij}\left(\partial_t\rho+\frac{\partial(\rho)}{\partial x^k}v_k+\frac{\partial(v_k)}{\partial x^k}\rho\right)dx^i\wedge dx^j+\frac{1}{2!}\rho\partial_t J_{ij}dx^i\wedge dx^j-\frac{1}{2!}\varepsilon^l{}_{mn}\varepsilon^{ijk}\rho J_{ij}\frac{\partial(v_l)}{\partial x^k}dx^m\wedge dx^n$$
$$+\frac{1}{2!}\frac{\partial(J_{ij})}{\partial x^k}\rho v_k dx^i\wedge dx^j=0 \qquad .$$

(1.74)

However, ρ satisfies the continuity equation

$$\partial_t\rho+\frac{\partial(\rho)}{\partial x^k}v_k+\frac{\partial(v_k)}{\partial x^k}\rho=0\,, \qquad (1.75)$$

which cancels the first term. This leaves

$$\frac{1}{2!}\rho\partial_t J_{ij}dx^i \wedge dx^j - \frac{1}{2!}\varepsilon^l{}_{mn}\varepsilon^{ijk}\rho J_{ij}\frac{\partial(v_l)}{\partial x^k}dx^m \wedge dx^n + \frac{1}{2!}\frac{\partial(J_{ij})}{\partial x^k}\rho v_k dx^i \wedge dx^j = 0\,. \tag{1.76}$$

Now, we take the dual of the above, by using equations (1.57), (1.58) and (1.59). This produces

$$\partial_t J^k dx_k - J^k\frac{\partial(v_l)}{\partial x^k}dx_l + \frac{\partial(J^l)}{\partial x^k}v_k dx_l = 0 \tag{1.77}$$

or

$$\frac{\partial\vec{J}}{\partial t} + (\vec{v}\cdot\nabla)\vec{J} = (\vec{J}\cdot\nabla)\vec{v}\,, \tag{1.78}$$

which is exactly the form of a material line element, (1.22). Therefore, a vector field, $\vec{J}$, whose 2-form dual satisfies $\partial_t\tilde{\omega} + \mathcal{L}_{\vec{v}}\tilde{\omega} = 0$ will evolve as a material line element.

As a corollary to the above, since, for a vector field, the Lie derivative is given by the Lie bracket,

$$\mathcal{L}_{\vec{v}}\vec{J} = [\vec{v},\vec{J}]\,, \tag{1.79}$$

whose coordinate representation is

$$\mathcal{L}_{\vec{v}}\vec{J} = v_i \frac{\partial J_j}{\partial x_i} - J_i \frac{\partial v_j}{\partial x_i} = (\vec{v}\cdot\nabla)\vec{J} - (\vec{J}\cdot\nabla)\vec{v}\,, \tag{1.80}$$

and, since the 2-form dual to the vector field demands that

$$\frac{\partial \vec{J}}{\partial t} + (\vec{v}\cdot\nabla)\vec{J} = (\vec{J}\cdot\nabla)\vec{v}\,, \tag{1.81}$$

we must have

$$\partial_t \vec{J} + \mathcal{L}_{\vec{v}}\vec{J} = 0\,. \tag{1.82}$$

That is to say, if the dual 2-form satisfies $\partial_t \tilde{\omega} + \mathcal{L}_{\vec{v}}\tilde{\omega} = 0$, then so must $\vec{J}$.

So, we have seen that all four equations describing motions of dynamical invariants are merely coordinate representations of the equation $\partial_t \tilde{\omega} + \mathcal{L}_{\vec{v}}\tilde{\omega} = 0$ applied to forms of different rank, with the assignments

$$
\begin{aligned}
&I \Rightarrow I \\
&\vec{S} \Rightarrow S_i dx^i \\
&\vec{J} \Rightarrow \frac{\rho}{2!}\varepsilon_{ijk} J^k dx^i \wedge dx^j \\
&f \Rightarrow \frac{1}{3!} f_{ijk} dx^i \wedge dx^j \wedge dx^k
\end{aligned}
\quad . \qquad (1.83)
$$

1.7 Interaction of Convection Operator with other Differential Geometry Operations

Now that we know that all dynamical invariants, when expressed as forms, satisfy the same convection operator equation, we can investigate creation of new dynamical invariants using differential geometry. There are a variety of operations that exist within differential geometry. If we could show that any of these operations commute with the convection operator, then we could construct new invariants by applying these operations to known invariants. More rigorously, suppose that Ψ is an exterior calculus operation that commutes with the operator $\partial_t + \mathcal{L}_{\vec{v}}$ and having $\Psi(0) = 0$. Then, if $\tilde{\omega}$ is a solution to $\partial_t \tilde{\omega} + \mathcal{L}_{\vec{v}} \tilde{\omega} = 0$ we have

$$
\begin{aligned}
&(\partial_t + \mathcal{L}_{\vec{v}})\left(\Psi(\tilde{\omega})\right) \\
&= \Psi\left((\partial_t + \mathcal{L}_{\vec{v}})(\tilde{\omega})\right) \\
&= \Psi(\partial_t \tilde{\omega} + \mathcal{L}_{\vec{v}} \tilde{\omega}) \\
&= \Psi(0) = 0
\end{aligned}
\quad . \qquad (1.84)
$$

Thus, $\Psi(\tilde{\omega})$ is also a solution to $\partial_t \tilde{\omega} + \mathcal{L}_{\vec{v}} \tilde{\omega} = 0$. Therefore, if $\tilde{\omega}$ is a dynamical invariant, so is $\Psi(\tilde{\omega})$.

Let us examine some specific operations. Notice that the operator, $\partial_t \tilde{\omega} + \mathcal{L}_{\vec{v}} \tilde{\omega} = 0$, is linear. Any linear combination of solutions which are of the same rank will also be a solution. Therefore, the solutions at least form a series of vector spaces, one for each rank.

Consider the differential operator as defined in (1.35). The differential commutes with ∂_t since ∂_t is a time derivative and the differential is made up of spatial derivatives. For the Lie derivative we have

$$\begin{aligned} \mathcal{L}_{\vec{v}}(d\tilde{\omega}) &= d[(d\tilde{\omega})(\vec{v})] + \left(d(d\tilde{\omega})\right)(\vec{v}) \\ &= d[(d\tilde{\omega})(\vec{v})] \end{aligned}, \tag{1.85}$$

since $\left(d(d\tilde{\omega})\right)(\vec{v}) = 0$. Then, employing the same formula again gives

$$\begin{aligned} d[(d\tilde{\omega})(\vec{v})] &= d[\mathcal{L}_{\vec{v}}\tilde{\omega} - d[\tilde{\omega}(\vec{v})]] \\ &= d[\mathcal{L}_{\vec{v}}\tilde{\omega}] - d[d[\tilde{\omega}(\vec{v})]] \\ &= d[\mathcal{L}_{\vec{v}}\tilde{\omega}] \end{aligned}. \tag{1.86}$$

Thus, the differential commutes with $\partial_t + \mathcal{L}_{\vec{v}}$ and $d\tilde{\omega}$ is an invariant if $\tilde{\omega}$ is an invariant.

We can also investigate the interaction of $\partial_t + \mathcal{L}_{\vec{v}}$ with binary operators. Suppose $\tilde{\omega}_i$ and $\tilde{\omega}_j$ both solve $\partial_t\tilde{\omega}+\mathcal{L}_{\vec{v}}\tilde{\omega}=0$. If we construct the wedge product, $\tilde{\omega}_i \wedge \tilde{\omega}_j$, then, by the product rule,

$$
\begin{aligned}
&(\partial_t + \mathcal{L}_{\vec{v}})\left(\tilde{\omega}_{i1} \wedge \tilde{\omega}_j\right) \\
&= \partial_t\left(\tilde{\omega}_i \wedge \tilde{\omega}_j\right) + \mathcal{L}_{\vec{v}}\left(\tilde{\omega}_i \wedge \tilde{\omega}_j\right) \\
&= (\partial_t\tilde{\omega}_i) \wedge \tilde{\omega}_j + \tilde{\omega}_i \wedge (\partial_t\tilde{\omega}_j) + (\mathcal{L}_{\vec{v}}\tilde{\omega}_i) \wedge \tilde{\omega}_j + \tilde{\omega}_i \wedge (\mathcal{L}_{\vec{v}}\tilde{\omega}_j), \\
&= (\partial_t\tilde{\omega}_i + \mathcal{L}_{\vec{v}}\tilde{\omega}_i) \wedge \tilde{\omega}_j + \tilde{\omega}_i \wedge (\partial_t\tilde{\omega}_j + \mathcal{L}_{\vec{v}}\tilde{\omega}_j) \\
&= 0 \wedge \tilde{\omega}_j + \tilde{\omega}_i \wedge 0 \\
&= 0 + 0 = 0
\end{aligned}
\tag{1.87}
$$

where the last steps used the multilinearity of the wedge product. Thus, $\tilde{\omega}_i \wedge \tilde{\omega}_j$ is also a solution of $\partial_t\tilde{\omega}+\mathcal{L}_{\vec{v}}\tilde{\omega}=0$.

Consider, also, a contraction of a form, $\tilde{\omega}$, which solves $\partial_t\tilde{\omega}+\mathcal{L}_{\vec{v}}\tilde{\omega}=0$, with a vector, $\vec{J}$, whose dual also solves the equation. This operation is also known as the interior product. Applying the contraction gives,

$$
\begin{aligned}
&(\partial_t + \mathcal{L}_{\vec{v}})\left(\tilde{\omega}(\vec{J})\right) \\
&= \partial_t\left(\tilde{\omega}(\vec{J})\right) + \mathcal{L}_{\vec{v}}\left(\tilde{\omega}(\vec{J})\right) \\
&= (\partial_t\tilde{\omega})(\vec{J}) + \tilde{\omega}(\partial_t\vec{J}) + (\mathcal{L}_{\vec{v}}\tilde{\omega})(\vec{J}) + \tilde{\omega}(\mathcal{L}_{\vec{v}}\vec{J}) \\
&= (\partial_t\tilde{\omega} + \mathcal{L}_{\vec{v}}\tilde{\omega})(\vec{J}) + \tilde{\omega}(\partial_t\vec{J} + \mathcal{L}_{\vec{v}}\vec{J}) \\
&= 0(\vec{J}) + \tilde{\omega}(0) \\
&= 0
\end{aligned} \quad . \tag{1.88}
$$

The fifth line uses the result from (1.82) . Thus we conclude that $\tilde{\omega}(\vec{J})\alpha\tilde{\omega}_i + \beta\tilde{\omega}_j$ is also an invariant.

We have now found exterior calculus operators that, when they act on solutions of $\partial_t\tilde{\omega} + \mathcal{L}_{\vec{v}}\tilde{\omega} = 0$, create objects which are also solutions. Thus, with a few generator solutions we can construct a whole algebra of infinitely many new solutions.

1.8 Coordinate Representations of Differential Geometry Operators

A number of dynamical invariants were discovered before differential geometry was applied to fluid and plasma equations. All of these except for the few invariants used as generators can be derived using the algebraic operations detailed, above. However, since the application of differential geometry is so recent, most of the literature about such invariants is written in the language of partial differential equations. Thus, it will be useful to know how to represent the algebraic operations in traditional vector calculus and vector analysis, as detailed by **[Tur & Yanovsky]**, so that we may identify any derived invariants with those that are already known, and to translate new invariants into the more traditional style.

First, we note that linear combinations work the same in coordinate form as in the abstract form. Further, we note that the algebra contains four different k-forms, while traditional vector calculus equations of fluid mechanics include only scalars and vectors.

Using the dual, we transform 2-forms into vectors and 3-forms into scalars, and vice versa. In differential equations this corresponds to multiplying by, or dividing by, ρ, the mass density. However, we do not need always need to multiply by ρ to identify invariants. This is due to the fact that the dual does not commute with the convective operator, and thus, if a particular transformation demands that the dual be applied last, it must be applied last. This has the effect of changing forms into vectors, and multiplying both sides of the equation by ρ or its reciprocal. Such a multiplication does not change the function that solves the equation, and factors of ρ will be omitted in situations where the dual is applied last, in particular where it applies to the 3-form. The proper substitutions are given in (1.83).

The differential, as defined in (1.35), when converted into traditional vector calculus gives divergence, gradient or curl when applied to the proper k-form and accompanied by proper use of the dual. This leads to the new invariants

$$
\begin{aligned}
&d(I) = \nabla I = \vec{S}\,' \\
&*(d\vec{S}) = \frac{1}{\rho}\nabla\times\vec{S} = \vec{J}\,' \\
&*(d(*\vec{J})) = \nabla\cdot\rho\vec{J} = \rho\,' \\
&d(*\rho) = 0
\end{aligned}
\quad , \qquad (1.89)
$$

where the unprimed expressions are known invariants, the primed expressions are new invariants of the appropriate type, and the star is the dual. In each of the above cases, the action of the differential takes an invariant and returns an invariant of the next higher degree. For instance, ∇I is a surface form. Examples of invariants of these types are the entropy gradient, ∇S and the potential vorticity $\vec{\omega} = \dfrac{\nabla\times\vec{v}}{\rho}$.

The wedge product takes a k-form and a m-form and returns a (k+m)-form. As with the differential, the coordinate representation of the wedge product depends upon which form it acts. Combining a 0-form with any other form corresponds to a simple multiplication of an invariant by the scalar field corresponding to a Lagrangian invariant. For other forms we have

$$
\begin{aligned}
&*(\vec{S}_1\wedge\vec{S}_2) = \frac{1}{\rho}\vec{S}_1\times\vec{S}_2 = \vec{J}\,' \\
&*((*\vec{J})\wedge\vec{S}) = \rho\vec{J}\cdot\vec{S} = \rho\,' \\
&(*\vec{J})\wedge(*\vec{J}) = (*\rho)\wedge\vec{S} = (*\rho)\wedge(*\vec{J}) = (*\rho)\wedge(*\rho) = 0
\end{aligned}
\quad . \qquad (1.90)
$$

Since we are in 3-space, we cannot combine anything that would give us a total degree greater than three. A familiar example of such an invariant is the magnetic helicity, $\vec{A}\cdot\vec{B}$, in MHD. A less familiar example is $\frac{\vec{q}\times\nabla S}{\rho}$ in hydrodynamics, which corresponds to a nonclosed 2-form and has interesting physical properties. That is to say, it is a frozen-in field with sources.

The interior product of a k-form with a vector, $\vec{J}$ (whose dual satisfies the equation associated with a 2-form), will be a (k-1)-form. The representation again depends upon the starting form. If we begin with a 0-form, the contraction gives us zero. If we begin with a 3-form, the interior product gives us the 2-form corresponding to $\vec{J}$. Thus, the only nontrivial cases are those where we begin with either 1-form or 2-form. The resulting representations are

$$\begin{aligned} I' &= \vec{J}\cdot\vec{S} \\ S' &= (\vec{J}\times\rho\vec{J}\,') \end{aligned}. \qquad (1.91)$$

Familiar examples of invariants from this category include the potential helicity, $\frac{\vec{A}\cdot\vec{B}}{\rho}$, in MHD and the Ertel invariant, $\frac{\vec{\omega}\cdot\nabla S}{\rho}$, in hydrodynamics. **[Ertel]**

With the representations of differential geometry operators on forms satisfying $\partial_t\tilde{\omega}+\mathcal{L}_{\vec{v}}\tilde{\omega}=0$ given in (1.89), (1.90) and (1.91), we can construct new invariants, or recognize existing ones, without needing to convert all equations into equations on forms. Construction of new geometrical invariants will be extended to new plasma models in **Chapter 2**. In **Chapter 4**, we build on these geometric invariants to construct new integral invariants, and build, further, to topological invariants in **Chapter 5**.

2 GEOMETRIC INVARIANTS FOR HYDRODYNAMIC AND PLASMA MODELS

2.1 Dynamical Invariants for the Equations of Compressible Hydrodynamics

We now begin identification of geometrical invariants and generation of invariants by examining barotropic, compressible hydrodynamics as a model. Recognition of invariants in incompressible hydrodynamics and a limited construction of new invariants using nongeneral techniques was discussed by **[Kuzmin]**. The barotropic, compressible case, and general invariant generation was described by **[Tur and Yanovsky]**.

Consider, first, the equations of ideal hydrodynamics:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \\ &\frac{D\vec{v}}{Dt} = -\frac{\nabla p}{\rho} \end{aligned} . \qquad (2.1)$$

Clearly, ρ satisfies the equation of associated with a 3-form. The vorticity, $\vec{\omega} = \nabla \times \vec{v}$, is known to be a frozen-in field, and, thus, is associated with a 2-form. To obtain a dynamical invariant from the velocity equation, we utilize the gauge transformation introduced earlier,

$$\vec{q} = \vec{v} + \nabla \varphi \qquad (2.2)$$

[Kuzmin]. Substituting the above into the velocity equation gives

$$\frac{D}{Dt}(\vec{q}-\nabla\varphi)=-\frac{\nabla p}{\rho}. \tag{2.3}$$

Solving for the material derivative of $\vec{q}$ gives

$$\begin{aligned}\frac{D\vec{q}}{Dt}&=\frac{D}{Dt}(\nabla\varphi)-\frac{\nabla p}{\rho}\\ &=\frac{\partial\nabla\varphi}{\partial t}+(\vec{v}\cdot\nabla)\nabla\varphi-\frac{\nabla p}{\rho}\\ &=(\vec{v}\cdot\nabla)\nabla\varphi-\nabla\left(-\frac{\partial\varphi}{\partial t}\right)-\frac{\nabla p}{\rho}\end{aligned}. \tag{2.4}$$

Now, using the product rule,

$$\begin{aligned}\nabla\left(\vec{v}\cdot\nabla\varphi\right)&=\nabla_{\vec{v}}\left(\vec{v}\cdot\nabla\varphi\right)+\nabla_{\nabla\varphi}\left(\vec{v}\cdot\nabla\varphi\right)\\ &=\left(\nabla v\right)^{T}\cdot\nabla\varphi+(\vec{v}\cdot\nabla)\nabla\varphi\end{aligned}, \tag{2.5}$$

we get

$$\begin{aligned}\frac{D\vec{q}}{Dt} &= -\left(\nabla v\right)^{T}\cdot\nabla\varphi+\nabla\left((\vec{v}\cdot\nabla)\varphi\right)-\nabla\left(-\frac{\partial\varphi}{\partial t}\right)-\frac{\nabla p}{\rho} \\ &= -\left(\nabla v\right)^{T}\cdot\nabla\varphi-\nabla\left(-\frac{D\varphi}{Dt}\right)-\frac{\nabla p}{\rho} \\ &= -\left(\nabla v\right)^{T}\cdot(\vec{q}-\vec{v})-\nabla\left(-\frac{D\varphi}{Dt}\right)-\frac{\nabla p}{\rho} \\ &= -\left(\nabla v\right)^{T}\cdot\vec{q}+\left(\nabla v\right)^{T}\vec{v}-\nabla\left(-\frac{D\varphi}{Dt}\right)-\frac{\nabla p}{\rho}\end{aligned} \quad . \qquad (2.6)$$

Using the product rule again on the second term gives

$$\frac{D\vec{q}}{Dt} = -(\nabla\vec{v})^{T}\cdot\vec{q}-\nabla\left(-\frac{D\varphi}{Dt}-\frac{1}{2}\left|\vec{v}\right|^{2}\right)-\frac{\nabla p}{\rho}. \qquad (2.7)$$

Now, let us assume that the fluid is barotropic, i.e. $p = p(\rho)$. Then, taking the curl of $\frac{\nabla p}{\rho}$ gives

$$\nabla\times\frac{\nabla p}{\rho} = \frac{1}{\rho}\nabla\times(\nabla p)+\nabla\left(\frac{1}{\rho}\right)\times\nabla p\,. \qquad (2.8)$$

The first term goes to zero, and the second gives

$$\nabla\times\frac{\nabla p}{\rho} = \frac{1}{\rho^{2}}\nabla\rho\times\frac{dp}{d\rho}\nabla\rho = \vec{0}\,. \qquad (2.9)$$

Since the curl of $\frac{\nabla p}{\rho}$ is zero and the density is nonzero, there must exist, over suitable domains, a potential, P, such that

$$\nabla P = \frac{\nabla p}{\rho}. \tag{2.10}$$

We can find P via integration over density, giving

$$P = \int \frac{\nabla p(\rho)}{\rho} d\rho. \tag{2.11}$$

We can now properly include the pressure term in the gradient in the equation for the material derivative, (2.7), giving

$$\frac{D\vec{q}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{q} - \nabla\left(P - \frac{D\varphi}{Dt} - \frac{1}{2}|\vec{v}|^2\right). \tag{2.12}$$

This is the equation which we will use to choose our gauge. We choose φ such that we have

$$\frac{D\varphi}{Dt} = P - \frac{1}{2}|\vec{v}|^2. \tag{2.13}$$

This particular choice is the geometric gauge **[Kuzmin]**, and reduces (2.12) to

$$\frac{D\vec{q}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{q}\,. \tag{2.14}$$

We see that, even though the velocity did not have the form of a dynamical invariant, this new equation has the structure of a 1-form.

Now that we have a partial set of basis forms, we can construct additional dynamical invariants through combinations of this basis set. For instance, we deduce that $\frac{\nabla\times\vec{v}}{\rho}$ should act as a material line element, and, with the inclusion of the entropy density Lagrangian invariant, we can construct the Ertel invariant, $\frac{\nabla\times\vec{v}\cdot\nabla S}{\rho}$, which is also a Lagrangian invariant. **[Ertel]**

2.2 Dynamical Invariants for Compressible MHD

Tur and Yanovsky described how to construct invariants for compressible MHD plasma **[Tur & Yanovsky]**. The relevant equations are

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla p}{\rho} + \frac{1}{\rho}(\nabla \times \vec{B}) \times \vec{B}$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$
$$\frac{DS}{Dt} = 0 \qquad . \qquad (2.15)$$
$$\frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right) = \left(\frac{\vec{B}}{\rho} \cdot \nabla\right)\vec{v}$$

We have included the equation for entropy density, $\frac{DS}{Dt} = 0$. We can see that the entropy density is a Lagrangian invariant (0-form) the mass density is a conserved quantity (a 3-form) and $\frac{\vec{B}}{\rho}$ is a line element (2-form). We would like to find a material surface. If we attempt a gauge transformation, (2.2), of the equation of motion we get

$$\frac{D\vec{q}}{Dt} = -(\nabla \vec{v})^T \cdot \vec{q} - \nabla\left(-\frac{D\varphi}{Dt} - \frac{1}{2}|\vec{v}|^2\right) - \frac{\nabla p}{\rho} + \frac{1}{\rho}(\nabla \times \vec{B}) \times \vec{B} \, . \qquad (2.16)$$

If the fluid is barotropic, the pressure term can be absorbed into the gradient. However, only under very special circumstances, a force-free plasma, for instance, will we be able to absorb or eliminate the magnetic term. Instead, let us return to the equation for $\frac{\vec{B}}{\rho}$. We express the magnetic field in terms of its vector potential, $\vec{A}$,

$$\vec{B} = \nabla \times \vec{A}. \tag{2.17}$$

Substituting into the equation for $\frac{\vec{B}}{\rho}$ gives

$$\frac{D}{Dt}\left(\frac{\nabla \times \vec{A}}{\rho}\right) = \left(\frac{\nabla \times \vec{A}}{\rho} \cdot \nabla\right)\vec{v} \tag{2.18}$$

which leads to

$$\frac{D\vec{A}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{A} + \nabla(\vec{v} \cdot \vec{A}) + \nabla\psi. \tag{2.19}$$

Where ψ is a new gauge field due to the magnetic field. If we gauge ψ such that

$$\psi = -\vec{v} \cdot \vec{A}, \tag{2.20}$$

then, (2.19) will become

$$\frac{D\vec{A}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{A}. \tag{2.21}$$

So, we see that it is $\vec{A}$ that behaves as the dual to a material surface element.

We now have a full, though not exhaustive, basis set of invariants which can be used to generate additional ones. Furthermore, since the equation of motion had not yet been used, the question of whether the pressure can be expressed as a gradient is, as of now, irrelevant, and any invariants generated from the current set will be trivially baroclinic. An infinite number of new invariants are now available, of ever increasing complexity. A few of the simpler examples are conserved quantities such as

$$\vec{A}\cdot\vec{B} \text{ and } \vec{B}\cdot\nabla S \,. \qquad (2.22)$$

The first of these is a projection of the vector potential onto its axis of rotation, and, thus, is an analog of the helicity called the magnetic helicity. We have new Lagrangian invariants such as

$$\vec{A}\cdot\frac{\vec{B}}{\rho} \qquad (2.23)$$

which can be interpreted as a potential magnetic helicity. Also, we have new material line elements such as

$$\frac{1}{\rho}\left(\nabla S\times\vec{B}\right) \qquad (2.24)$$

and, also, an analog of the Ertel invariant

$$\frac{\vec{B}}{\rho}\cdot\nabla S\ . \tag{2.25}$$

2.3 Dynamical Invariants for Compressible Hall MHD Equations

Using the equations of hydrodynamics as a model, let us perform the more novel task of finding dynamical invariants for the equations of Hall MHD. A few such invariants have be identified by **[Shivamoggi (2009)]**. Here, we extend the existing results to find a complete basis set, and use then use them to construct a fully general infinite set.

Restating the Hall MHD equations for a compressible, barotropic plasma:

$$\begin{aligned}
&\rho \frac{D\vec{v}}{Dt} = -\nabla p + \frac{1}{c}\vec{J} \times \vec{B} \\
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \\
&\vec{E} + \frac{1}{c}\vec{v} \times \vec{B} = \frac{1}{cne}\vec{J} \times \vec{B} \\
&\frac{DS}{Dt} = 0 \\
&\nabla \cdot \vec{B} = 0 \\
&\nabla \times \vec{B} = \frac{1}{c}\vec{J} \\
&\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\
&p = p(\rho)
\end{aligned} \quad . \tag{2.26}$$

Again, we have augmented the system of equations with the entropy equation $\frac{DS}{Dt} = 0$. Again, the entropy density is a Lagrangian invariant and the mass density is a conserved quantity. Unlike the hydrodynamic and MHD case, Hall MHD will require a different approach involving two gauge fields. To find the remaining basis form we again introduce the gauge transformation,

$$\vec{q} = \vec{v} + \nabla \varphi , \tag{2.27}$$

into the velocity equation. Assuming that the plasma is barotropic yields

$$\frac{D\vec{q}}{Dt} = -(\nabla \vec{v})^T \cdot \vec{q} - \nabla \left(\int \frac{\nabla p(\rho)}{\rho} d\rho - \frac{D\varphi}{Dt} - \frac{1}{2}|v|^2 \right) + \frac{1}{c}\left(\vec{J} \times \frac{\vec{B}}{\rho} \right). \tag{2.28}$$

We can now gauge φ as

$$\frac{D\varphi}{Dt} = P - \frac{1}{2}|v|^2 , \tag{2.29}$$

where P is defined as in the hydrodynamic case, (2.11). This yields

$$\frac{D\vec{q}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{q} + \frac{1}{c}\left(\vec{J} \times \frac{\vec{B}}{\rho}\right). \tag{2.30}$$

The equation for $\vec{q}$ does not behave as a dynamical invariant due to the electromagnetic term. Fortunately, for plasma equations, we have a second gauge freedom; the one associated with the electromagnetic field. Letting

$$\vec{B} = \nabla \times (\vec{A} + \nabla\theta) \tag{2.31}$$

and substituting into the Maxwell-Faraday equation and then eliminating the electric field gives

$$\frac{\partial(\nabla \times (\vec{A} + \nabla\theta))}{\partial t} = \nabla \times \left(\vec{v} \times (\nabla \times (\vec{A} + \nabla\theta)) - \frac{1}{e}\left(\vec{J} \times \frac{\vec{B}}{n}\right)\right). \tag{2.32}$$

Interchanging the curl with the time derivative and removing the curl from both sides introduces the gradient of another arbitrary scalar field, ψ . This is our other gauge field.

$$\frac{\partial \vec{A}}{\partial t} = \vec{v} \times (\nabla \times \vec{A}) - \frac{1}{e}\left(\vec{J} \times \frac{\vec{B}}{n} \right) + \nabla \psi \tag{2.33}$$

After application of vector identities we have

$$\frac{\partial \vec{A}}{\partial t} = (\nabla \vec{A})^T \cdot \vec{v} - (\vec{v} \cdot \nabla)\vec{A} - \frac{1}{e}\left(\vec{J} \times \frac{\vec{B}}{n} \right) + \nabla \psi \, . \tag{2.34}$$

And, finally,

$$\frac{D\vec{A}}{Dt} = (\nabla \vec{A})^T \cdot \vec{v} - \frac{1}{e}\left(\vec{J} \times \frac{\vec{B}}{n} \right) + \nabla \psi \, . \tag{2.35}$$

Now, using the fluid impulse equation together with the equation for the vector potential, we form the equation for the combination

$$\vec{q} + \frac{e\vec{A}}{mc} \tag{2.36}$$

[Shivamoggi (2009)] giving

$$\frac{D\left(\vec{q}+\frac{e\vec{A}}{mc}\right)}{Dt}=-(\nabla\vec{v})^T\cdot\vec{q}+\frac{1}{c}\left(\vec{J}\times\frac{\vec{B}}{\rho}\right)+\frac{e}{mc}(\nabla\vec{A})^T\cdot\vec{v}-\frac{1}{c}\left(\vec{J}\times\frac{\vec{B}}{\rho}\right)+\frac{e}{mc}\nabla\psi \; . \tag{2.37}$$

$$=-(\nabla\vec{v})^T\cdot\vec{q}+\frac{e}{mc}(\nabla\vec{A})^T\cdot\vec{v}+\frac{e}{mc}\nabla\psi$$

Now, employing the gauge

$$\psi=-\vec{A}\cdot\vec{v} \tag{2.38}$$

gives

$$\frac{D\left(\vec{q}+\frac{e\vec{A}}{mc}\right)}{Dt}=-(\nabla\vec{v})^T\cdot\vec{q}+\frac{e}{mc}(\nabla\vec{A})^T\cdot\vec{v}-\frac{e}{mc}(\nabla\vec{A})^T\cdot\vec{v}-\frac{e}{mc}(\nabla\vec{v})^T\cdot\vec{A}\,, \tag{2.39}$$

which simplifies to

$$\frac{D\left(\vec{q}+\frac{e\vec{A}}{mc}\right)}{Dt}=-(\nabla\vec{v})^T\cdot\left(\vec{q}+\frac{e\vec{A}}{mc}\right). \tag{2.40}$$

Thus, we see that the combination $\vec{q}+\frac{e\vec{A}}{mc}$ is our material surface invariant. Other than dimensionality, it closely parallels the generalized momentum of a particle in an electromagnetic field, and is, thus, a generalized velocity. Furthermore, we know that we can construct a material line element by taking the curl and the dual, giving $\frac{\vec{\omega}}{\rho}+\frac{e\vec{B}}{mc\rho}$ which is a generalized potential vorticity. We notice that the vorticity has been modified by the Larmor frequency $\frac{e\vec{B}}{mc}$.

Now that we have a basis set of dynamical invariants, we can begin constructing new invariants. For instance, we have that

$$\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\cdot\left(\frac{\vec{\omega}}{\rho}+\frac{e\vec{B}}{mc\rho}\right) \tag{2.41}$$

should be a Lagrangian invariant, confirming a result derived by **[Shivamoggi (2007)]** by other means. Since this quantity is the dot product of the generalized velocity with its curl divided by mass, it acts as a generalized potential helicity. Additionally, we have new conserved quantities such as

$$\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\cdot\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right) \text{ and } \left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)\cdot\nabla S\,, \tag{2.42}$$

the first of which is generalized helicity. We also gain new material line elements like

$$\frac{1}{\rho}\left(\nabla S\times\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\right), \tag{2.43}$$

$$\nabla\times\left(\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)\times\left(\nabla S\times\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\right)\right) \tag{2.44}$$

and

$$\nabla\times\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)\times\left(\nabla S\times\nabla\left(\left(\frac{\vec{\omega}}{\rho}+\frac{e\vec{B}}{pmc}\right)\cdot\nabla S\right)\right) \tag{2.45}$$

and an analog of the Ertel invariant

$$\left(\frac{\vec{\omega}}{\rho}+\frac{e\vec{B}}{mc\rho}\right)\cdot\nabla S\,. \tag{2.46}$$

As before, infinitely many new invariants are now available. The list, above, includes only a few examples of the lowest order and simplest.

2.4 Dynamical Invariants for eMHD Equations

Besides the frozen-in field, $\vec{B}_e$, dynamical invariants for electron MHD have not been considered. We develop them, here.

The equations of collisionless electron MHD are

$$\begin{aligned} \frac{\partial \vec{B}_e}{\partial t} &= \nabla \times (\vec{\mathrm{v}}_\mathrm{e} \times \vec{B}_e) \\ \vec{\mathrm{v}}_\mathrm{e} &= -\frac{c}{4\pi ne}(\nabla \times \vec{B}) \end{aligned}, \tag{2.47}$$

where $\vec{B}_e = \vec{B} - d_e{}^2 \nabla^2 \vec{B}$ is the generalized magnetic field, $\vec{\mathrm{v}}_\mathrm{e}$ is the electron velocity and d_e is the electron skin depth, the maximum depth radiation can penetrate into the plasma. We see that we must adjust our approach, again, since this model lacks a velocity evolution equation entirely. Application of a vector calculus identity gives

$$\begin{aligned} &\frac{\partial \vec{B}_e}{\partial t} = \nabla \times (\vec{\mathrm{v}}_\mathrm{e} \times \vec{B}_e) = \\ &\vec{\mathrm{v}}_\mathrm{e}(\nabla \cdot \vec{B}_e) - \vec{B}_e(\nabla \cdot \vec{\mathrm{v}}_\mathrm{e}) + (\vec{B}_e \cdot \nabla)\vec{\mathrm{v}}_\mathrm{e} - (\vec{\mathrm{v}}_\mathrm{e} \cdot \nabla)\vec{B}_e \end{aligned}. \tag{2.48}$$

Since $\nabla \cdot \vec{B}_e = \vec{0}$ and $\nabla \cdot \vec{\mathrm{v}}_\mathrm{e} = -\nabla \cdot \frac{c}{4\pi ne}(\nabla \times \vec{B}) = \vec{0}$, we are left with

$$\frac{\partial \vec{B}_e}{\partial t} + (\vec{v}_e \cdot \nabla)\vec{B}_e = (\vec{B}_e \cdot \nabla)\vec{v}_e . \qquad (2.49)$$

Thus, the generalized magnetic field is a frozen-in field.

To find the dual of a co-moving surface, we let $\vec{B}_e = \nabla \times \vec{A}_e$, where $\vec{A}_e$ is the generalized magnetic vector potential. Using the original equation for the generalized magnetic field and uncurling gives

$$\begin{aligned} \frac{\partial \nabla \times \vec{A}_e}{\partial t} &= \nabla \times [\vec{v}_e \times (\nabla \times \vec{A}_e)] \\ \frac{\partial \vec{A}_e}{\partial t} &= \vec{v}_e \times (\nabla \times \vec{A}_e) + \nabla \varphi \end{aligned} , \qquad (2.50)$$

where φ is our gauge field. Now,

$$\begin{aligned} \frac{\partial \vec{A}_e}{\partial t} &= \vec{v}_e \times (\nabla \times \vec{A}_e) + \nabla \varphi \\ &= \nabla(\vec{v}_e \cdot \vec{A}_e) - \nabla_{\vec{v}_e}(\vec{v}_e \cdot \vec{A}_e) - (\vec{v}_e \cdot \nabla)\vec{A}_e + \nabla \varphi . \\ &= -(\nabla \vec{v}_e)^T \cdot \vec{A}_e - (\vec{v}_e \cdot \nabla)\vec{A}_e + \nabla(\varphi + \vec{v}_e \cdot \vec{A}_e) \end{aligned} \qquad (2.51)$$

Moving $(\vec{v}_e \cdot \nabla)\vec{A}_e$ to the left hand side, and gauging φ as $\varphi = -\vec{v}_e \cdot \vec{A}_e$ gives

$$\frac{D\vec{A}_e}{Dt} = -(\nabla \vec{\mathrm{v}}_\mathrm{e})^T \cdot \vec{A}_e \,. \tag{2.52}$$

Thus, the generalized magnetic vector potential with the appropriate choice of gauge is the dual of a co-moving surface.

For our remaining local invariants, we can, firstly, employ the entropy density, S, as a Lagrangian invariant. Secondly, we notice that the electron fluid is incompressible, since

$$\nabla \cdot \vec{\mathrm{v}}_\mathrm{e} = -\frac{c}{4\pi ne} \nabla \cdot (\nabla \times \vec{B}) = 0 \,. \tag{2.53}$$

This incompressibility provides a continuity equation for the electron density, thereby providing a density invariant. EMHD makes the stronger assumption that electron density is uniform.

Now, we can construct and interpret new invariants for the EMHD system analogous to those for Hall MHD. For instance, due to uniform density

$$\vec{A}_e \cdot \vec{B}_e \tag{2.54}$$

[Shivamoggi] is both Lagrangian invariant and a conserved quantity. It is a generalized magnetic helicity.

Additionally,

$$\vec{B}_e \cdot \nabla S \tag{2.55}$$

is the analog of the Ertel invariant, and

$$\nabla S \times \vec{A}_e \tag{2.56}$$

is a new frozen-in field.

2.5 Invariants for a Generalized, Nonuniform EMHD

The construction of the electron MHD equations assumes the uniformity of the electron density. However, we can construct local invariants even when this condition is violated. In the interest of generality, we develop these new invariants, here. Such invariants may be useful in applications which require additional flexibility, or seek to understand phenomena near the eMHD regime, or during transitions to or from such a regime.

The derivation of EMHD utilizes the generalized momentum,

$$\vec{p}_e = m_e \vec{\mathrm{v}}_e - \frac{e\vec{A}}{c}. \text{ **[Kingsep]**} \tag{2.57}$$

Substituting the unnormalized Ampere's Law,

$$\vec{v}_e = -\frac{c}{4\pi n_e e}\nabla\times\vec{B}\,, \tag{2.58}$$

into the expression for the generalized momentum, and taking the curl gives

$$\begin{aligned}\nabla\times\vec{p}_e &= -\nabla\times\left[\frac{cm_e}{4\pi n_e e}\nabla\times\vec{B}+\frac{e\vec{A}}{c}\right] \\ &= -\frac{cm_e}{4\pi e}\nabla\times\left[\frac{1}{n_e}\nabla\times\vec{B}\right]+\nabla\times\frac{e\vec{A}}{c}\end{aligned}. \tag{2.59}$$

The curl of the generalized momentum is called the generalized potential vorticity. Now, if n_e is constant, this gives

$$\begin{aligned}&-\frac{cm_e}{4\pi n_e e}\nabla\times\left[\nabla\times\vec{B}\right]-\frac{e\vec{B}}{c} \\ &= \frac{cm_e}{4\pi n_e e}\nabla^2\vec{B}-\frac{e\vec{B}}{c} \\ &= \frac{e}{c}(d_e{}^2\vec{B}-\vec{B}) = \vec{B}_e\end{aligned}\quad, \tag{2.60}$$

where d_e is the electron skin depth. Substituting the above into the evolution equation for the generalized potential vorticity, (2.59), gives the familiar eMHD equations. However, if the electron density is not constant, we have

$$-\frac{cm_e}{4\pi e}\nabla\times\left[\frac{1}{n_e}\nabla\times\vec{B}\right]-\nabla\times\frac{e\vec{A}}{c}=$$
$$-\frac{cm_e}{4\pi e}\left[\nabla\frac{1}{n_e}\right]\times\vec{B}+\frac{cm_e}{4\pi n_e e}\nabla^2\vec{B}-\frac{e\vec{B}}{c}\neq\vec{B}_e \quad (2.61)$$

Thus, the equation for the generalized magnetic field does not have the structure of a frozen-in field in this case. However, the generalized potential vorticity evolves, in the collisionless case, according to

$$\frac{\partial}{\partial t}(\nabla\times\vec{p}_e)=\nabla\times[\vec{\mathrm{v}}_\mathrm{e}\times(\nabla\times\vec{p}_e)]$$
$$=\vec{\mathrm{v}}_\mathrm{e}(\nabla\cdot(\nabla\times\vec{p}_e))-(\nabla\times\vec{p}_e)(\nabla\cdot\vec{\mathrm{v}}_\mathrm{e})+((\nabla\times\vec{p}_e)\cdot\nabla)\vec{\mathrm{v}}_\mathrm{e}-(\vec{\mathrm{v}}_\mathrm{e}\cdot\nabla)(\nabla\times\vec{p}_e) \quad (2.62)$$

This yields

$$\frac{D}{Dt}(\nabla\times\vec{p}_e)=((\nabla\times\vec{p}_e)\cdot\nabla)\vec{\mathrm{v}}_\mathrm{e}\,. \quad (2.63)$$

Thus, in the nonuniform case, it is the generalized potential vorticity that plays the role of the material line element. Similarly, by uncurling the time evolution equation and setting the gauge to

$$\varphi = -\vec{\mathrm{v}}_{\mathrm{e}} \cdot \vec{p}_e , \tag{2.64}$$

we find that the generalized momentum is a convected surface element. Again, the entropy density provides a base Lagrangian invariant. As in the standard EMHD case, Ampere's Law ensures that electron is incompressible; therefore, electron density provides a density invariant. In this case, even though the density is not uniform, it can still be canceled from equations for the various invariants, and will not appear. Thus, we have

$$\vec{p}_e \cdot \nabla \times \vec{p}_e \tag{2.65}$$

as both a Lagrangian invariant and a conserved quantity. This quantity is another generalized helicity. We also have

$$\nabla \times \vec{p}_e \cdot \nabla S \tag{2.66}$$

as an analog of the Ertel invariant, and

$$\nabla S \times \vec{p}_e \tag{2.67}$$

as a new frozen-in field.

2.6 Recognizing Gauge Transformations

Given the ease with which tools based upon differential geometry can generate new invariants from a known basis set, it is natural to ask how such a basis set can be identified when encountering a new fluid model. While physical quantities whose equations already match those of invariants are easy to recognize, invariants can be extracted from equations of a variety of forms. Historically, differential equations, rather than differential geometry, have been the preferred approach to the construction of fluid models. As such, this section and the next three sections develop techniques for identifying differential equations that can yield basis invariants and extraction of these invariants.

In the previous sections, gauge transformations have been used to convert equations into those for material surface invariants which were then used as a basis invariant. For a given equation, we would like to know whether a gauge transformation would be a useful approach. In response to this, let us find a general set of equations that can be transformed into that of a surface invariant through the use of a gauge transformation.

Consider the evolution equation for the dual to a known surface invariant,

$$\frac{D\vec{S}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{S} \,. \qquad (2.68)$$

Now, consider a new vector quantity, $\vec{z}$, that can be obtained from $\vec{S}$ by a gauge transformation. That is to say,

$$\vec{S} = \vec{z} + \nabla\varphi . \tag{2.69}$$

Substituting into the evolution equation for $\vec{S}$ to get the evolution equation for $\vec{z}$ gives

$$\frac{D(\vec{z} + \nabla\varphi)}{Dt} = -(\nabla\vec{v})^T \cdot (\vec{z} + \nabla\varphi) , \tag{2.70}$$

which yields

$$\frac{D\vec{z}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{z} - \frac{D(\nabla\varphi)}{Dt} - (\nabla\vec{v})^T \cdot \nabla\varphi . \tag{2.71}$$

Now, expanding the material derivative of the gradient gives

$$\begin{aligned} &\frac{D(\nabla\varphi)}{Dt} = \frac{\partial(\nabla\varphi)}{\partial t} + (\vec{v}\cdot\nabla)(\nabla\varphi) \\ &= \nabla\frac{\partial(\nabla\varphi)}{\partial t} + \nabla\left((\vec{v}\cdot\nabla)\varphi\right) - (\nabla\vec{v})^T \cdot \nabla\varphi . \\ &= \nabla\frac{D\varphi}{Dt} - (\nabla\vec{v})^T \cdot \nabla\varphi \end{aligned} \tag{2.72}$$

Substituting this into the equation for $\vec{z}$ gives

$$\begin{aligned}\frac{D\vec{z}}{Dt} &= -(\nabla\vec{v})^T \cdot \vec{z} - \nabla\frac{D\varphi}{Dt} + (\nabla\vec{v})^T \cdot \nabla\varphi - (\nabla\vec{v})^T \cdot \nabla\varphi \\ &= -(\nabla\vec{v})^T \cdot \vec{z} - \nabla\frac{D\varphi}{Dt}\end{aligned} \quad . \tag{2.73}$$

Therefore, the equation for $\vec{z}$ looks like the equation for a material surface, but contains an extra term which is the gradient of the material derivative of a scalar field. Alternatively, since

$$\nabla(\vec{v}\cdot\vec{z}) = (\nabla\vec{v})^T \cdot \vec{z} + (\nabla\vec{z})^T \cdot \vec{v}\,, \tag{2.74}$$

We can have

$$\frac{D\vec{z}}{Dt} = -\nabla(\vec{v}\cdot\vec{z}) + (\nabla\vec{z})^T \cdot \vec{v} - \nabla\frac{D\varphi}{Dt}\,. \tag{2.75}$$

Finally, we can eliminate the gradient term by absorbing it into the existing gradient term. As long as a scalar field is integrable in all of its variables, it will be the material derivative of another scalar field. Therefore, the material derivative does not restrict the nature of the scalar field, and the only remaining requirement is that additional term must be conservative, which can be easily determined by using the curl. So, we can conclude that if the material derivative of a vector field gives a material surface-like term or $(\nabla\vec{z})^T\cdot\vec{v}$, with the remainder being conservative, then it can be transformed into a material surface element by using a gauge transformation, and a general vector field must conform one of these two forms in order for such a transformation to be possible.

Even though the above is true for a general vector field, we can get a broader range of equations that can be transformed if we put restrictions on the nature of $\vec{z}$. Specifically, we require there to be a relationship between $\vec{z}$ and the velocity field, $\vec{v}$. If we let

$$\vec{z}=\overrightarrow{F}(\vec{v}) \tag{2.76}$$

then,

$$\nabla(\vec{v}\cdot\vec{z})=(\nabla\vec{v})^T\cdot\vec{z}+v_iF_{i,j}v_{j,k}\,. \tag{2.77}$$

Using this in the evolution equation for $\vec{z}$, above, gives

$$\begin{aligned}\frac{D\vec{z}}{Dt}&=-(\nabla\vec{v})^T\cdot\vec{z}-\nabla\frac{D\varphi}{Dt}\\&=-\nabla(\vec{v}\cdot\vec{z})+v_iF_{i,j}v_{j,k}-\nabla\frac{D\varphi}{Dt}\end{aligned}\,. \tag{2.78}$$

The gradient terms can be combined together with the result that any function of velocity whose material derivative is of the form $v_i F_{i,j} v_{j,k}$ plus a conservative term can be transformed into a material surface element.

The most useful case is $\vec{z} = \vec{v}$. In this situation, we have

$$(\nabla \vec{v})^T \cdot \vec{z} = \frac{1}{2} \nabla (\vec{v} \cdot \vec{v}) \,, \tag{2.79}$$

which is already conservative and can be entirely absorbed into the gradient. So, for a velocity field, anything that evolves like

$$\frac{D\vec{v}}{Dt} = -(\nabla \vec{v})^T \cdot \vec{v} + \nabla \psi \tag{2.80}$$

or

$$\frac{D\vec{v}}{Dt} = \nabla \psi \tag{2.81}$$

can be transformed into a material surface element. An example of this final form is that of the velocity field for a barotropic hydrodynamic system, see (2.1) and (2.10), which we know can be transformed into a material surface element via the geometric gauge developed by **[Kuzmin]** and **[Oseledets]**.

2.7 Ambiguity of Invariants in Incompressible Systems and Difficulties of Extending to Compressible Analogs

When working with a compressible model, it is a common technique to analyze the simpler, incompressible, model and then extend the results to the compressible analog. This approach can be applied when seeking new basis invariants for a compressible model when basis invariants for the incompressible analog are known. Though some invariants have been known, in literature, for a long time **[Ertel]**, **[Kuzmin]** and others, their extensions to compressible analogs have proven difficult. A possible cause of this difficulty is a natural ambiguity in the nature of invariants in the incompressible case. It is possible for a physical quantity to satisfy the equation for an invariant in the incompressible model, but not generate a corresponding invariant when extending the model to the compressible. This ambiguity may lead to the misidentification of invariants for incompressible systems. It is only in the compressible case that the full nature of the invariant structures is revealed. This section details this ambiguity, and its effects, so that it may more readily avoided when seeking basis invariants in future compressible models.

Consider a conserved quantity obeying

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 , \qquad (2.82)$$

or

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \vec{v}) \tag{2.83}$$

in material form. In the incompressible case, $\nabla \cdot \vec{v} = 0$, and so

$$\frac{D\rho}{Dt} = 0\,. \tag{2.84}$$

Thus, ρ is constant in the co-moving frame and is both a conserved quantity and a Lagrangian invariant for the incompressible model. This, of course, can be taken as the definition of incompressibility. The same reasoning applies to Lagrangian invariants, which have both the properties of Lagrangian invariants and conserved quantities in an incompressible system. In a compressible system, $\nabla \cdot \vec{v} \neq 0$, and this degeneracy is broken. This, now, leads to Lagrangian invariants and conserved quantities having distinct physical behavior. Problems may arise during the extension of solutions from an incompressible system to a compressible analog if invariant quantities are misidentified and assigned to the wrong category during the extension process.

A potentially greater problem is that, since conserved quantities have the properties of Lagrangian invariants in incompressible systems, and since multiplication by a Lagrangian invariant does not change the category of an invariant, it is possible to construct invariants in the incompressible system which have no analog in the analogous compressible system. The use of such an invariant during an attempt to extend incompressible properties to the compressible case will cause the extension to fail. Consider, for instance, the quantity ρ^2 where ρ is known to obey the continuity equation. Then,

$$\frac{D\rho^2}{Dt} = 2\rho \frac{D\rho}{Dt} = -2\rho^2 (\nabla \cdot \vec{v}) . \tag{2.85}$$

In the incompressible case, the right hand side equals zero, giving ρ^2 the properties of both a Lagrangian invariant and a conserved quantity. In the compressible case, however, ρ^2 is not an invariant of any kind.

The ambiguity of invariants for incompressible systems manifests itself differently for material line elements and material surfaces. Even in the incompressible case, material lines and surfaces maintain distinct physical behavior. However, the construction of new invariants requires the multiplication and division by ρ when taking the dual. In the incompressible case, these extra factors of ρ can be omitted, leading to difficulties in the compressible case, where they are required. By way of example, consider a material surface, $\vec{S}$, satisfying

$$\frac{D\vec{S}}{Dt} = -(\nabla \vec{v})^T \cdot \vec{S} . \tag{2.86}$$

Examining the behavior of $\nabla\times\vec{S}$ gives

$$\begin{aligned}
&\frac{D(\nabla\times\vec{S})}{Dt}=\frac{\partial(\nabla\times\vec{S})}{\partial t}+(\vec{v}\cdot\nabla)(\nabla\times\vec{S})\\
&=\frac{\partial(\nabla\times\vec{S})}{\partial t}-\nabla\times(\vec{v}\times(\nabla\times\vec{S}))+\vec{v}\left(\nabla\cdot(\nabla\times\vec{S})\right)-(\nabla\times\vec{S})(\nabla\cdot\vec{v})+\left((\nabla\times\vec{S})\cdot\nabla\right)\vec{v}\\
&=\frac{\partial(\nabla\times\vec{S})}{\partial t}-\nabla\times(\vec{v}\times(\nabla\times\vec{S}))-(\nabla\times\vec{S})(\nabla\cdot\vec{v})+\left((\nabla\times\vec{S})\cdot\nabla\right)\vec{v}\\
&=\frac{\partial(\nabla\times\vec{S})}{\partial t}-\nabla\times\left(\nabla(\vec{v}\cdot\vec{S})-(\nabla\vec{v})^{T}\cdot\vec{S}-(\vec{v}\cdot\nabla)\vec{S}\right)-(\nabla\times\vec{S})(\nabla\cdot\vec{v})+\left((\nabla\times\vec{S})\cdot\nabla\right)\vec{v}\quad. \qquad (2.87)\\
&=\nabla\times\left(\frac{\partial\vec{S}}{\partial t}+(\vec{v}\cdot\nabla)\vec{S}+(\nabla\vec{v})^{T}\cdot\vec{S}\right)-\nabla\times\left(\nabla(\vec{v}\cdot\vec{S})\right)-(\nabla\times\vec{S})(\nabla\cdot\vec{v})+\left((\nabla\times\vec{S})\cdot\nabla\right)\vec{v}\\
&=\nabla\times\left(\frac{D\vec{S}}{Dt}+(\nabla\vec{v})^{T}\cdot\vec{S}\right)-(\nabla\times\vec{S})(\nabla\cdot\vec{v})+\left((\nabla\times\vec{S})\cdot\nabla\right)\vec{v}\\
&=-(\nabla\times\vec{S})(\nabla\cdot\vec{v})+\left((\nabla\times\vec{S})\cdot\nabla\right)\vec{v}
\end{aligned}$$

Thus, $\nabla\times\vec{S}$ has the properties of a material line element, but only in the incompressible case. Simply using the same the same invariant for the compressible analog will not be successful. If, instead, we look at $\dfrac{\nabla\times\vec{S}}{\rho}$, we get

$$\frac{D}{Dt}\left(\frac{\nabla\times\vec{S}}{\rho}\right)=\frac{1}{\rho}\frac{D(\nabla\times\vec{S})}{Dt}-\frac{1}{\rho^2}(\nabla\times\vec{S})\frac{D\rho}{Dt}$$

$$=-\frac{1}{\rho}(\nabla\times\vec{S})(\nabla\cdot\vec{v})+\frac{1}{\rho}\left((\nabla\times\vec{S})\cdot\nabla\right)\vec{v}+\frac{1}{\rho^2}(\nabla\times\vec{S})\rho(\nabla\cdot\vec{v})\,. \tag{2.88}$$

$$=\left(\left(\frac{\nabla\times\vec{S}}{\rho}\right)\cdot\nabla\right)\vec{v}$$

This behaves as a material line element in both the incompressible and compressible systems, and is, thus, the proper combination.

The above consideration allows us to perform a better study of the magnetic helicity invariant, $\vec{A}\cdot\vec{B}$, that is encountered in the MHD system or the generalized magnetic helicity invariant, $\vec{A}_e\cdot\vec{B}_e$, of electron MHD. The equations that govern the magnetic field and its vector potential are

$$\frac{D\vec{B}}{Dt}=-\vec{B}(\nabla\cdot\vec{v})+\left(\vec{B}\cdot\nabla\right)\vec{v} \tag{2.89}$$

and

$$\frac{D\vec{A}}{Dt}=-(\nabla\vec{v})^T\cdot\vec{A}. \tag{2.90}$$

Then,

$$
\begin{aligned}
\frac{D(\vec{A}\cdot\vec{B})}{Dt} &= \frac{D\vec{A}}{Dt}\cdot\vec{B}+\vec{A}\cdot\frac{D\vec{B}}{Dt} \\
&= -\left((\nabla\vec{v})^T\cdot\vec{A}\right)\cdot\vec{B}-(\vec{A}\cdot\vec{B})(\nabla\cdot\vec{v})+\vec{A}\cdot\left((\vec{B}\cdot\nabla)\vec{v}\right). \\
&= -v_{i,j}A_iB_j-(\vec{A}\cdot\overrightarrow{B})(\nabla\cdot\vec{v})+A_iB_jv_{i,j} \\
&= -(\vec{A}\cdot\vec{B})(\nabla\cdot\vec{v})
\end{aligned}
\qquad (2.91)
$$

So, $\vec{A}\cdot\vec{B}$ is a conserved quantity in both the compressible and incompressible MHD systems, but a Lagrangian invariant in the incompressible system, only. On the other hand, if we consider $\frac{\vec{A}\cdot\vec{B}}{\rho}$, we have

$$
\begin{aligned}
\frac{D}{Dt}\frac{\vec{A}\cdot\vec{B}}{\rho} &= \frac{1}{\rho}\frac{D\vec{A}}{Dt}\cdot\vec{B}+\frac{1}{\rho}\vec{A}\cdot\frac{D\vec{B}}{Dt}-\frac{\vec{A}\cdot\vec{B}}{\rho^2}\frac{D\rho}{Dt} \\
&= -\frac{1}{\rho}(\vec{A}\cdot\overrightarrow{B})(\nabla\cdot\vec{v})+\frac{\vec{A}\cdot\vec{B}}{\rho^2}\rho(\nabla\cdot\vec{v}) \qquad . \\
&= 0
\end{aligned}
\qquad (2.92)
$$

Therefore, $\frac{\vec{A}\cdot\vec{B}}{\rho}$ is a Lagrangian invariant in both systems, and a conserved quantity in the incompressible system, only.

Thus, we see that construction of basis invariants in compressible models requires specific placement of factors of ρ. Such factors may be missing in the incompressible case, or may be misplaced, if present.

2.8 A New Kind of Gauge Transformation (Non-conservative Body Forces on an Incompressible Flow)

In this section, we continue our development of techniques for identification of basis invariants by developing a new gauge of geometrical interest analogous to the geometric gauge, but applicable to a different category of physical equations; those with non-conservative body forces.

Consider a quantity governed by the equation

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{v} + \nabla \times \vec{F} \qquad (2.93)$$

in an incompressible flow. This equation is of the same form as the vorticity equation for an ideal, incompressible, hydrodynamic system in the presence of non-conservative body forces. In the absence of the $\vec{F}$ term, $\vec{\omega}$ would have the form of a material line element. The presence of $\vec{F}$ means that $\vec{\omega}$ is no longer of the proper form. Though the symbol $\vec{\omega}$ is used for this vector field, the vorticity itself is a poor choice for the following transformation due to its zero divergence. A field with a nonzero divergence, but satisfying the same equation, would be a suitable choice. In terms of differential geometry, one that is dual to an nonclosed 2-form, examples of which can be constructed use the representation of the wedge product in (1.90); $\frac{\vec{q}\cdot\nabla S}{\rho}$, for example.

For this category of equation, we choose the gauge transformation

$$\vec{q} = \vec{\omega} + \nabla\times\vec{A}, \tag{2.94}$$

where $\vec{A}$ is an unknown vector field to be determined. Notice that this transformation preserves the divergence of $\vec{\omega}$.

Substituting $\vec{q}$ into the equation for $\vec{\omega}$, (2.93), gives

$$\begin{aligned}
\frac{D\vec{q}}{Dt} &= \frac{D}{Dt}(\nabla\times\vec{A}) + \nabla\times\vec{F} + (\vec{\omega}\cdot\nabla)\vec{v} \\
&= \nabla\times\frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\nabla)(\nabla\times\vec{A}) + \nabla\times\vec{F} + (\vec{\omega}\cdot\nabla)\vec{v} \\
&= \nabla\times\frac{\partial\vec{A}}{\partial t} - \nabla\times(\vec{v}\times(\nabla\times\vec{A})) + \vec{v}(\nabla\cdot(\nabla\times\vec{A})) - (\nabla\times\vec{A})(\nabla\cdot\vec{v}) + \left[(\nabla\times\vec{A})\cdot\nabla\right](\vec{v}) + \nabla\times\vec{F} + (\vec{\omega}\cdot\nabla)\vec{v}
\end{aligned}, \tag{2.95}$$

where the last equality uses the product rule for the curl of a cross product. Now, we eliminate the curl of the divergence, and the divergence of the velocity due to the incompressibility of the medium, giving

$$\begin{aligned}
\frac{D\vec{q}}{Dt} &= \nabla\times\frac{\partial\vec{A}}{\partial t} - \nabla\times(\vec{v}\times(\nabla\times\vec{A})) + \left[(\nabla\times\vec{A})\cdot\nabla\right](\vec{v}) + \nabla\times\vec{F} + (\vec{\omega}\cdot\nabla)\vec{v} \\
&= \nabla\times\frac{\partial\vec{A}}{\partial t} - \nabla\times\left[\nabla(\vec{v}\cdot\vec{A}) - \nabla_{\vec{v}}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}\right] + \left[(\nabla\times\vec{A})\cdot\nabla\right](\vec{v}) + \nabla\times\vec{F} + (\vec{\omega}\cdot\nabla)\vec{v} \\
&= \nabla\times\left[\frac{\partial\vec{A}}{\partial t} + (\nabla\vec{v})^T\cdot\vec{A} + (\vec{v}\cdot\nabla)\vec{A} + \vec{F}\right] - \nabla\times\left[\nabla(\vec{v}\cdot\vec{A})\right] + \left[(\nabla\times\vec{A})\cdot\nabla\right](\vec{v}) + (\vec{\omega}\cdot\nabla)\vec{v} \\
&= \nabla\times\left[\frac{D\vec{A}}{Dt} + (\nabla\vec{v})^T\cdot\vec{A} + \vec{F}\right] + \left[(\nabla\times\vec{A})\cdot\nabla\right](\vec{v}) + (\vec{\omega}\cdot\nabla)\vec{v}
\end{aligned} \quad . \qquad (2.96)$$

Reusing the gauge condition $\nabla\times\vec{A} = \vec{q} - \vec{\omega}$ gives

$$\begin{aligned}
\frac{D\vec{q}}{Dt} &= \nabla\times\left[\frac{D\vec{A}}{Dt} + (\nabla\vec{v})^T\cdot\vec{A} + \vec{F}\right] + \left[\vec{q}\cdot\nabla\right](\vec{v}) - (\vec{\omega}\cdot\nabla)\vec{v} + (\vec{\omega}\cdot\nabla)\vec{v} \\
&= \nabla\times\left[\frac{D\vec{A}}{Dt} + (\nabla\vec{v})^T\cdot\vec{A} + \vec{F}\right] + \left[\vec{q}\cdot\nabla\right](\vec{v})
\end{aligned} \quad . \qquad (2.97)$$

We see that if we gauge $\vec{A}$ such that

$$\frac{D\vec{A}}{Dt} + (\nabla\vec{v})^T\cdot\vec{A} + \vec{F} = \nabla\varphi, \qquad (2.98)$$

then $\vec{q}$ will behave as a frozen-in field. In fact, with $\varphi = 0$, we recognize that the gauge field is a modified material surface element. This frozen-in field can now be used as a basis invariant for generation of further invariants.

2.9 Non-conservative Body Forces in Compressible Systems

We now extend the new gauge developed in the previous section to its analog for compressible fluids.

Suppose that a quantity in a compressible fluid governed by the equation

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{v} - \vec{\omega}(\nabla\cdot\vec{v}) + \nabla\times\vec{F}\,. \tag{2.99}$$

This may appear as the vorticity equation for the ideal, hydrodynamic flow of a compressible fluid, though, again, only an unclosed form satisfying the same equation would result in a nontrivial transformation. Again, we notice that $\vec{\omega}$ does not behave as a material line element, and we would like to find a related quantity that does.

Introduce the transformation

$$\vec{q} = \frac{\vec{\omega}}{\rho} + \frac{\nabla\times\vec{A}}{\rho}\,. \tag{2.100}$$

This transformation can be found by introducing the ansatz

$$\vec{\omega} = \rho^m\vec{q} + \rho^n(\nabla\times\vec{A}) \tag{2.101}$$

and matching powers and coefficients. We notice the presence of $\frac{\vec{\omega}}{\rho}$, which is a material line element for the case where no body forces are present. We also notice that, unfortunately, this transformation is not technically a gauge transformation since it does not preserve the divergence of the original quantity. Despite this sacrifice, we can still use it to find invariant structures.

Upon substitution into the equation for $\vec{\omega}$, we have

$$\rho\frac{D\vec{q}}{Dt}=\frac{D}{Dt}(\nabla\times\vec{A})-\vec{q}\frac{D\rho}{Dt}+\nabla\times\vec{F}+(\vec{\omega}\cdot\nabla)\vec{v}-\vec{\omega}(\nabla\cdot\vec{v})\,. \tag{2.102}$$

Now, treating $\frac{D}{Dt}(\nabla\times\vec{A})$ as detailed in the incompressible case in **section 2.8** gives

$$\rho\frac{D\vec{q}}{Dt}=\nabla\times\left[\frac{D\vec{A}}{Dt}+(\nabla\vec{v})^{T}\cdot\vec{A}+\vec{F}\right]+\left[(\nabla\times\vec{A})\cdot\nabla\right](\vec{v})-(\nabla\times\vec{A})(\nabla\cdot\vec{v})-\vec{q}\frac{D\rho}{Dt}+(\vec{\omega}\cdot\nabla)\vec{v}-\vec{\omega}(\nabla\cdot\vec{v})\,. \tag{2.103}$$

Notice that, in this case, the $-(\nabla\times\vec{A})(\nabla\cdot\vec{v})$ term does not cancel. Reusing the transformation equation, we get

$$\rho\frac{D\vec{q}}{Dt} = \nabla\times\left[\frac{D\vec{A}}{Dt} + (\nabla\vec{v})^T\cdot\vec{A} + \vec{F}\right] + \left[\left((\nabla\times\vec{A}) + \vec{\omega}\right)\cdot\nabla\right](\vec{v}) - \left((\nabla\times\vec{A}) + \vec{\omega}\right)(\nabla\cdot\vec{v}) - \vec{q}\frac{D\rho}{Dt}$$
$$= \nabla\times\left[\frac{D\vec{A}}{Dt} + (\nabla\vec{v})^T\cdot\vec{A} + \vec{F}\right] + \left[\rho\vec{q}\cdot\nabla\right](\vec{v}) - \rho\vec{q}(\nabla\cdot\vec{v}) - \vec{q}\frac{D\rho}{Dt} \quad .(2.104)$$

Finally, using the density continuity equation in Lagrangian form,

$$\frac{D\rho}{Dt} = -\rho(\nabla\cdot\vec{v})\,, \quad (2.105)$$

gives

$$\rho\frac{D\vec{q}}{Dt} = \nabla\times\left[\frac{D\vec{A}}{Dt} + (\nabla\vec{v})^T\cdot\vec{A} + \vec{F}\right] + \rho\left(\vec{q}\cdot\nabla\right)(\vec{v}) - \rho\vec{q}(\nabla\cdot\vec{v}) + \vec{q}\rho(\nabla\cdot\vec{v})$$
$$= \nabla\times\left[\frac{D\vec{A}}{Dt} + (\nabla\vec{v})^T\cdot\vec{A} + \vec{F}\right] + \rho\left(\vec{q}\cdot\nabla\right)(\vec{v}) \quad . \quad (2.106)$$

Thus, again, choosing

$$\frac{D\vec{A}}{Dt} + (\nabla\vec{v})^T\cdot\vec{A} + \vec{F} = \nabla\varphi \quad (2.107)$$

and canceling ρ, gives $\vec{q}$ the behavior of a material line element.

The development in this and the previous sections may allow for the finding of invariant structures for hydrodynamic systems, and, perhaps with extension, plasma systems, in the presence of nonconservative body forces. Please note that, while a barotropy condition was necessary to construct surface invariants from the velocity equation in **section 2.1**, no assumptions have been made as to the nature of the forces in these two sections, other than that their curl appears in the equation for $\vec{\omega}$.

2.10 Other Gauges

Besides the geometric gauge, (2.13), hydrodynamics literature contains a few other gauges that have been used for specialized purposes in incompressible hydrodynamics. We now briefly describe those gauges and develop their analogs for plasma models.

Returning to the equation that governs the evolution of the gauge field for hydrodynamics,

$$\frac{D\vec{q}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{q} - \nabla\left(P - \frac{D\varphi}{Dt} - \frac{1}{2}|\vec{v}|^2 \right), \qquad (2.108)$$

consider φ chosen such that

$$\frac{\partial \varphi}{\partial t} = P + \frac{1}{2}|\vec{v}|^2 . \qquad (2.109)$$

This is the zero gauge described by **[Russo and Smereka]**. With this choice, the equation for $\vec{q}$, (2.108), takes on the form

$$\frac{D\vec{q}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{q} - \nabla\left(-(\vec{v}\cdot\nabla)\varphi - |\vec{v}|^2\right). \qquad (2.110)$$

This gauge is called the zero gauge because the equation for $\vec{q}$, with this choice of gauge, can be rewritten as

$$\frac{\partial \vec{q}}{\partial t} - \vec{v} \times (\nabla \times \vec{q}) = 0 . \qquad (2.111)$$

Alternatively, E and Liu **[E and Lui]** developed the EL gauge with the choice

$$\frac{\partial \varphi}{\partial t} = P . \qquad (2.112)$$

This gives

$$\frac{D\vec{q}}{Dt} = -(\nabla \vec{v})^T \cdot \vec{q} - \nabla \left(-(\vec{v} \cdot \nabla)\varphi - \frac{1}{2}|\vec{v}|^2 \right) \tag{2.113}$$

for the $\vec{q}$ equation. This choice of gauge results in an equation that is more numerically stable than the original velocity equation. Finally, Maddocks and Pego **[Maddocks and Pego]** defined the impetus gauge with the choice

$$\frac{\partial \varphi}{\partial t} = P + |\vec{v}|^2 - \vec{v} \cdot \vec{q} \tag{2.114}$$

from a variational approach to solving the velocity equation. This transforms the equation for $\vec{q}$ into

$$\begin{aligned}
&\frac{D\vec{q}}{Dt} = -(\nabla \vec{v})^T \cdot \vec{q} - \nabla \left(-\frac{1}{2}|\vec{v}|^2 \right) \\
&= -(\nabla \vec{v})^T \cdot \nabla \varphi - (\nabla \vec{v})^T \cdot \vec{v} - \nabla \left(-\frac{1}{2}|\vec{v}|^2 \right). \\
&= -(\nabla \vec{v})^T \cdot \nabla \varphi - \frac{1}{2} \nabla |\vec{v}|^2 - \nabla \left(-\frac{1}{2}|\vec{v}|^2 \right) \\
&= -(\nabla \vec{v})^T \cdot \nabla \varphi
\end{aligned} \tag{2.115}$$

For the incompressible case, it is possible to now determine $\nabla \varphi$ by solving the Poisson's equation

$$\nabla^2\varphi = \nabla\cdot\vec{q}\ . \tag{2.116}$$

For the MHD and Hall MHD models, we can construct analogous equations for the zero and EL gauges. The impetus gauge is more problematic, and can only be used in the incompressible case, since $\nabla\varphi$ cannot be determined independently, otherwise. For MHD the zero gauge, EL gauge and impetus equations are

$$\begin{aligned}
\frac{D\vec{q}}{Dt} &= -(\nabla\vec{v})^T\cdot\vec{q} - \nabla\left(-(\vec{v}\cdot\nabla)\varphi - \left|\vec{v}\right|^2\right) + (\nabla\times\vec{B})\times\vec{B} \\
\frac{D\vec{q}}{Dt} &= -(\nabla\vec{v})^T\cdot\vec{q} - \nabla\left(-(\vec{v}\cdot\nabla)\varphi - \frac{1}{2}\left|\vec{v}\right|^2\right) + (\nabla\times\vec{B})\times\vec{B}\,, \\
\frac{D\vec{q}}{Dt} &= -(\nabla\vec{v})^T\cdot\nabla\varphi + (\nabla\times\vec{B})\times\vec{B}
\end{aligned} \tag{2.117}$$

respectively. For Hall MHD, they are

$$\begin{aligned}
\frac{D\vec{q}}{Dt} &= -(\nabla\vec{v})^T\cdot\vec{q} - \nabla\left(-(\vec{v}\cdot\nabla)\varphi - \left|\vec{v}\right|^2\right) + \frac{1}{c}\left(\vec{J}\times\frac{\vec{B}}{\rho}\right) \\
\frac{D\vec{q}}{Dt} &= -(\nabla\vec{v})^T\cdot\vec{q} - \nabla\left(-(\vec{v}\cdot\nabla)\varphi - \frac{1}{2}\left|\vec{v}\right|^2\right) + \frac{1}{c}\left(\vec{J}\times\frac{\vec{B}}{\rho}\right). \\
\frac{D\vec{q}}{Dt} &= -(\nabla\vec{v})^T\cdot\nabla\varphi + \frac{1}{c}\left(\vec{J}\times\frac{\vec{B}}{\rho}\right)
\end{aligned} \tag{2.118}$$

For electron MHD, there is no gauge transformation for the velocity field, and, thus, such modified gauges are meaningless.

We should note, here, that all three plasma models have gauge freedom in their magnetic vector potential. While we have only considered those magnetic gauges that yield useful geometric properties, gauge transformations analogous to the zero, EL and impetus gauges are possible for the magnetic vector potential, as well, and should result in similar benefits. For instance, the EL version of the evolution equation for the magnetic vector potential may have improved numerical stability.

3 THE REPRESENTATION AND INVARIANCE OF THE FLUID IMPULSE

3.1 Fluid Impulse and Gauge Transformations

We have introduced the gauge transformation on the velocity field,

$$\vec{q} = \vec{v} + \nabla\varphi\,, \tag{3.1}$$

for the purpose of constructing geometrical invariants. We would like to find an interpretation for this new quantity. **[Batchelor]** and **[Saffman]** have showed that, for an incompressible hydrodynamic fluid of uniform density, this quantity can be connected to the fluid impulse density per unit mass density. We now describe that connection, and extend the interpretation to the plasma models.

The fluid impulse, $\vec{I}$, is the total momentum change required to generate, from rest, a given fluid motion created by external, non-conservative forces. That is to say,

$$\frac{\partial\vec{I}}{\partial t} = \iiint_V \rho\vec{F}dV\,. \tag{3.2}$$

Here, $\vec{F}$ is the external non-conservative force per unit density, ρ is the mass density and V is the fluid volume.

It has been shown by **[Batchelor]** that, for an incompressible hydrodynamic fluid whose velocity falls off as r^{-3}, or faster, at large distances, that the total momentum of a vorticity distribution is given by

$$\vec{I} = \frac{\rho}{2}\iiint_V \vec{r}\times\vec{\omega}dV \ , \tag{3.3}$$

where $\vec{r}$ is the position vector, $\vec{\omega}$ is the fluid vorticity, and the integral is taken over the entire fluid volume, V. This, then, is a part of the fluid impulse called the vortex momentum. To continue, we derive some less familiar vector calculus identities for our use, now and later.

Consider two vectors $\vec{a}$ and $\vec{b}$ in the combination $(\vec{a}\cdot\nabla)\vec{b}$. Converting to index notation and applying the product rule gives the identity

$$\begin{aligned}(\vec{a}\cdot\nabla)\vec{b} &= a_i b_{j,i}\\ &= (a_i b_j)_{,i} - a_{i,i} b_j \\ &= \nabla\cdot\left(\vec{a}\otimes\vec{b}\right) - (\nabla\cdot\vec{a})\vec{b}\end{aligned} \quad . \tag{3.4}$$

In the last expression, the first object is a vector whose components are the divergences of the tensor along the first index. Additionally, from the above result, we get

$$\vec{a}\times(\nabla\times\vec{b}) = (\nabla\vec{b})^T\cdot\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$$
$$= \nabla(\vec{a}\cdot\vec{b}) - (\nabla\vec{a})^T\cdot\vec{b} - \nabla\cdot\left(\vec{a}\otimes\vec{b}\right) + (\nabla\cdot\vec{a})\vec{b} \quad (3.5)$$

Now, integrating the above over a volume, V , gives

$$\iiint_V \vec{a}\times(\nabla\times\vec{b})dV = \oiint_{\partial V}(\vec{a}\cdot\vec{b})d\vec{S} - \iiint_V (\nabla\vec{a})^T\cdot\vec{b}dV - \iiint_V \nabla\cdot\left(\vec{a}\otimes\vec{b}\right)dV + \iiint_V (\nabla\cdot\vec{a})\vec{b}dV \ . \quad (3.6)$$

Looking at the integral

$$\iiint_V \nabla\cdot\left(\vec{a}\otimes\vec{b}\right)dV \ , \quad (3.7)$$

we see that its integrand is a vector whose components are $\nabla\cdot(\vec{a}b_j)$. Thus, applying the Divergence Theorem to each component yields

$$\left(\iiint_V \nabla\cdot\left(\vec{a}\otimes\vec{b}\right)dV\right)_j = \oiint_{\partial V}(\vec{a}b_j)\cdot\hat{n}dS = \oiint_{\partial V} b_j(\vec{a}\cdot\hat{n})dS \ . \quad (3.8)$$

Reassembling the above into a vector gives

$$\iiint_V \nabla \cdot \left(\vec{a} \otimes \vec{b}\right) dV = \oiint_{\partial V} \vec{b}(\vec{a} \cdot \hat{n}) dS \,. \tag{3.9}$$

Thus, we have the identity

$$\iiint_V \vec{a} \times (\nabla \times \vec{b}) dV = \oiint_{\partial V} (\vec{a} \cdot \vec{b}) d\vec{S} - \iiint_V (\nabla \vec{a})^T \cdot \vec{b} dV - \oiint_{\partial V} \vec{b}(\vec{a} \cdot \hat{n}) dS + \iiint_V (\nabla \cdot \vec{a}) \vec{b} dV \,. \tag{3.10}$$

Now, for the position vector, $\vec{r}$, we have

$$\nabla \vec{r} = I \tag{3.11}$$

and,

$$\nabla \cdot \vec{r} = 3 \,. \tag{3.12}$$

So, in the specific case that $\vec{a} = \vec{r}$,

$$\begin{aligned} \iiint_V \vec{r} \times (\nabla \times \vec{b}) dV &= \oiint_{\partial V} (\vec{r} \cdot \vec{b}) d\vec{S} - \iiint_V \vec{b} dV - \oiint_{\partial V} \vec{b}(\vec{r} \cdot \hat{n}) dS + \iiint_V 3\vec{b} dV \\ &= 2\iiint_V \vec{b} dV + \oiint_{\partial V} (\vec{r} \cdot \vec{b}) d\vec{S} - \oiint_{\partial V} \vec{b}(\vec{r} \cdot \hat{n}) dS \end{aligned} \,. \tag{3.13}$$

With these identities in place, we now return to the expression for the vortex momentum, (3.3). For an incompressible hydrodynamic fluid the velocity field and the field $\vec{q}$ have the same vorticity. Therefore, we have

$$\begin{aligned}\vec{I} &= \frac{\rho}{2}\iiint_V \vec{r}\times\vec{\omega}dV \\ &= \frac{\rho}{2}\iiint_V \vec{r}\times(\nabla\times\vec{q})dV\end{aligned}. \tag{3.14}$$

Application of the identity, (3.13), gives

$$\begin{aligned}\vec{I} &= \frac{\rho}{2}\iiint_V \vec{r}\times(\nabla\times\vec{q})dV \\ &= \rho\iiint_V \vec{q}dV + \frac{\rho}{2}\oiint_{\partial V}(\vec{r}\cdot\vec{q})d\vec{S} - \frac{\rho}{2}\oiint_{\partial V}\vec{q}(\vec{r}\cdot\hat{n})dS\end{aligned}, \tag{3.15}$$

where the volume integral is over the entire fluid volume. If the portion of the fluid with nonzero $\vec{q}$ has compact support, as would be the case if vorticity was localized, then the surface of integration can be placed outside the nonzero region, the surface integrals will vanish, and there will be no contribution to the volume integral from outside the enclosed region. Thus, for this case, we have

$$\vec{I} = \frac{\rho}{2}\iiint_V \vec{r} \times (\nabla \times \vec{q})dV = \rho \iiint_V \vec{q}dV \,, \tag{3.16}$$

and, so, $\vec{q}$ can be interpreted as the vortex momentum density per unit mass density, which we will call $\vec{p}$, the fluid impulse density, as a shorthand.

For the plasma models, (3.3) still defines the vortex momentum, and, therefore, $\vec{p}$ is still the fluid impulse density. For compressible, barotropic models the field, $\vec{q}$, may be defined to have the same curl as the velocity field, but the expression for the vortex momentum, (3.3), that must be handled carefully. Since mass density is not constant, ρ must be moved inside the integral. However, for these cases, we can still work with an analog of $\vec{p}$, based upon the incompressible definition, even though it may not be the full vortex momentum.

3.2 Why Fluid Impulse Density

Beyond its geometrical properties, it is natural to ask whether defining the fluid impulse density in this way is a useful thing to do. Clearly, the fluid impulse density is not the velocity. Since the velocity is a more easily measurable quantity, why would we want to use the impulse density? We discuss here a few types of situations where the impulse density may be preferable to the velocity, and the disadvantages of each.

When we look at the integral that gives the vortex momentum,

$$\vec{I} = \frac{\rho}{2}\iiint_V \vec{r} \times \vec{\omega} dV \ , \tag{3.17}$$

we see that, although the derivation of this integral required that the magnitude of velocity fall off as r^{-3} at infinity, the integral, itself, makes no reference to velocity. Its convergence is only dependent upon the behavior of the vorticity. If the vorticity is well behaved, this integral can be computed even if the velocity has undesirable properties. Since the fluid impulse density has the same vorticity as the velocity field, if it has more desirable properties it can used in place of the velocity in applications that would normally require velocity without altering any vorticity related results.

For example, suppose that $\Re$ is a compact region. Suppose that the vector field, $\vec{F}$, is such that it has no divergence and nonzero curl on parts of $\Re$ and vanishes outside of $\Re$. For continuity and differentiability, $\vec{F}$ can be chosen such that its magnitude and its curl disappear on the boundary of $\Re$. Consider the case of a velocity field given by

$$\vec{v} = < x, y, 0 > + \vec{F} \ . \tag{3.18}$$

This velocity field is incompressible and has zero curl outside of $\Re$. We notice that the magnitude of this velocity does not diminish for large distances. Thus, any integrals involving $\vec{v}$ taken over the entire fluid or over any surface terms taken to infinity will not converge.

However, in this case, integrals of $\vec{v}$ can be broken into two parts: the integral over the conservative portion, which contains no vorticity information, and the integral over $\vec{F}$, where any vorticity effects will be found. Therefore, if we replace $\vec{v}$ with

$$\begin{aligned}\vec{p} &= \vec{v} + \nabla\varphi \\ &= \vec{v} - < x, y, 0 >, \\ &= \vec{F}\end{aligned} \tag{3.19}$$

then, similar integrals on $\vec{p}$ may converge, while maintaining vorticity. On the other hand, though $\vec{p}$ and its integrals are well behaved in this case, we have a nonzero velocity at infinity, which some, though not all, applications consider to be nonphysical.

A more physically meaningful example may be to consider

$$\vec{v} = \left\langle \frac{-y}{|\vec{r}|}, \frac{x}{|\vec{r}|}, 0 \right\rangle \tag{3.20}$$

on $\Re$, and

$$\vec{v} = \vec{F}, \tag{3.21}$$

otherwise. Where $\Re$ and $\vec{F}$ are defined as above and with the values of the function and its derivatives chosen so that they match on the boundary. In this case, we do have the velocity falling to zero at infinity. However, the magnitude does not decrease sufficiently quickly to ensure the convergence of integrals over the entire fluid. This time, if we choose

$$\begin{aligned}\vec{p} &= \vec{v} + \nabla\varphi \\ &= \vec{v} - \left\langle \frac{-y}{|\vec{r}|}, \frac{x}{|\vec{r}|}, 0 \right\rangle,\end{aligned} \tag{3.22}$$

we again have $\vec{p} = \vec{0}$ outside of $\Re$. Unfortunately, we see that we now have a singularity at the origin, impacting the physical relevance of $\vec{p}$. Furthermore, we see that we have exchanged the divergence at infinity of integrals of $\vec{v}$ with possible divergence at the origin. This may or may not be an improvement.

Finally, consider a velocity field of the form

$$\vec{v} = \left\langle \frac{-y}{|\vec{r}|}, \frac{x}{|\vec{r}|}, 0 \right\rangle + \overrightarrow{F} \tag{3.23}$$

Where $\Re$ and $\overrightarrow{F}$ are defined as above except that $\overrightarrow{F}$ is zero near the origin. In this case, the original velocity field has a singularity, which, though nonphysical, may still arise in hydrodynamic computations. In this case,

$$\begin{aligned}\vec{p} &= \vec{v} + \nabla\varphi \\ &= \vec{v} - \left\langle \frac{-y}{|\vec{r}|}, \frac{x}{|\vec{r}|}, 0 \right\rangle\end{aligned} \quad (3.24)$$

will remove the singularity, with the remaining system being well behaved.

3.3 Fluid Impulse for an Incompressible Hydrodynamic Fluid

Having defined the fluid impulse and fluid impulse density, we would like to have a sense of their behaviors. Specific solutions for the behavior of the fluid impulse density will be derived in **Chapter 6.** However, in the remainder of this chapter, we discuss a useful, general result regarding the fluid impulse.

[Batchelor] showed that, for an ideal, incompressible hydrodynamic system whose velocity and vorticity fields fall off sufficiently quickly, the fluid impulse is a global invariant. We rederive this result, including details of steps omitted by Batchelor, so that we may develop similar results for the plasma systems.

We begin by taking the material derivative of the expression for the fluid impulse, giving

$$\begin{aligned}\frac{D\vec{I}}{Dt} &= \frac{D}{Dt}\left(\frac{\rho}{2}\iiint_V \vec{r}\times\vec{\omega}dV\right)\\ &= \frac{\rho}{2}\frac{D}{Dt}\left(\iiint_V \vec{r}\times\vec{\omega}dV\right) \quad . \\ &= \frac{\rho}{2}\frac{\partial}{\partial t}\left(\iiint_V \vec{r}\times\vec{\omega}dV\right)+\frac{\rho}{2}(\vec{v}\cdot\nabla)\left(\iiint_V \vec{r}\times\vec{\omega}dV\right)\end{aligned} \tag{3.25}$$

In the above, we used incompressibility to pull the density out of the derivative. Now, since the integral is over volume, the integral's result has all spatial dependence integrated out, making the second, spatial, derivative equal to zero. Continuing, we have

$$\begin{aligned}\frac{D\vec{I}}{Dt} &= \frac{\rho}{2}\frac{\partial}{\partial t}\left(\iiint_V \vec{r}\times\vec{\omega}dV\right)\\ &= \frac{\rho}{2}\iiint_V \frac{\partial\vec{r}}{\partial t}\times\vec{\omega}+\vec{r}\times\frac{\partial\vec{\omega}}{\partial t}dV\end{aligned} \quad . \tag{3.26}$$

The position vector is a fixed vector, and has no time derivative. Therefore,

$$\frac{D\vec{I}}{Dt} = \frac{\rho}{2}\iiint_V \vec{r}\times\frac{\partial\vec{\omega}}{\partial t}dV \; . \tag{3.27}$$

We now substitute the vorticity equation for hydrodynamics to get

$$\frac{D\vec{I}}{Dt} = \frac{\rho}{2}\iiint_V \vec{r} \times \left(\nabla \times (\vec{v} \times \vec{\omega})\right) dV \ . \tag{3.28}$$

Our objective, here, is to transform this integral into a sum of surface integrals, which we can then evaluate by use of the falloff of the velocity and vorticity.

Now, utilizing the integral identity, (3.13), we have

$$\begin{aligned} \frac{D\vec{I}}{Dt} &= \frac{\rho}{2}\iiint_V \vec{r} \times \left(\nabla \times (\vec{v} \times \vec{\omega})\right) dV \\ &= \rho \iiint_V \vec{v} \times \vec{\omega} dV + \frac{\rho}{2} \oiint_{\partial V} (\vec{r} \cdot (\vec{v} \times \vec{\omega})) d\vec{S} - \frac{\rho}{2} \oiint_{\partial V} (\vec{v} \times \vec{\omega})(\vec{r} \cdot \hat{n}) dS \end{aligned} \ . \tag{3.29}$$

In the above, the integrals are over the entire fluid volume, and the surface is allowed to go to infinity. As before, if vorticity has compact support, the surface integrals can be made to vanish. However, if the vorticity is not bounded, we must let the surface extend towards infinity. However, by assumption, velocity is $O(r^{-3})$, therefore $\vec{v} \times \vec{\omega}$ is $O(r^{-7})$, and the surface integral is $O(r^{-4})$ and vanishes for large $\left|\vec{r}\right|$.

Continuing, we have

$$
\begin{aligned}
\frac{D\vec{I}}{Dt} &= \rho\iiint_V \vec{v}\times\vec{\omega}dV \\
&= \rho\iiint_V \vec{v}\times(\nabla\times\vec{v})dV \\
&= \rho\iiint_V \nabla\left(\frac{1}{2}\left|\vec{v}\right|^2\right)-(\vec{v}\cdot\nabla)\vec{v}dV
\end{aligned}
\quad . \qquad (3.30)
$$

Using the identity in (3.4) and the incompressibility of the fluid gives

$$
(\vec{v}\cdot\nabla)\vec{v} = \nabla\cdot\left(\vec{v}\otimes\vec{v}\right)-(\nabla\cdot\vec{v})\vec{v} = \nabla\cdot\left(\vec{v}\otimes\vec{v}\right). \qquad (3.31)
$$

This, upon substitution into (3.30), use of the Divergence Theorem and of the identity in (3.8), gives

$$
\begin{aligned}
\frac{D\vec{I}}{Dt} &= \rho\iiint_V \nabla\left(\frac{1}{2}\left|\vec{v}\right|^2\right)-\nabla\cdot\left(\vec{v}\otimes\vec{v}\right)dV \\
&= \frac{\rho}{2}\oiint_{\partial V}\left|\vec{v}\right|^2 d\vec{S}-\rho\oiint_{\partial V}\vec{v}(\vec{v}\cdot\hat{n})dS
\end{aligned}
\quad . \qquad (3.32)
$$

For a velocity field with compact support these surface integrals can easily be made to vanish. For a noncompact region, our assumption that velocity is $O(r^{-3})$ ensures that each integral is $O(r^{-4})$ and goes to zero at large distances.

Thus, we have

$$\frac{D\vec{I}}{Dt} = 0\,, \tag{3.33}$$

and the fluid impulse is a global invariant.

It is useful to note that, while we require velocity to be $O(r^{-3})$ and vorticity to be $O(r^{-4})$, we do not have to determine both of these conditions for the incompressible case. If velocity is known to be $O(r^{-3})$, then $\vec{\omega} = \nabla \times \vec{v}$ implies that vorticity will be $O(r^{-4})$. On the other hand, if vorticity is known to be $O(r^{-4})$, and the fluid is incompressible, then, $\vec{v} = \nabla \times \vec{A}$ and

$$\vec{\omega} = \nabla \times (\nabla \times \vec{A})) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\,. \tag{3.34}$$

If we gauge $\vec{A}$ such that $\nabla \cdot \vec{A} = 0$ then, we have

$$\vec{\omega} = -\nabla^2 \vec{A} \tag{3.35}$$

which implies that $\vec{A}$ is at most $O(r^{-2})$ and the velocity is at most $O(r^{-3})$.

3.4 Fluid Impulse for Incompressible MHD

We now extend **[Batchelor]**'s result to magnetohydrodynamics. Working with the fluid impulse analog, we have, as before,

$$\vec{I} = \frac{\rho}{2}\iiint_V \vec{r}\times\vec{\omega}\,dV \tag{3.36}$$

and

$$\begin{aligned}\frac{D\vec{I}}{Dt} &= \frac{D}{Dt}\left(\frac{\rho}{2}\iiint_V \vec{r}\times\vec{\omega}\,dV\right)\\ &= \frac{\rho}{2}\iiint_V \vec{r}\times\frac{\partial\vec{\omega}}{\partial t}\,dV\end{aligned}. \tag{3.37}$$

Once again, we have utilized incompressibility. Now, substituting the vorticity equation for MHD gives

$$\begin{aligned}\frac{D\vec{I}}{Dt} &= \frac{\rho}{2}\iiint_V \vec{r}\times\left(\nabla\times(\vec{v}\times\vec{\omega})+\frac{1}{4\pi}\nabla\times\left((\nabla\times\vec{B})\times\vec{B}\right)\right)dV\\ &= \frac{\rho}{2}\iiint_V \vec{r}\times\left(\nabla\times(\vec{v}\times\vec{\omega})\right)dV+\frac{\rho}{8\pi}\iiint_V \vec{r}\times\left(\nabla\times\left((\nabla\times\vec{B})\times\vec{B}\right)\right)dV\end{aligned}. \tag{3.38}$$

The first integral is already known to go to zero for compact support or for sufficiently rapid falloff of velocity or vorticity. Applying the identities from the previous section to the second integral gives

$$\begin{aligned}\frac{D\vec{I}}{Dt} &= \frac{\rho}{8\pi}\iiint_V \vec{r}\times\left(\nabla\times\left((\nabla\times\vec{B})\times\vec{B}\right)\right)dV \\ &= \frac{\rho}{4\pi}\iiint_V (\nabla\times\vec{B})\times\vec{B}dV + \frac{\rho}{8\pi}\oiint_{\partial V}(\vec{r}\cdot((\nabla\times\vec{B})\times\vec{B}))d\vec{S} - \frac{\rho}{8\pi}\oiint_{\partial V}((\nabla\times\vec{B})\times\vec{B})(\vec{r}\cdot\hat{n})dS\end{aligned} \quad . \qquad (3.39)$$

As before, we must eliminate the surface terms. Also, as before, one of the options is for the magnetic field to have compact support. However, since the magnetic field can permeate empty space, terminating the fluid at a boundary is not a sufficient condition for terminating the magnetic field. Instead, we can imagine a plasma filling a cavity within a superconducting medium. Such a medium will block magnetic field lines at the interface. Embedding the plasma in various magnetized media does allow us control of the falloff of the magnetic field. However, the more physically useful arrangements would involve a bounded region of current surrounded by an unbounded plasma or surrounded by free space. In both of these situations, the dominant term in the magnetic field at large distances comes from the dipole moment, which falls off as r^{-3}. In these cases, the integral is $O(r^{-4})$ as it had been in the hydrodynamic case.

Thus, we have

$$\frac{D\vec{I}}{Dt} = \frac{\rho}{4\pi}\iiint_V (\nabla\times\vec{B})\times\vec{B}dV$$
$$= \frac{\rho}{4\pi}\iiint_V -\nabla\left(\frac{1}{2}\left|\vec{B}\right|^2\right) + (\vec{B}\cdot\nabla)\vec{B}dV \quad . \tag{3.40}$$
$$= -\frac{\rho}{8\pi}\oiint_{\partial V}\left|\vec{B}\right|^2 d\vec{S} + \frac{\rho}{4\pi}\oiint_{\partial V}\vec{B}(\vec{B}\cdot\hat{n})dS$$

In the dipole case, these surface terms disappear, and we have

$$\frac{D\vec{I}}{Dt} = 0 . \tag{3.41}$$

Thus, we have shown that the fluid impulse is a global invariant in the MHD model, as well. A similar analysis can be applied to the fluid impulse in Hall MHD and we obtain the proof of invariance for Hall MHD.

3.5 Fluid Impulse for EMHD

We now extend the results of the previous section even further to show that a similar global invariant exists for the electron MHD system. First, we begin with Ampere's Law which gives the electron velocity equation for eMHD,

$$\vec{\mathrm{v}}_e = -\frac{c}{4\pi ne}\nabla \times \vec{B}. \tag{3.42}$$

Taking the curl of both sides of (3.42), yields the electron vorticity

$$\vec{\omega}_e = -\frac{c}{4\pi ne}\nabla \times (\nabla \times \vec{B}). \tag{3.43}$$

This gives the vorticity evolution equation

$$\frac{\partial \vec{\omega}_e}{\partial t} = -\nabla \times (\nabla \times \frac{\partial \vec{B}}{\partial t}). \tag{3.44}$$

Now, computing the material derivative of the fluid impulse,

$$\begin{aligned} \frac{D\vec{I}}{Dt} &= \frac{D}{Dt}\left(\frac{\rho}{2}\iiint_V \vec{r} \times \vec{\omega}_e dV\right) \\ &= \frac{\rho}{2}\iiint_V \vec{r} \times \frac{\partial \vec{\omega}_e}{\partial t} dV \end{aligned}, \tag{3.45}$$

gives

$$\frac{D\vec{I}}{Dt} = -\frac{\rho}{2}\iiint_V \vec{r} \times \left(\nabla \times (\nabla \times \frac{\partial \vec{B}}{\partial t}) \right) dV \ . \tag{3.46}$$

Now, using the identity detailed in (3.13), we have

$$\frac{D\vec{I}}{Dt} = -\rho \iiint_V \nabla \times \frac{\partial \vec{B}}{\partial t} dV - \frac{\rho}{2} \oiint_{\partial V} (\vec{r} \cdot (\nabla \times \frac{\partial \vec{B}}{\partial t})) d\vec{S} + \frac{\rho}{2} \oiint_{\partial V} (\nabla \times \frac{\partial \vec{B}}{\partial t})(\vec{r} \cdot \hat{n}) dS \ . \tag{3.47}$$

Here, we must restrict not the magnitude of the magnetic field, but its rate of variation. If the magnetic field varies sufficiently slowly at large distances, we are left with

$$\begin{aligned} \frac{D\vec{I}}{Dt} &= -\rho \iiint_V \nabla \times \frac{\partial \vec{B}}{\partial t} dV \\ &= -\rho \oiint_{\partial V} \hat{n} \times \frac{\partial \vec{B}}{\partial t} dS \end{aligned} \ . \tag{3.48}$$

Enforcing the same slowness in variation at infinity yields

$$\frac{D\vec{I}}{Dt} = 0 \ . \tag{3.49}$$

Therefore, for a slowly varying magnetic field, the fluid is, again, a global invariant.

4 INTEGRAL INVARIANTS AND CIRCULATION THEOREMS

In addition to local invariant structures, there are other properties of a fluid which we can be determined without solving the fluid equations. Among these are integral invariants and topological invariants. In **this chapter**, we will construct integral invariants based on the local invariants developed in **Chapter 2**, then, in **Chapter 5**, we continue to topological invariants.

Whereas local, geometric invariants maintain a particular geometric structure at any given point, integral invariants are quantities whose total value over a particular region remains unchanged. The fluid impulse, as demonstrated in **sections 3.3** through **section 3.5** is an integral invariant. Circulation theorems, which detail the invariance of fluid properties around closed material curves, are another variety. The most famous example of such an invariant is given by Kelvin's Circulation Theorem, whose derivation follows **[Batchelor]**:

The equation of motion for an incompressible, viscous hydrodynamic fluid is

$$\rho \frac{D\vec{v}}{Dt} = \rho\vec{F} - \nabla p + \mu\nabla^2\vec{v} \tag{4.1}$$

If we assume that that the external force is conservative (it is often gravitational) then it can be written as a gradient, and, dividing through by mass density, we get

$$\frac{D\vec{v}}{Dt} = \nabla\left(\vec{F} - \frac{p}{\rho}\right) + \upsilon\nabla^2\vec{v} \tag{4.2}$$

where υ is the kinematic viscosity.

Now, consider the circulation of flow around a closed material curve

$$C(t) = \oint_{L(t)} \vec{v}\cdot d\vec{l}\ . \tag{4.3}$$

Taking the time derivative we have

$$\frac{dC}{dt} = \frac{d}{dt}\left(\oint_{L(t)} \vec{v}\cdot d\vec{l}\right). \tag{4.4}$$

Now, since the integral is taken along a material curve, pushing the time derivative into the integral changes it into a material derivative. The line element evolves as $\frac{d}{dt}(d\vec{l}\,) = d\vec{l}\cdot\nabla\vec{v}$, as any material line element does, (1.22). From these facts, and the product rule, we have

$$\frac{dC}{dt} = \oint_{L(t)} \frac{D\vec{v}}{Dt}\cdot d\vec{l} + \oint_{L(t)} \vec{v}\cdot(d\vec{l}\cdot\nabla\vec{v})\,. \tag{4.5}$$

Now, substituting the equation for $\frac{D\vec{v}}{Dt}$ and applying Stokes' Theorem gives

$$\begin{aligned}
\frac{dC}{dt} &= \oint_{L(t)} \frac{D\vec{v}}{Dt}\cdot d\vec{l} + \oint_{L(t)} \vec{v}\cdot(d\vec{l}\cdot\nabla\vec{v}) \\
&= \oint_{L(t)} \left(\nabla\left(\vec{F} - \frac{p}{\rho}\right) + \upsilon\nabla^2\vec{v}\right)\cdot d\vec{l} + \oint_{L(t)} \vec{v}\cdot(d\vec{l}\cdot\nabla\vec{v})\,. \\
&= \upsilon \oint_{L(t)} \nabla^2\vec{v}\cdot d\vec{l}
\end{aligned} \tag{4.6}$$

In the case that the fluid is inviscid, we get Kelvin's Circulation Theorem,

$$\frac{dC}{dt} = 0\,. \tag{4.7}$$

Thus, Kelvin's Circulation Theorem is an example of an integral invariant. We see that integral invariants come in two broad categories: those that are integrated over material sets, as is the case, here, and global invariants which are integrated over the entire space as demonstrated by the fluid impulse.

4.1 Varieties of Integral Invariants over Material Sets

In this section, we discuss generalities of the construction of integral invariants over material sets. There are three varieties of integral invariants over material sets: those integrated over material lines, material surfaces or material volumes. **[Tur & Yanovsky]** Each local invariant has a corresponding integral invariant. **[Tur & Yanovsky]** The different varieties are detailed, below.

First, consider a vector field, $\vec{S}$, that evolves as the dual to a material surface element. That is to say,

$$\frac{D\vec{S}}{Dt} = -(\nabla\vec{v})^T \cdot \vec{S} \,. \tag{4.8}$$

Let us integrate this vector field around a closed material loop, $L(t)$. The meaning of this integral is the circulation of the vector field around the loop. Taking the time derivative of the integral gives,

$$\frac{d}{dt}\oint_{L(t)} \vec{S}\cdot d\vec{l} = \oint_{L(t)} \frac{D\vec{S}}{Dt}\cdot d\vec{l} + \vec{S}\cdot\left((d\vec{l}\cdot\nabla)\vec{v}\right). \tag{4.9}$$

Here, again since the integral is over a material domain, the time derivative, when pushed inside the integral, becomes a material derivative. Continuing,

$$\begin{aligned}&\frac{d}{dt}\oint_{L(t)} \vec{S}\cdot d\vec{l} = \oint_{L(t)} -v_{i,j}S_i dl_j + S_i dl_j v_{i,j}\,. \\ &= 0\end{aligned} \tag{4.10}$$

Thus, the total circulation of the vector field around the material loop is invariant.

Now, consider at vector field that behaves as a material line element. That is to say,

$$\frac{D\vec{J}}{Dt} = (\vec{J}\cdot\nabla)\vec{v}\,. \tag{4.11}$$

Let us integrate the field $\rho\vec{J}$ over a material surface, $S(t)$. This gives us the flux of the field through the surface.

$$\iint_{S(t)} \rho\vec{J}\cdot d\vec{S} \tag{4.12}$$

Here, we must note that $d\vec{S}$ is a vector normal to a material surface element, not the dual as we have used up to this point. Multiplying $d\vec{S}$ by ρ yields a new quantity that evolves as the dual. That is to say,

$$\frac{D\rho d\vec{S}}{Dt} = -(\nabla\vec{v})^T \cdot \rho d\vec{S}\,. \tag{4.13}$$

Using this fact, we find time evolution of the flux, (4.12). This gives,

$$\begin{aligned}
&\frac{d}{dt}\iint\limits_{S(t)} \rho\vec{J}\cdot d\vec{S} = \iint\limits_{S(t)} \frac{D\vec{J}}{Dt}\cdot\rho d\vec{S} + \vec{J}\cdot\frac{\partial\rho d\vec{S}}{\partial t}\\
&== \iint\limits_{S(t)} \left((\vec{J}\cdot\nabla)\vec{v}\right)\cdot\rho d\vec{S} - \vec{J}\cdot\left((\nabla\vec{v})^T\cdot\rho d\vec{S}\right).\\
&= \iint\limits_{S(t)} J_i v_{j,i}\rho dS_j - J_i v_{j,i}\rho dS_j\\
&= 0
\end{aligned} \tag{4.14}$$

Therefore, the flux of $\rho\vec{J}$ through the material surface is conserved.

Finally, consider a conserved quantity, ρ, satisfying

$$\frac{D\rho}{Dt} = -\rho(\nabla\cdot\vec{v})\,. \tag{4.15}$$

We will investigate the time evolution of the integral of this quantity over a material volume, $V(t)$. Here, again, we must point out that dV evolves as a volume 3-vector, not a volume form. Thus, its change in time is given by the divergence of the velocity field. This gives,

$$
\begin{aligned}
&\frac{d}{dt}\iiint_{V(t)} \rho dV = \iiint_{V(t)} \frac{D\rho}{Dt}dV + \rho\frac{DdV}{Dt} \\
&= \iiint_{V(t)} -\rho(\nabla\cdot\vec{v})dV + \rho(\nabla\cdot\vec{v})dV \quad . \\
&= 0
\end{aligned}
\tag{4.16}
$$

Thus, as expected, the total amount of the conserved quantity in the material volume is conserved.

An alternative means to derive the above results would be to consider the advection of the material set as a time dependent change of coordinates. In this case, the evolution operator for the local invariant will cancel with the Jacobian of the transformation, leaving an integral that is time independent. **[Tur & Janovsky]**

4.2 Generalization of Integral Invariants

In the previous section, we saw that each local invariant, other than the Lagrange invariant, has a corresponding integral invariant. However, we can do more. Suppose we begin with a material surface invariant, $\vec{S}$, and add the gradient of a scalar field, φ. Then, obviously,

$$\frac{d}{dt}\oint_{L(t)}(\vec{S}+\nabla\varphi)\cdot d\vec{l}=\frac{d}{dt}\oint_{L(t)}\vec{S}\cdot d\vec{l}+\frac{d}{dt}\oint_{L(t)}\nabla\varphi\cdot d\vec{l} \qquad (4.17)$$
$$=0$$

by the result from the previous section and application of Stokes' Theorem. Similarly, adding the curl of a vector potential to a material line element gives

$$\frac{d}{dt}\iint_{S(t)}(\rho\vec{J}+\nabla\times\vec{A})\cdot d\vec{S}=\frac{d}{dt}\iint_{S(t)}\rho\vec{J}\cdot d\vec{S}+\frac{d}{dt}\iint_{S(t)}(\nabla\times\vec{A})\cdot d\vec{S} \qquad (4.18)$$
$$=0$$

by the results from the previous section and the Divergence Theorem. What we notice from these two simple proofs is that the integrands are exactly the set of quantities that can be transformed into a material surface or material line element by a gauge transformation, and, thus, all such quantities have an associated integral invariant. (see **section 2.6**) Since **[Kuzmin]** has shown that, for a hydrodynamic system, the velocity can be transformed into a material surface element via a gauge transformation, we have a quick means to determine that Kelvin's Circulation Theorem must be true. Also, we note that the set of integral invariants is much larger than the set of local invariants.

Beyond this, as always, we can add to the integrand, or multiply the integrand by, any Lagrangian invariants or function of Lagrangian invariants since these do not participate in the material derivative.

4.3 Integral Invariants for Compressible Hall MHD

We can now generate a variety of new integral invariants for the Hall MHD system by using the local invariants derived, previously. For instance, we have circulation theorems such as

$$\frac{d}{dt}\left[\oint_{L(t)}\left((\vec{q}+\frac{e\vec{A}}{mc})f+\nabla\varphi\right)\cdot d\vec{l}\right]=0\,. \tag{4.19}$$

Here, f is an arbitrary function of Lagrangian invariants, and φ is an arbitrary scalar field. We also have conserved flux integrals such as

$$\frac{d}{dt}\left[\iint_{S(t)}\left((\vec{\omega}+\frac{e\vec{B}}{mc})f+\nabla\times\vec{A}\right)\cdot d\vec{S}\right]=0 \tag{4.20}$$

and

$$\frac{d}{dt}\left[\iint_{S(t)}\left(f\nabla S\times(\vec{q}+\frac{e\vec{A}}{mc})+\nabla\times\vec{A}\right)\cdot d\vec{S}\right]=0 \tag{4.21}$$

and also integrals corresponding to conserved quantities such as

$$\frac{d}{dt}\iiint\limits_{V(t)} f\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\cdot\left(\vec{\varpi}+\frac{e\vec{B}}{mc}\right)dV=0 \tag{4.22}$$

and

$$\frac{d}{dt}\iiint\limits_{V(t)} f\left(\vec{\varpi}+\frac{e\vec{B}}{mc}\right)\cdot\nabla S dV=0\,. \tag{4.23}$$

Again, we note that these are merely the simplest examples of an infinite set of invariants.

4.4 Integral Invariants for Electron MHD

We can repeat a similar process for eMHD, (2.47), giving new circulation theorems

$$\frac{d}{dt}\left[\oint\limits_{L(t)}\left(f\vec{A}_e+\nabla\varphi\right)\cdot d\vec{l}\right]=0\,, \tag{4.24}$$

conserved flux integrals such as

$$\frac{d}{dt}\left[\iint_{S(t)}\left(f\vec{B}_e+\nabla\times\vec{A}\right)\cdot d\vec{S}\right]=0 \tag{4.25}$$

and

$$\frac{d}{dt}\left[\iint_{S(t)}\left(f\nabla S\times\vec{A}_e+\nabla\times\vec{A}\right)\cdot d\vec{S}\right]=0\,, \tag{4.26}$$

and integrals of conserved quantities such as

$$\frac{d}{dt}\iiint_{V(t)} f\vec{A}_e\cdot\vec{B}_e dV=0 \tag{4.27}$$

and

$$\frac{d}{dt}\iiint_{V(t)} f\vec{B}_e\cdot\nabla S dV=0\,. \tag{4.28}$$

For these examples, multiplication by ρ is unnecessary since ρ is constant.

4.5 Integral Invariants for Generalized, Nonuniform EMHD

For the previously discussed generalized EMHD, our new circulation theorems include

$$\frac{d}{dt}\left[\oint_{L(t)}\left(f\,\vec{p}_e+\nabla\varphi\right)\cdot d\vec{l}\right]=0\,. \tag{4.29}$$

Flux integral invariants include

$$\frac{d}{dt}\left[\iint_{S(t)}\left(f\,\rho\nabla\times\vec{p}_e+\nabla\times\vec{A}\right)\cdot d\vec{S}\right]=0 \tag{4.30}$$

and

$$\frac{d}{dt}\left[\iint_{S(t)}\left(f\,\rho\nabla S\times\vec{p}_e+\nabla\times\vec{A}\right)\cdot d\vec{S}\right]=0\,. \tag{4.31}$$

Finally, we have examples of integrals of conserved quantities such as

$$\frac{d}{dt}\iiint_{V(t)} f\,\vec{p}_e \cdot \nabla \times \vec{p}_e dV \tag{4.32}$$

and

$$\frac{d}{dt}\iiint_{V(t)} f(\nabla \times \vec{p}_e) \cdot \nabla S dV \ . \tag{4.33}$$

Again, we can ignore mass density, but, this time, not because it is a constant, but because it is a Lagrangian invariant due to incompressibility.

5 TOPOLOGICAL INVARIANTS

Another, even broader, category of invariants is topological invariants. Topological invariants arise naturally from the geometrical invariants described earlier in **Chapter 2**. Consider, for instance, physical quantities that correspond to material line elements. These come in two varieties: those that have sources and those that are sourceless. Quantities which behave like material line elements with sources arising from nonclosed two-forms and have point charges which can act as sources or sinks of the material line elements.**[Tur, Yanovsky]** The more familiar frozen-in fields are those that are sourceless, and all known physically meaningful frozen-in fields are of this type. These come from exact 2-forms, and are, therefore, closed. That is to say, they have zero divergence, and, therefore, no sources or sinks. Without charges to act as sources or sinks, the field lines of such physical quantities can only form closed loops or terminate at fluid boundaries. Loops cannot break, recombine, or diminish to zero magnitude as long as the proper physical conditions are maintained (for a simple fluid, as long as it remains inviscid). The fact that closed loops must remain closed, and cannot pass through each other, means that they behave as mathematical knots. Therefore, any topological invariants of knots, such as the self-winding number **[Moffatt]**, become invariants of the frozen-in field. On the other hand, field lines of a sourceless frozen-in field that extend to infinity or terminate at fluid boundaries behave like mathematical braids, and inherit all invariants of that field of topology.

Consider, on the other hand, a quantity that behaves as a dual to a material surface. Whereas material line elements can always be assembled into material curves, material surface elements must obey the Frobenius condition,

$$\vec{S} \cdot \nabla \times \vec{S} = 0 , \tag{5.1}$$

in order to allow a consistent assembly into a material surface. Once a material surface is constructed, however, it will have properties analogous to the frozen-in fields. Surfaces corresponding to unclosed one-forms will have sources in the form of closed material loops that bound the surface. Exact one-forms will yield unbounded surfaces which must form closed surfaces or terminate at fluid boundaries. This will lead to a different set of topological restrictions and invariants. For instance, closed surfaces that are nested must remain nested.

Finally, though the topic is not as rich, we mention, for completeness, the unclosed 3-form which will be a volume bounded by a material surface.

Though much has been said of the topology of the exact forms, the rarity of unclosed forms in existing applications makes them an interesting target of study as well. If a physically useful unclosed form could be found, then the fact that it will behave so much unlike the closed forms may give rise to interesting new physics and novel research. Potentially useful unclosed two forms can be constructed using the wedge product operation discussed in (1.90).

5.1 Linking Number of Closed Material Curves

We would now like to discuss the construction of a variety of topological invariant. The discussion, here, is a generalization of a derivation in **[Moffatt]**, and a variation of **[Tur and Yanovsky]**, who describe a related proof in differential geometry for a more restricted category of quantities.

Consider a thin knot (closed filament) of a quantity $\rho\vec{J}$ where $\vec{J}$ behaves as a material line element derived from an exact 2-form. Since the two-form associated with $\vec{J}$ is exact, $\rho\vec{J}$ is sourceless and has zero divergence as can be deduced from equation (1.89). Consider a material surface arranged as a cross-section of this filament. We can find the total flux of $\rho\vec{J}$ through the material surface. This is called the circulation strength, Γ. Thus,

$$\Gamma = \iint_{S(t)} \rho\vec{J} \cdot d\vec{S} \,. \tag{5.2}$$

Now, let us construct a second cross-section further down the filament, and let us join these two surfaces to form a closed surface. Since the divergence of $\rho\vec{J}$ inside the closed surface is zero, we have, by the Divergence Theorem, that the total flux through the second cross-section must be the same as that through the first. Thus, the circulation strength remains the same. If we continue this process around the entire knot, we see that the circulation strength must be the same everywhere along the filament.

Now, consider two linked, circular loops, γ_1 and γ_2, of quantities $\rho\vec{J}_1$ and $\rho\vec{J}_2$, respectively, where $\vec{J}_1$ and $\vec{J}_2$ are both material line elements derived from exact 2-forms. We can compute the total flux of $\rho\vec{J}_2$ through the loop γ_1 be the surface integral

$$\iint_{S_1(t)} \rho\vec{J}_2 \cdot d\vec{S}\,, \tag{5.3}$$

where $S_1(t)$ is the surface bounded by γ_1. This, of course, is just the circulation strength of $\rho\vec{J}_2$, Γ_2. Now, since Γ_2 is constant along γ_2, if γ_2 links γ_1 multiple times, the total flux will be

$$\iint_{\gamma_1(t)} \rho\vec{J}_2 \cdot d\vec{S} = L(\gamma_2,\gamma_1)\Gamma_2\,, \tag{5.4}$$

where $L(\gamma_2,\gamma_1)$ is the linking number of the two loops, and, by symmetry,

$$\iint_{\gamma_2(t)} \rho\vec{J}_1 \cdot d\vec{S} = L(\gamma_1,\gamma_2)\Gamma_1\,. \tag{5.5}$$

Since γ_1 and γ_2 are both material line elements, $S_1(t)$ and $S_2(t)$ are both material surfaces, integrals (5.4) and (5.5) are both integral invariants by equation (4.14); topological invariants of the link have become integral invariants of the fluid.

To continue, we would like to express an integral similar to (5.4) in terms of physical quantities. Since $\rho\vec{J}_1$ and $\rho\vec{J}_2$ are both exact, we know that the inverse of their curls will behave as a material surface element plus the gradient of a gauge field. If we define a material surface, $\vec{S}_2$, to be such that

$$\nabla \times \vec{S}_2 = \rho\vec{J}_2, \tag{5.6}$$

the we have, by Stokes' Theorem, that

$$\begin{aligned} L(\gamma_2,\gamma_1)\Gamma_2 &= \iint_{S_1(t)} \rho\vec{J}_2 \cdot d\vec{S} \\ &= \oint_{L(t)} curl^{-1}(\rho\vec{J}_2)\cdot d\vec{l} \\ &= \oint_{L(t)} \left(\vec{S}_2 + \nabla\varphi\right)\cdot d\vec{l} \\ &= \oint_{L(t)} \vec{S}_2 \cdot d\vec{l} \end{aligned} \tag{5.7}$$

is an integral invariant (see equation (4.14)). Now, let us embed these filaments in a three dimensional material volume, $V(t)$, where the fields $\rho\vec{J}_1$ and $\rho\vec{J}_2$ are zero except near the filaments. We will impose a material coordinate system on this volume where one axis lies parallel to the filaments and the other two are orthogonal. Since $\rho \vec{J}_1$ is parallel to γ_1, if we partially integrate the quantity $\rho \vec{J}_1 dV$ along the two directions orthogonal to γ_1 we will get

$$\rho \vec{J}_1 dV = \Gamma_1 d\vec{l} \ . \tag{5.8}$$

Therefore, for filament γ_1, the integral invariant $\iiint\limits_{V(t)} \left(curl^{-1}(\rho \vec{J}_2) \right) \cdot \rho \vec{J}_1 dV$ gives

$$\begin{aligned} &\iiint\limits_{V_1(t)} \left(curl^{-1}(\rho \vec{J}_2) \right) \cdot \rho \vec{J}_1 dV \\ &= \Gamma_1 \oint\limits_{L(t)} curl^{-1}(\rho \vec{J}_2) \cdot d\vec{l} \quad . \\ &= \Gamma_1 \Gamma_2 L(\gamma_2, \gamma_1) \end{aligned} \tag{5.9}$$

Since all arguments, above, are symmetric, we have

$$\iiint\limits_{V_1(t)} \left(curl^{-1}(\rho \vec{J}_2) \right) \cdot \rho \vec{J}_1 dV = \iiint\limits_{V_2(t)} \left(curl^{-1}(\rho \vec{J}_1) \right) \cdot \rho \vec{J}_2 dV \ . \tag{5.10}$$

Thus, we can take the integral over the space containing both filaments to give

$$\iiint_{V(t)} \left(curl^{-1}(\rho \vec{J}_2) \right) \cdot \rho \vec{J}_1 dV = 2\Gamma_1 \Gamma_2 L(\gamma_2, \gamma_1) . \qquad (5.11)$$

Suppose, now, that we have multiple loops of $\rho \vec{J}_2$ winding the loop of $\rho \vec{J}_1$. Then, expression (5.3) gives

$$\iint_{S_1(t)} \rho \vec{J}_2 \cdot d\vec{S} = \sum_i L(\gamma_i, \gamma_1) \Gamma_i . \qquad (5.12)$$

Therefore, the integral over the entire volume will give

$$\iiint_{V(t)} \left(curl^{-1}(\rho \vec{J}_2) \right) \cdot \rho \vec{J}_1 dV = \sum_i \sum_j \Gamma_j \Gamma_i L(\gamma_i, \gamma_j) . \qquad (5.13)$$

That is to say, if we disregard circulation strength, the volume provides a measure of the total interlinking of the fields $\rho \vec{J}_1$ and $\rho \vec{J}_2$. The integrand can, therefore, be interpreted as the average interlinking of the two fields near a given point weighted by circulation strength.

Of course, even though we have used simple circular loops as a model for deriving the above results, since the quantities obtained are topological invariants, the integrals still hold for homotopically deformed loops. Resulting configurations can be anything from a few simple loops, a large number of highly distorted loops, or even space-filling loops. **[Moffatt]**

5.2 Specific Linking Cases of Interest

We now discuss a few particular useful cases of frozen-in field linking. First, we notice that the linking number in equation (5.13) is difficult to extract. However, if there is only one loop of each field, or if the loops of each field all have the same circulation strength, then the linking number can be extracted. In this case, the integrand can be interpreted as the average interlinking of the two fields per unit volume **[Moffatt]**. While this simplification may seem to remove much of the richness of this approach, in the case of complicated, perhaps space-filling, fields, the result is far from trivial.

Secondly, consider the case where the integral (5.13) has $\rho\vec{J}_1 = \rho\vec{J}_2$. Here, the integral can be interpreted as a measure of the interlinking of different field lines of the same field. However, consider a single field line that is self-linked. Near each location of self-linking, we can insert two segments of opposite orientation and having zero field strength. These added segments will have the effect of splitting the field line into two field lines, one linked with the other, while removing the self-linkage. Since the field strength is zero along these segments, their insertion does not affect the result of the linking integral (5.13) **[Moffatt]**. Thus, the integral

$$\iiint_{V(t)} \left(curl^{-1}(\rho\vec{J})\right)\cdot\rho\vec{J}dV \qquad (5.14)$$

is a measure of both the interlinking and self-linking of the field $\rho\vec{J}$. **[Tur and Yanovsky]** If only a single field line is involved, then there is no interlinking and the circulation strength is constant, so the integral produces the self-linking of the field line, directly.

Another situation worth considering comes from the fact that, during the derivation of the linking integral, we assumed that $curl^{-1}(\rho\vec{J})$ had the behavior of a material surface element, and we discarded the gauge field, $\nabla\varphi$, by use of Stokes' Theorem. However, in some cases we are interested in the behavior of a gauge transformation of a surface element instead of the behavior of the surface element, itself. This is the case when dealing with the velocity field, which is a gauge transform of the fluid impulse density. In this case, the gauge field cannot be discarded, and the convergence of the integral depends upon the behavior of the gauge field at the fluid boundary or at infinity. Specifically, the normal component of $\varphi_1\rho\vec{J}_2$ must disappear at the boundary, or at infinity. Alternatively, if the integral is not taken over a material volume, this same condition must be met. **[Moffatt]** This allows the integral to be taken over the entire fluid, for example. Obviously, this condition is trivially true if the fluid is contained within a boundary that is impermeable to the field.

Finally, the integrals in the last two sections are, of course, not the only way to find linking number. They are chosen because they can be computed in terms of physical quantities. Certainly, the integrals above differ from the linking number in that they include the circulation strength, and, thus, are not limited to integer values. Since the loops in the above examples are determined by the fact that they lie parallel to the fields $\rho\vec{J}_i$, and since multiplying by a nowhere zero scalar field does not change these directions, we see that if we had a means to compute the linking number more directly, then the result would not change upon multiplication by such a nowhere zero scalar field. Thus, we see that the variety of linking topological invariants is much broader than both integral invariants, which were equivalent up to multiplication by a function of Lagrangian invariants and addition of an exact term, and geometric invariants, which were equivalent up to multiplication by functions of Lagrangian invariants, only.

5.3 Specific Topological Invariants for Hydrodynamics and Plasma Models

Now that we have a method of constructing topological invariants from integral invariants, we give some known examples of such invariants from literature, and extend to newer plasma models. One of the most famous topological invariants for hydrodynamics is Moffatt's

$$\iiint_{V(t)} \vec{v} \cdot \vec{\omega} dV \text{ , } \textbf{[Moffatt]} \qquad (5.15)$$

the total helicity. Here, the vorticity is a sourceless frozen-in field, and the velocity is a gauge transform of the material surface element which is the inverse curl of the vorticity. Thus this integral yields the knottedness of vortex lines but requires appropriate boundary conditions.

Two well-known examples from MHD are Woltjer's magnetic helicity,

$$\iiint_{V(t)} \vec{A}\cdot\vec{B}dV \ , \tag{5.16}$$

and cross helicity,

$$\iiint_{V(t)} \vec{v}\cdot\vec{B}dV \tag{5.17}$$ **[Woltjer (1958a) (1958b)]**.

Boundary conditions for the magnetic helicity only need to be considered if the choice of gauge is something other than the geometric gauge.

To construct new topological linking invariants, we first construct a new sourceless frozen-in field and then combine it with a material surface element, or a gauge transform, into a conserved volume integral. **[Tur and Yanovsky]** used a similar technique to construct topological invariants for MHD, though they consider only self-linking invariants.

Examples of topological invariants for barotropic, compressible MHD would include self-linking invariants such as

$$\begin{gathered}\iiint_{V(t)} \frac{1}{\rho}(\vec{B}\times(\nabla S\times\vec{A}))\cdot\nabla\times(\vec{B}\times(\nabla S\times\vec{A}))dV \\ \iiint_{V(t)} \frac{1}{\rho}\vec{B}\times\left(\nabla S\times\nabla\left(\frac{\vec{B}\cdot\nabla S}{\rho}\right)\right)\cdot\nabla\times\left(\vec{B}\times\left(\nabla S\times\nabla\left(\frac{\vec{B}\cdot\nabla S}{\rho}\right)\right)\right)dV\end{gathered} \ . \tag{5.18}$$

[Tur and Yanovsky] These represent the self-linking of the fields $\frac{1}{\rho}(\vec{B}\times(\nabla S\times\vec{A}))$ and $\frac{1}{\rho}\vec{B}\times\left(\nabla S\times\nabla\left(\frac{\vec{B}\cdot\nabla S}{\rho}\right)\right)$, respectively. However, we can construct new cross-linking invariants such as

$$\begin{aligned}&\iiint_{V(t)}\nabla S\cdot\vec{B}dV\\&\iiint_{V(t)}\nabla S\cdot\nabla\times(\vec{B}\times(\nabla S\times\vec{A}))dV\ ,\\&\iiint_{V(t)}\vec{A}\cdot\nabla\times(\vec{B}\times(\nabla S\times\vec{A}))dV\end{aligned}\tag{5.19}$$

all of which represent inter-linkings of various frozen-in fields.

For compressible, barotropic Hall MHD, the generalized magnetic helicity integral,

$$\iiint_{V(t)}\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\cdot\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)dV\ ,\tag{5.20}$$

gives the knottedness of the generalized magnetic field. Further, we can construct additional self-linking invariants such as

$$\iiint_{V(t)} \frac{1}{\rho}\left(\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)\times\left(\nabla S\times\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\right)\right)\cdot\nabla\times\left(\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)\times\left(\nabla S\times\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\right)\right)dV$$

$$\iiint_{V(t)} \frac{1}{\rho}\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)\times\left(\nabla S\times\nabla\left(\left(\frac{\vec{\omega}}{\rho}+\frac{e\vec{B}}{pmc}\right)\cdot\nabla S\right)\right)\cdot\nabla\times\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)\times\left(\nabla S\times\nabla\left(\left(\frac{\vec{\omega}}{\rho}+\frac{e\vec{B}}{pmc}\right)\cdot\nabla S\right)\right)dV,$$

(5.21)

and cross-linking invariants such as

$$\iiint_{V(t)} \nabla S\cdot\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)dV$$

$$\iiint_{V(t)} \nabla S\cdot\nabla\times\left(\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)\times\nabla S\times\left(\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\right)\right)dV\ . \qquad (5.22)$$

$$\iiint_{V(t)} \vec{A}\cdot\nabla\times\left(\left(\vec{\omega}+\frac{e\vec{B}}{mc}\right)\times\left(\nabla S\times\left(\vec{q}+\frac{e\vec{A}}{mc}\right)\right)\right)dV$$

Similarly, for electron MHD, we have a measure to the knottedness of the generalized magnetic field,

$$\iiint_{V(t)} \vec{A}_e\cdot\vec{B}_e dV\ , \qquad (5.23)$$

integrals that measure self-linking for various fields such as

$$\iiint_{V(t)} \vec{B}_e \times (\nabla S \times \vec{A}_e)) \cdot \nabla \times (\vec{B}_e \times (\nabla S \times \vec{A}_e))dV$$

$$\iiint_{V(t)} \vec{B}_e \times \left(\nabla S \times \nabla \left(\frac{\vec{B}_e \cdot \nabla S}{\rho} \right) \right) \cdot \nabla \times \left(\vec{B}_e \times \left(\nabla S \times \nabla \left(\frac{\vec{B}_e \cdot \nabla S}{\rho} \right) \right) \right) dV, \tag{5.24}$$

and integrals that measure inter-linking between different fields, such as

$$\iiint_{V(t)} \nabla S \cdot \vec{B}_e dV$$

$$\iiint_{V(t)} \nabla S \cdot \nabla \times (\vec{B}_e \times (\nabla S \times \vec{A}_e))dV\ . \tag{5.25}$$

$$\iiint_{V(t)} \vec{A}_e \cdot \nabla \times (\vec{B}_e \times (\nabla S \times \vec{A}_e))dV$$

Again, we see that these techniques allow of rapid construction of new invariants instead of deriving them individually, as had been the typical approach.

5.4 Applications of Topological Invariants

One of Moffatt's insights in investigating topological invariants was the application of the Schwarz inequality to the helicity integral. That is to say, we have

$$\left(\iiint\limits_{V(t)} \vec{v}\cdot\vec{\omega}dV\right)^2 \le \left(\iiint\limits_{V(t)} |\vec{v}|^2\, dV\right)\left(\iiint\limits_{V(t)} |\vec{\omega}|^2\, dV\right). \tag{5.26}$$

We already know that the first term is constant. If we restrict ourselves to an incompressible, inviscid hydrodynamic fluid, then conservation of energy demands that the second integral be constant as well. That is to say, the lack of viscosity prevents loss of kinetic energy. The total kinetic energy is determined by the total of the square of the velocity times the mass density. Since the fluid is incompressible, the mass density is a Lagrangian invariant, and thus, moves with the flow. Since the total local mass density cannot change, neither can the total square of the velocity. Because of the constancy of the first two integrals, we have a lower bound on the value of the third. That is to say, the total vorticity cannot decrease below a certain point.

Of course, since we have access to a much larger selection of integral invariants, we could construct many more similar inequalities, and place a lower bound on any term if we have a means to find an upper bound for the other.

For a novel application, we consider magnetic reconnection. During magnetic reconnection, magnetic field topology changes. Tracking topology through a reconnection event is one of the goals of magnetic reconnection research. Electron inertia provides a mechanism for reconnection, and thus, it is hoped that the electron MHD model can be used as tool for its understanding. While the ideal would be a precise knowledge of the behavior of the magnetic field during reconnection via electron inertia, we can utilize the topological invariants for the MHD and electron MHD systems to make some general statements about this behavior. In

MHD, electron inertia is ignored, since it is assumed that ions and electrons have enough time to react to electric fields. Over sufficiently short time scales, particle reaction time becomes important, and the magnetic field is augmented by an electron inertia term to give the generalized magnetic field

$$\vec{B}_e = \vec{B} - {d_e}^2 \nabla^2 \vec{B}, \tag{5.27}$$

where d_e is the electron skin depth and is proportional to electron mass. Even though the generalized magnetic field does not correspond to a physical field in the same way as the magnetic field, we can still use this mathematical construct as a means to track topology within the eMHD regime. Now, in a regime where the MHD model is valid, the magnetic helicity,

$$\iiint_{V(t)} \vec{A} \cdot \vec{B} dV , \tag{5.28}$$

is an integral invariant, and measures the self-knottedness of magnetic field lines. Upon transition to an electron MHD regime, it is the generalized magnetic helicity,

$$\iiint_{V(t)} \vec{A}_e \cdot \vec{B}_e dV , \tag{5.29}$$

that is conserved. (This development is due to Shivamoggi, unpublished.) We would like to see how this difference impacts conservation of magnetic helicity.

For a pair of loops, we can use Gauss' integral formula,

$$L(\gamma_2,\gamma_1)=\frac{1}{4\pi}\oint_{\gamma_2}\oint_{\gamma_1}\frac{\vec{r}}{|\vec{r}|^3}\cdot d\vec{l}_1\times d\vec{l}_2 \text{ [Moffatt]} \tag{5.30}$$

to compute their linking number. If we assume that the magnetic field is nonzero only close to loops, we can turn the above into sequential volume integrals (see (5.9)). This gives, for the generalized helicity,

$$\begin{aligned}&\iiint_{V(t)}\vec{A}_e\cdot\vec{B}_e dV = 2\Gamma^2 L(\gamma_2,\gamma_1)\\ &=\frac{1}{2\pi}\iiint_{V_2(t)}\iiint_{V_1(t)}\frac{\vec{r}}{|\vec{r}|^3}\cdot\vec{B}_e(\vec{r}_1)\times\vec{B}_e(\vec{r}_2)dV_1 dV_2\end{aligned}\text{. [Moffatt]} \tag{5.31}$$

In the above, we have assumed that all field lines have the same circulation strength. This allows the resulting integral to be interpreted as the self-linking of a single field line. Now, utilizing the definition of the generalized magnetic field, we get,

$$\iiint_{V(t)} \vec{A}_e \cdot \vec{B}_e dV = \frac{1}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(\vec{B}(\vec{r}_1) - d_e{}^2 \nabla^2 \vec{B}(\vec{r}_1)\right) \times \left(\vec{B}(\vec{r}_2) - d_e{}^2 \nabla^2 \vec{B}(\vec{r}_2)\right) dV_1 dV_2$$
$$= \frac{1}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(\vec{B}(\vec{r}_1)\right) \times \left(\vec{B}(\vec{r}_2)\right) dV_1 dV_2 - \frac{1}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(d_e{}^2 \nabla^2 \vec{B}(\vec{r}_1)\right) \times \left(\vec{B}(\vec{r}_2)\right) dV_1 dV_2$$
$$- \frac{1}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(\vec{B}(\vec{r}_1)\right) \times \left(d_e{}^2 \nabla^2 \vec{B}(\vec{r}_2)\right) dV_1 dV_2 + \frac{1}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(d_e{}^2 \nabla^2 \vec{B}(\vec{r}_1)\right) \times \left(d_e{}^2 \nabla^2 \vec{B}(\vec{r}_2)\right) dV_1 dV_2$$
$$= \frac{1}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(\vec{B}(\vec{r}_1)\right) \times \left(\vec{B}(\vec{r}_2)\right) dV_1 dV_2 + \frac{d_e{}^4}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(\nabla^2 \vec{B}(\vec{r}_1)\right) \times \left(\nabla^2 \vec{B}(\vec{r}_2)\right) dV_1 dV_2$$
$$. \tag{5.32}$$

The first term, we notice, is the self-winding of the magnetic field times the square of its circulation strength

$$\frac{1}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(\vec{B}(\vec{r}_1)\right) \times \left(\vec{B}(\vec{r}_2)\right) dV_1 dV_2 = 2\Gamma^2 L_{\vec{B}}(\vec{B}_2, \vec{B}_1)$$
$$= \iiint_{V(t)} \vec{A} \cdot \vec{B} dV \quad . \tag{5.33}$$

For the second term, we employ a vector calculus identity giving

$$\frac{d_e{}^4}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(\nabla^2 \vec{B}(\vec{r}_1)\right) \times \left(\nabla^2 \vec{B}(\vec{r}_2)\right) dV_1 dV_2 =$$
$$\frac{d_e{}^4}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(\nabla(\nabla \cdot \vec{B}(\vec{r}_1)) - \nabla \times (\nabla \times \vec{B}(\vec{r}_1))\right) \times \left(\nabla(\nabla \cdot \vec{B}(\vec{r}_2)) - \nabla \times (\nabla \times \vec{B}(\vec{r}_2))\right) dV_1 dV_2 \;. \tag{5.34}$$
$$= \frac{d_e{}^4}{2\pi} \iiint_{V_2(t)} \iiint_{V_1(t)} \frac{\vec{r}}{|\vec{r}|^3} \cdot \left(\nabla \times (\nabla \times \vec{B}(\vec{r}_1))\right) \times \left(\nabla \times (\nabla \times \vec{B}(\vec{r}_2))\right) dV_1 dV_2$$

Now, recalling the electron velocity for electron MHD, (2.58), and taking the curl, gives the electron vorticity which is

$$\begin{aligned}\vec{\omega}_e &= -\frac{c}{4\pi n_e e}\nabla\times(\nabla\times\vec{B}) \\ &= -d_e{}^2\frac{e}{cm_e}\nabla\times(\nabla\times\vec{B})\end{aligned}. \tag{5.35}$$

Substituting this into (5.34) gives

$$\begin{aligned}&\frac{d_e{}^4}{2\pi}\iiint\limits_{V_2(t)}\iiint\limits_{V_1(t)}\frac{\vec{r}}{|\vec{r}|^3}\cdot\left(\nabla^2\vec{B}(\vec{r}_1)\right)\times\left(\nabla^2\vec{B}(\vec{r}_2)\right)dV_1dV_2 \\ &= \frac{c^2m_e{}^2}{2\pi e^2}\iiint\limits_{V_2(t)}\iiint\limits_{V_1(t)}\frac{\vec{r}}{|\vec{r}|^3}\cdot\left(\vec{\omega}_e(\vec{r}_1)\right)\times\left(\vec{\omega}_e(\vec{r}_2)\right)dV_1dV_2\end{aligned}, \tag{5.36}$$

which we identify as the self-winding of the electron vorticity times twice its circulation strength. That is to say, we have,

$$\frac{c^2m_e{}^2}{2\pi e^2}\iiint\limits_{V_2(t)}\iiint\limits_{V_1(t)}\frac{\vec{r}}{|\vec{r}|^3}\cdot\left(\vec{\omega}_e(\vec{r}_1)\right)\times\left(\vec{\omega}_e(\vec{r}_2)\right)dV_1dV_2 = \frac{c^2m_e{}^2}{2\pi e^2}2\Gamma^2L_{\vec{\omega}_e}(\vec{\omega}_{e2},\vec{\omega}_{e1}) \tag{5.37}$$

Combining the results from (5.33) and (5.37) into (5.32) gives

$$\Gamma_{\vec{B}e}{}^2 L_{\vec{B}e} = \Gamma_{\vec{B}}{}^2 L_{\vec{B}} + \frac{c^2 m_e{}^2}{2\pi e^2} \Gamma_{\vec{\omega}e}{}^2 L_{\vec{\omega}e}, \tag{5.38}$$

where $\Gamma_{\vec{B}e}$, $\Gamma_{\vec{B}}$ and $\Gamma_{\vec{\omega}e}$ are the circulation strengths of the generalized magnetic field, the magnetic field and the electron vorticity, respectively, and $L_{\vec{B}e}$, $L_{\vec{B}}$, and $L_{\vec{\omega}e}$ are the total self-linkages of their respective fields. Grouping MHD quantities with electron MHD quantities gives

$$\Gamma_{\vec{B}}{}^2 L_{\vec{B}} = \Gamma_{\vec{B}e}{}^2 L_{\vec{B}e} - \frac{c^2 m_e{}^2}{2\pi e^2} \Gamma_{\vec{\omega}e}{}^2 L_{\vec{\omega}e}. \tag{5.39}$$

This form allows us to draw some conclusions about the behavior of the magnetic field as it becomes the generalized magnetic field during a transition from a MHD regime to an electron MHD regime. First, in the MHD regime, electron mass is neglected and the generalized magnetic field and magnetic field are identical. This corresponds to setting m_e equal to zero in the above equation. Upon transition to the electron MHD regime, m_e is nonzero, and the third term must be considered. Also, both the circulation strength and the self-linking of the generalized magnetic field are constant in the eMHD regime, due to zero divergence and topological invariance, respectively.

We can now make several possible predictions for the behavior of the various fields, which can be supported or refuted by physical experiment:

First, we note that the circulation strength of the electron vorticity cannot be zero, since that would put the system back into the MHD regime.

Second, suppose that the self-linking of the electron vorticity is always zero. This allows the magnetic field and generalized magnetic field to maintain the same circulation strength and topology, but it implies that electron vorticity is never self-linked after a transition from MHD. On the other hand, this also seems to imply that, if a configuration with non-zero electron vorticity self-linking could be created, it would by quasi-stable, preventing a transition back to the MHD until another mechanism of reconnection can eliminate the self-linking. Since the electron vorticity is not a frozen-in field without additional conditions, this option is unlikely.

Third, if electron vorticity and its self-linking are both nonzero then the magnetic field properties must change during transition to the generalized magnetic field. If we assume that generalized magnetic field topology must be the same as the original magnetic field, then circulation strength must change in order to absorb the contribution from the electron vorticity term.

The fourth, and most interesting, option is when we require circulation strength to remain approximately constant. In this case, the self-linking of the original magnetic field is distributed between the generalized magnetic field and the electron vorticity. Magnetic field topology must change during the transition to the generalized magnetic field. Upon return to the MHD domain, the magnetic field may assume a topology distinct from the original, as long as total helicity is conserved. Of course, since self-linking is quantized, such topological alterations would be quantized as well, implying that the circulation strength must be allowed to vary at least a small amount.

Of course, it is possible that a combination of effects exist, either simultaneously or at different energy levels. The case where the self-linking of electron vorticity is forced to be zero, in particular, implies the existence of a secondary mechanism. However, any combination that allows the difference in topology between the magnetic and generalized magnetic fields would seem to be a boon to the understanding of reconnection. Current experiments seem to support a change in the circulation strength.

6 SPECIFIC SOLUTIONS OF THE FLUID IMPULSE EQUATION

6.1 Solutions for Incompressible Hydrodynamics

The final goal of this research is to determine the behavior of the fluid impulse density in particular geometries in order to provide insight into its significance. We detail below two solutions to the impulse equation for a hydrodynamic fluid in a cylindrical geometry to use as a model for future solutions:

First, we generalize a solution of **[Russo and Smereka]**. Let us look for a solution with cylindrical symmetry and with a steady flow. Let vorticity have the form

$$\vec{\omega} = f(r)\hat{z} = < 0, 0, f(r) > . \tag{6.1}$$

Then, the velocity field will satisfy the equation

$$\vec{\omega} = \nabla \times \vec{v} = \left\langle \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \frac{1}{r}\frac{\partial r v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \right\rangle = \left\langle \omega_r, \omega_\theta, \omega_z \right\rangle . \tag{6.2}$$

In order to maintain cylindrical symmetry, the velocity field must have no θ dependence and no time dependence by the assumption of steadiness:

$$\vec{v} = \vec{v}(r, z)\,. \tag{6.3}$$

This gives

$$\frac{\partial v_z}{\partial \theta} = \frac{\partial v_r}{\partial \theta} = 0\,, \tag{6.4}$$

and thus

$$\begin{aligned} \omega_r &= 0 = -\frac{\partial v_\theta}{\partial z} \\ \omega_z &= f(r) = \frac{1}{r}\frac{\partial r v_\theta}{\partial r} \end{aligned}\,. \tag{6.5}$$

The first of these two equations implies that v_θ is a function of r, only. Because of this, the second equation is an ordinary differential equation which can be integrated to give

$$v_\theta = \frac{1}{r}\int_0^r s f(s)\,ds\,. \tag{6.6}$$

This leaves v_r and v_z to satisfy

$$\frac{\partial v_r}{\partial z} = \frac{\partial v_z}{\partial r}. \tag{6.7}$$

If we further assume that our flow is effectively two dimensional and incompressible then we have

$$\begin{aligned} v_r &= \frac{c}{r} \\ v_\theta &= \frac{1}{r}\int_0^r sf(s)ds \\ v_z &= 0 \end{aligned} \qquad \textbf{(6.8)}$$

for an arbitrary constant, c. This is a generalization of the **[Russo and Smereka]** result. For a flow to be "two dimensional", in this context, means that the fluid has no motion or variation in the z direction. The fluid, itself, of course, remains three dimensional. One must verify that this assumption is valid, but the fluid equations, in this case, leave us sufficient latitude to make this simplification.

This velocity field, (6.8), contains a singularity at the origin. If the region occupied by our fluid does not include the origin, then this may be acceptable. However, we choose $c = 0$ to remove the singularity. This leaves

$$v_r = 0$$
$$v_\theta = \frac{1}{r}\int_0^r sf(s)ds \qquad (6.9)$$
$$v_z = 0$$

Now, let us use this result in the fluid impulse equation with geometric gauge

$$\frac{\partial \vec{p}}{\partial t} + (\vec{v} \cdot \nabla)\vec{p} = -(\nabla \vec{v})^T \vec{p} \,. \qquad (6.10)$$

The conversion of this equation into cylindrical coordinates is somewhat involved, so the process is detailed here for later use.

When taking the spatial derivative of a vector field in a curvilinear coordinate system we must consider not only the change in the field with position, but also the change of the unit vectors with position. The derivatives of the unit vectors are vectors, themselves. For an orthogonal curvilinear coordinate system, the vector components of the derivatives of the unit vectors are given by:

$$\frac{\partial \hat{e}_i}{\partial x_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial x_i}\hat{e}_j \qquad (6.11)$$

for $i \neq j$ and

$$\frac{\partial \hat{e}_i}{\partial x_i} = -\sum_{k \neq i} \frac{1}{h_k} \frac{\partial h_i}{\partial x_k} \hat{e}_k \, , \tag{6.12}$$

otherwise. *In the above expressions, repeated indices are not summed*, and the h_i , called the scale factors, are the dependencies of differential changes in arclength on differential changes along the coordinate curves. Now, to compute $\nabla \vec{F}$, we apply the generalized gradient operator giving:

$$\nabla \vec{F} = \sum_i \frac{1}{h_i} \frac{\partial \vec{F}}{\partial x_i} \otimes \hat{e}_i \; . \tag{6.13}$$

When computing the spatial derivative, we must use the product rule to differentiate both the components and the unit vectors, and employ the unit vector derivative identities above. For instance, if

$$\vec{F} = \sum_i F_i \hat{e}_i \tag{6.14}$$

then,

$$\frac{\partial \vec{F}}{\partial x_j} = \sum_i \frac{1}{h_j}\frac{\partial F_i}{\partial x_j}\hat{e}_i + \frac{1}{h_j}F_i\frac{\partial \hat{e}_i}{\partial x_j}. \qquad (6.15)$$

Each of the derivatives of the unit vectors will have their own components. After grouping the new components by dimension, we have the tensor

$$(\nabla\vec{F})_{ij} = \frac{1}{h_j}\frac{\partial F_i}{\partial x_j} + \delta_{ij}\left(\sum_{k\neq i}\frac{F_k}{h_i h_k}\frac{\partial h_i}{\partial x_k}\right) - (\vec{1}-\delta_{ij})\frac{F_j}{h_i h_j}\frac{\partial h_j}{\partial x_i} \qquad (6.16)$$

where $\vec{1}$ is a matrix of ones.

For cylindrical coordinates specifically, the scale factors are

$$\begin{aligned} h_r &= h_z = 1 \\ h_\theta &= r \end{aligned} \qquad (6.17)$$

giving

$$\nabla\vec{F} = \begin{bmatrix} \frac{\partial F_r}{\partial r} & \frac{1}{r}\frac{\partial F_r}{\partial \theta} - \frac{F_\theta}{r} & \frac{\partial F_r}{\partial z} \\ \frac{\partial F_\theta}{\partial r} & \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{F_r}{r} & \frac{\partial F_\theta}{\partial z} \\ \frac{\partial F_z}{\partial r} & \frac{1}{r}\frac{\partial F_z}{\partial \theta} & \frac{\partial F_z}{\partial z} \end{bmatrix}. \qquad (6.18)$$

Using this in the fluid impulse equation, (6.10), and grouping by components gives

$$
\begin{aligned}
&\frac{\partial p_r}{\partial t}+\frac{\partial p_r}{\partial r}v_r+\frac{1}{r}\frac{\partial p_r}{\partial \theta}v_\theta-\frac{p_\theta}{r}v_\theta+\frac{\partial p_r}{\partial z}v_z=-\frac{\partial v_r}{\partial r}p_r-\frac{\partial v_\theta}{\partial r}p_\theta-\frac{\partial v_z}{\partial r}p_z\\
&\frac{\partial p_\theta}{\partial t}+\frac{\partial p_\theta}{\partial r}v_r+\frac{1}{r}\frac{\partial p_\theta}{\partial \theta}v_\theta+\frac{p_r}{r}v_\theta+\frac{\partial p_\theta}{\partial z}v_z=-\frac{1}{r}\frac{\partial v_r}{\partial \theta}p_r+\frac{v_\theta}{r}p_r-\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}p_\theta-\frac{v_r}{r}p_\theta-\frac{1}{r}\frac{\partial v_z}{\partial \theta}p_z\ .\\
&\frac{\partial p_z}{\partial t}+\frac{\partial p_z}{\partial r}v_r+\frac{1}{r}\frac{\partial p_z}{\partial \theta}v_\theta+\frac{\partial p_z}{\partial z}v_z=-\frac{\partial v_r}{\partial z}p_r-\frac{\partial v_\theta}{\partial z}p_\theta-\frac{\partial v_z}{\partial z}p_z
\end{aligned}
\tag{6.19}
$$

Substituting the velocity field,

$$\vec{v}=<0,\frac{1}{r}\int_0^r sf(s)ds,0>=<0,U(r),0>, \tag{6.20}$$

into (6.19) gives:

$$
\begin{aligned}
&\frac{\partial p_r}{\partial t}+\frac{1}{r}\frac{\partial p_r}{\partial \theta}U-\frac{p_\theta}{r}U=-U'p_\theta\\
&\frac{\partial p_\theta}{\partial t}+\frac{1}{r}\frac{\partial p_\theta}{\partial \theta}U=0\qquad .\\
&\frac{\partial p_z}{\partial t}+\frac{1}{r}\frac{\partial p_z}{\partial \theta}U=0
\end{aligned}
\tag{6.21}
$$

Now, enforcing cylindrical symmetry, there must be no θ dependence:

$$
\begin{aligned}
&\frac{\partial p_r}{\partial t} - \frac{p_\theta}{r} U = -U' p_\theta \\
&\frac{\partial p_\theta}{\partial t} = 0 \\
&\frac{\partial p_z}{\partial t} = 0
\end{aligned} \tag{6.22}
$$

From the second and third equations we know that p_θ and p_z must be unknown functions of r and z only. Since p_θ is time independent, we can integrate the first equation in terms of t to get

$$p_r = -U' p_\theta(r,z)t + U \frac{p_\theta}{r} t \,. \tag{6.23}$$

Thus, we have that the azimuthal and axial components of the fluid impulse density are time independent, while the radial component grows linearly in time. This gives the general solution

$$\vec{p} = < (\frac{U}{r} - U') p_\theta(r,z)t, p_\theta(r,z), p_z(r,z) > , \tag{6.24}$$

where p_θ and p_z are determined by initial conditions. If we again make the simplification that the fluid impulse density is effectively two-dimensional, we obtain a solution obtain a solution more similar to that of **[Russo and Smereka]**:

$$\vec{p} = <(\frac{U}{r} - U')\, p_\theta(r)t,\ p_\theta(r), 0>. \tag{6.25}$$

Now, consider the cylindrically symmetrical flow $\vec{v} = f(r)\hat{z} = <0, 0, f(r)>$. For this flow we can compute the vorticity by taking the curl directly giving $\vec{\omega} = \nabla \times \vec{u} = \left\langle 0, -f', 0 \right\rangle$. Upon substitution into the fluid impulse equation, we have

$$\begin{aligned}
\frac{\partial p_r}{\partial t} + f\frac{\partial p_r}{\partial z} &= -\frac{\partial f}{\partial r} p_z = -f' p_z \\
\frac{\partial p_\theta}{\partial t} + f\frac{\partial p_\theta}{\partial z} &= -\frac{\partial f}{\partial \theta} p_z = 0 \\
\frac{\partial p_z}{\partial t} + f\frac{\partial p_z}{\partial z} &= -\frac{\partial f}{\partial z} p_z = 0
\end{aligned} \quad . \tag{6.26}$$

Since the flow is translation invariant in the axial direction, it makes sense to choose an impulse density with the same property (though such a choice is certainly not required). This gives:

$$\begin{aligned}
\frac{\partial p_r}{\partial t} &= -f' p_z \\
\frac{\partial p_\theta}{\partial t} &= 0 \\
\frac{\partial p_z}{\partial t} &= 0
\end{aligned} \quad . \tag{6.27}$$

Again we have the azimuthal and axial components time independent, allowing us to integrate the first equation in time giving:

$$\vec{p} = < -f' p_z(r)t, p_\theta(r), p_z(r) > . \tag{6.28}$$

Again, we see that the radial component of the impulse density grows linearly in time while the azimuthal and axial components remain constant.

6.2 Solution for Compressible, Barotropic Hydrodynamics

We now derive a solution to a compressible, barotropic hydrodynamic system which extends the previous results. Suppose that the velocity field has the cylindrically symmetrical description,

$$\vec{v} = < V, U(r - Vt), 0 > . \tag{6.29}$$

Notice that, if we set V equal to zero, we get exactly the velocity profile which was used in the incompressible case, (6.20).

First, let us consider the mass conservation equation, (1.4) equation 1, which, when written in cylindrical coordinates gives

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 . \tag{6.30}$$

Substituting in the given velocity field gives

$$\frac{\partial \rho}{\partial t} + \frac{V}{r}\frac{\partial}{\partial r}(r\rho) + \frac{U}{r}\frac{\partial \rho}{\partial \theta} = 0\,. \tag{6.31}$$

If we assume cylindrical symmetry for the density distribution, the final term drops away, leaving

$$\frac{\partial \rho}{\partial t} + \frac{V}{r}\frac{\partial}{\partial r}(r\rho) = 0\,. \tag{6.32}$$

Multiplying through by r and rewriting gives

$$\begin{aligned} & r\frac{\partial \rho}{\partial t} + V\frac{\partial}{\partial r}(r\rho) = \\ & \frac{\partial}{\partial t}(r\rho) + V\frac{\partial}{\partial r}(r\rho) = 0 \end{aligned}\,. \tag{6.33}$$

Above, the last equation uses the fact that r and t are independent variables. We now make a substitution,

$$\overline{\rho} = r\rho\,, \tag{6.34}$$

and we see that our new variable satisfies the transport equation

$$\frac{\partial}{\partial t}(\overline{\rho}) + V\frac{\partial}{\partial r}(\overline{\rho}) = 0\,, \tag{6.35}$$

which has a traveling wave solution

$$\overline{\rho} = f(r - Vt)\,. \tag{6.36}$$

The unknown function is determined by boundary conditions. Thus, the mass density has the form

$$\rho = \frac{1}{r} f(r - Vt)\,. \tag{6.37}$$

Let us make some observations about this density function. First, we see that variations in density move outwards from the center with velocity V and their magnitude falls off as r^{-1}. Second, if we set V equal to zero, the density loses time dependency and becomes steady. Third, we observe that the divergence of a velocity field in cylindrical coordinates is

$$\nabla \cdot \vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z)\,. \tag{6.38}$$

Inputting our specific velocity field, (6.29), gives

$$\nabla \cdot \vec{v} = \frac{V}{r}\,. \tag{6.39}$$

If we assume that $r \neq 0$ and, again, let V equal zero, we get

$$\nabla \cdot \vec{v} = 0 \tag{6.40}$$

making the flow incompressible. We already remarked that setting V to zero gives the same velocity field as the incompressible problem. Thus, V can be taken to be a compressibility parameter, measuring both how much the flow deviates from an incompressible flow, and how much the velocity field deviates from the velocity field chosen for the incompressible problem. Finally, with V at zero we can choose $f = cr$. This leaves the density constant, as preferred in incompressible scenarios. This, then, is the proper choice for f for most applications.

One last detail of the density to consider is the boundary condition. Unlike many traveling wave problems, an initial condition may not be a valid approach to determining the density function. If V is negative, an initial condition must be independent of θ. Otherwise, the density function will become multivalued when the directionally dependent portion reaches the center. The situation is even worse if V is positive, since the portion of the density field near the axis immediately becomes undefined. Instead, we use a boundary condition. This presents difficulties of its own. Since we have a factor of r^{-1} in the density function, but our unknown function only has r appearing in the combination $r-Vt$, there is no way to construct a nontrivial density function that would be defined at $r=0$ for nonzero V. Thus, we cannot specify a boundary condition at that location, or consider a region that includes that location. We can, however, consider the portion of the fluid which lies outside of a cylinder of finite radius, $r=a$. Suppose that we have

$$\rho(a,\theta,z,t)=F(t)\,. \qquad (6.41)$$

where $F(t)$ is a known function at $r=a$. Then, we get

$$\rho=\frac{a}{r}F(r-a-Vt)\,. \qquad (6.42)$$

Again, we notice that the choice

$$\rho=\frac{a}{r}F(r-a-Vt)=\frac{ac}{r}(r-a-Vt) \qquad (6.43)$$

prescribed at a radius approaching zero, will give a constant density in the $V\to 0$ limit.

Having addressed the mass density, we now consider the components of the impulse density itself. Utilizing the evolution equations for the cylindrical components of the impulse density, (6.19), and substituting in the given velocity field, (6.29), we get

$$\begin{aligned}
&\frac{\partial p_r}{\partial t}+\frac{\partial p_r}{\partial r}V+\frac{1}{r}\frac{\partial p_r}{\partial \theta}U-\frac{p_\theta}{r}U=-U'\,p_\theta \\
&\frac{\partial p_\theta}{\partial t}+\frac{\partial p_\theta}{\partial r}V+\frac{1}{r}\frac{\partial p_\theta}{\partial \theta}U+\frac{p_r}{r}U=\frac{U}{r}\,p_r-\frac{V}{r}\,p_\theta\,. \\
&\frac{\partial p_z}{\partial t}+\frac{\partial p_z}{\partial r}V+\frac{1}{r}\frac{\partial p_z}{\partial \theta}U=0
\end{aligned} \tag{6.44}$$

Again, we assume that the fluid impulse density is cylindrically symmetrical, and effectively two-dimensional. This yields

$$\begin{aligned}
&\frac{\partial p_r}{\partial t}+\frac{\partial p_r}{\partial r}V-\frac{p_\theta}{r}U=-U'\,p_\theta \\
&\frac{\partial p_\theta}{\partial t}+\frac{\partial p_\theta}{\partial r}V=-\frac{V}{r}\,p_\theta \qquad . \\
&\frac{\partial p_z}{\partial t}+\frac{\partial p_z}{\partial r}V=0
\end{aligned} \tag{6.45}$$

We see that the z component is governed by the transport equation. Thus, it has a traveling wave solution in the r direction with speed V; for instance,

$$p_z=g_z(r-Vt)\,. \tag{6.46}$$

Multiplying the equation for the azimuthal component by r gives

$$r\frac{\partial p_\theta}{\partial t}+r\frac{\partial p_\theta}{\partial r}V+Vp_\theta=0\,. \tag{6.47}$$

Pulling the r into the time derivative, and using the reverse product rule on the r derivative gives

$$\frac{\partial(rp_\theta)}{\partial t}+V\frac{\partial(rp_\theta)}{\partial r}=0\,. \tag{6.48}$$

This is the same modified transport equation that is satisfied by the mass density (6.33) and, thus, has the same solution

$$p_\theta=\frac{1}{r}g_\theta(r-Vt)\,. \tag{6.49}$$

Finally, since we know the form of the azimuthal component, we can consider the radial equation. Rewriting the equation gives

$$\frac{\partial p_r}{\partial t}+V\frac{\partial p_r}{\partial r}=\frac{p_\theta}{r}U-U'\,p_\theta\,. \tag{6.50}$$

This is the nonhomogeneous transport equation which has the known solution

$$p_r = g_r(r - Vt) + \int_0^t H(r - V(t-s), s)ds \,, \qquad (6.51)$$

where $g_r(r)$ is the initial condition and $H(r,t)$ is the forcing term. Consider, though, when r and t appear in the combination $r - Vt$. Making the substitutions

$$\begin{aligned} r &\Rightarrow r - V(t-s) \\ t &\Rightarrow s \end{aligned} \,, \qquad (6.52)$$

we get

$$r - Vt \Rightarrow r - V(t-s) - Vs = r - Vt \,. \qquad (6.53)$$

Thus, we see that the traveling wave portion of the nonhomogeneous term is not altered by this substitution, and that all s dependency is removed from that portion of the integral. The azimuthal component also has a dependency on r outside the combination $r - Vt$, namely, in the multiple of r^{-1} that appears in (6.49). This will have to be separated so that it can be dealt with independently. So, we rewrite the nonhomogeneous term as

$$\frac{p_\theta}{r}U - U' p_\theta = \frac{g_\theta}{r^2}U - U'\frac{g_\theta}{r}. \tag{6.54}$$

So, substituting the nonhomogeneous term into the solution for the nonhomogeneous transport equation gives

$$\begin{aligned} p_r &= g_r(r-Vt) + \int_0^t \frac{g_\theta}{(r-V(t-s))^2}U - \frac{U' g_\theta}{r-V(t-s)}ds \\ &= g_r(r-Vt) + Ug_\theta \int_0^t \frac{1}{(r-V(t-s))^2}ds - U' g_\theta \int_0^t \frac{1}{r-V(t-s)}ds \end{aligned}. \tag{6.55}$$

This can be integrated by substitution to give

$$\begin{aligned} p_r &= g_r(r-Vt) + Ug_\theta \frac{1}{V}\int_{r-Vt}^{r} \frac{1}{u^2}du - U' g_\theta \frac{1}{V}\int_{r-Vt}^{r} \frac{1}{u}du \\ &= g_r(r-Vt) - Ug_\theta \frac{1}{V}\left(\frac{1}{r} - \frac{1}{r-Vt}\right) - U' g_\theta \frac{1}{V}\left(\ln|r| - \ln|r-Vt|\right). \\ &= g_r(r-Vt) + Up_\theta \frac{t}{r-Vt} - U' p_\theta r \frac{1}{V}\ln\left|\frac{r}{r-vt}\right| \end{aligned} \tag{6.56}$$

Thus, the components of the fluid impulse density are given by

$$
\begin{aligned}
p_r &= g_r(r - Vt) + Up_\theta \frac{t}{r - Vt} - U' p_\theta r \frac{1}{V} \ln\left|\frac{r}{r - vt}\right| \\
p_\theta &= \frac{1}{r} g_\theta(r - Vt) \\
p_z &= g_z(r - Vt)
\end{aligned}
\quad . \tag{6.57}
$$

Examining this result, we can make a few observations. First, we notice that, if we are to prescribe initial conditions, we encounter the same problem as we encountered with the mass density. We can solve this, again, by prescribing boundary conditions at a specific radius, instead. Secondly, if we take $\lim_{V \to 0}$ and assume two-dimensionality then it is clear that the azimuthal and axial components give exactly the solution for the incompressible problem (6.25). For the p_r component, it can be verified that the logarithmic term gives

$$
\lim_{V \to 0}\left(-U' p_\theta r \frac{1}{V} \ln\left|\frac{r}{r - vt}\right| \right) = -U' p_\theta t \tag{6.58}
$$

either by use of l’Hôpital’s Rule or by setting V equal to zero in the original integral (6.55). Thirdly, with V, the compressibility parameter, equal to zero, the azimuthal and axial components are static, and the radial component grows linearly in time. As the compressibility parameter is increased, the azimuthal and axial components propagate outwards from the axis at speed V, being attenuated at the rate of r^{-1} in the case of the azimuthal component, and unattenuated in the case of the axial component. The behavior of the radial component is more complex. The initial conditions propagate outwards as with the other two components. Additionally, if we ignore the effects of U, U' and p_θ, we see that, for sufficiently large time, the second term grows as t^{-1} and tends to the constant $-\frac{1}{V}$, while the third term grows logarithmically. Therefore, neglecting initial conditions, the asymptotic behavior for large time has several possibilities. If U, U' and p_θ are all bounded, then the third term will dominate, and growth will be logarithmic. If U and U' are bounded, but p_θ is not, then the third term will still dominate, but growth will be of the order of p_θ times a logarithm. If U is not bounded but of polynomial order or less, then U' will be of lesser order than U. Now, the second term will dominate, and growth will be of the order of U. If U is beyond polynomial order, the second or third term may dominate, depending upon other details. For smaller times, we see that the second and third terms in the radial component become undefined at $r = Vt$. In order to maintain a finite impulse density, Up_θ must go to zero at least linearly at $r - Vt = 0$, while $U' p_\theta r$ must tend towards zero at least as fast as the reciprocal of the logarithm at the same value. The logarithmic portion of the third term will also have a zero-crossing at

$$t = \frac{2r}{V}. \tag{6.59}$$

At extremely high compressibility, the propagation speeds of initial conditions continue to increase, while the magnitudes of the two nonhomogeneous terms in the radial component tend towards zero. This has the effect of decoupling the components while removing any nonhomogeneous effects.

6.3 Solution for Incompressible MHD

We now consider the behavior of the MHD analog of fluid impulse density for incompressible MHD. For ideal, incompressible MHD $\vec{p}$ satisfies the equation

$$\frac{\partial \vec{p}}{\partial t} + (\vec{v} \cdot \nabla)\vec{p} = -(\nabla \vec{v})^T \vec{p} + (\nabla \times \vec{B}) \times \vec{B} . \tag{6.60}$$

Consider a cylindrically symmetrical magnetic vector potential and velocity field which are given, in cylindrical coordinates, by

$$\begin{aligned} \vec{A} &= < A_r(r,t), A_\theta(r,t), 0 > \\ \vec{v} &= < 0, U(r), 0 > \end{aligned} . \tag{6.61}$$

The general magnetic field is given by

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ &= \left\langle \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right\rangle \end{aligned} . \tag{6.62}$$

Substituting the given conditions, (6.61), gives

$$\vec{B} = \left\langle 0, 0, \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} \right\rangle . \\ =< 0, 0, B_z > \tag{6.63}$$

In order to find the magnetic term in the equation for $\vec{p}$, we take the curl of the above to get

$$\nabla \times \vec{B} = \left\langle 0, -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r}\right), 0 \right\rangle . \tag{6.64}$$

Taking the cross product of the magnetic field with its curl gives

$$(\nabla \times \vec{B}) \times \vec{B} = \left\langle -\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r}\right), 0, 0 \right\rangle \\ = \left\langle -\frac{1}{r} B_z \frac{\partial B_z}{\partial r}, 0, 0 \right\rangle \quad , \\ =< W(r,t), 0, 0 > \tag{6.65}$$

where we let $W(r,t) = -\frac{1}{r} B_z \frac{\partial B_z}{\partial r}$. With this in place, the components of the fluid impulse density analog satisfy

$$\frac{\partial p_r}{\partial t} + \frac{\partial p_r}{\partial r} v_r + \frac{1}{r}\frac{\partial p_r}{\partial \theta} v_\theta - \frac{p_\theta}{r} v_\theta + \frac{\partial p_r}{\partial z} v_z = -\frac{\partial v_r}{\partial r} p_r - \frac{\partial v_\theta}{\partial r} p_\theta - \frac{\partial v_z}{\partial r} p_z + W(r,t)$$
$$\frac{\partial p_\theta}{\partial t} + \frac{\partial p_\theta}{\partial r} v_r + \frac{1}{r}\frac{\partial p_\theta}{\partial \theta} v_\theta + \frac{p_r}{r} v_\theta + \frac{\partial p_\theta}{\partial z} v_z = -\frac{1}{r}\frac{\partial v_r}{\partial \theta} p_r + \frac{v_\theta}{r} p_r - \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} p_\theta - \frac{v_r}{r} p_\theta - \frac{1}{r}\frac{\partial v_z}{\partial \theta} p_z \; .$$
$$\frac{\partial p_z}{\partial t} + \frac{\partial p_z}{\partial r} v_r + \frac{1}{r}\frac{\partial p_z}{\partial \theta} v_\theta + \frac{\partial p_z}{\partial z} v_z = -\frac{\partial v_r}{\partial z} p_r - \frac{\partial v_\theta}{\partial z} p_\theta - \frac{\partial v_z}{\partial z} p_z$$

(6.66)

Utilizing the given velocity field, (6.61), gives

$$\begin{aligned} &\frac{\partial p_r}{\partial t} + \frac{1}{r}\frac{\partial p_r}{\partial \theta} U - \frac{p_\theta}{r} U = -U' p_\theta + W(r,t) \\ &\frac{\partial p_\theta}{\partial t} + \frac{1}{r}\frac{\partial p_\theta}{\partial \theta} U = 0 \qquad . \\ &\frac{\partial p_z}{\partial t} + \frac{1}{r}\frac{\partial p_z}{\partial \theta} U = 0 \end{aligned} \tag{6.67}$$

This system is clearly an extension of the incompressible hydrodynamic system, (6.21).

Applying cylindrical symmetry and effective two-dimensionality yields a similar solution

$$\begin{aligned} &p_r = (\frac{U}{r} - U') p_\theta(r) t + \int_o^t W(r,s) ds \\ &p_\theta = p_\theta(r) \qquad . \\ &p_z = 0 \end{aligned} \tag{6.68}$$

To continue, let us look at the evolution equation for the magnetic field,

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \,. \tag{6.69}$$

Using the giving velocity field, (6.61) and magnetic field, (6.63) gives

$$\vec{v} \times \vec{B} = UB_z \hat{r} \tag{6.70}$$

and, therefore,

$$\nabla \times (\vec{v} \times \vec{B}) = 0 \,. \tag{6.71}$$

Thus, from the time evolution equation

$$\frac{\partial B_z}{\partial t} = 0 \,. \tag{6.72}$$

Therefore, B_z is a function of r, only. This also implies that

$$A_\theta = f(r) + \frac{1}{r} g(t) \,. \tag{6.73}$$

Now, since

$$W(r,t) = -\frac{1}{r} B_z \frac{\partial B_z}{\partial r}, \tag{6.74}$$

we have that W must be a function of r, only. And so, the integral in the radial component of $\vec{p}$ can be integrated to give

$$\begin{aligned} p_r &= \left(\frac{U}{r} - U'\right) p_\theta(r) t - \frac{1}{r} B_z B_z' t \\ p_\theta &= p_\theta(r) \\ p_z &= 0 \end{aligned} \quad . \tag{6.75}$$

We see that the radial component behaves similarly to the incompressible hydrodynamic analog, (6.25). There is also linear growth in time, now, but with a portion due to the magnetic field.

7 Conclusion and Discussion

7.1 Conclusion

Differential geometry offers an excellent framework for unification of apparently dissimilar concepts. Unfortunately, much translation is required to adapt these powerful tools to the differential equations that have been the traditionally preferred approach. When viewed with differential geometry, the relationships between the dynamical invariants in fluids are made clear. Using such techniques, infinite sets of new invariants can be generated from a basis set. In Chapter 2 we use translated versions of differential geometry operations to generate infinite sets of new invariants for the Hall and eMHD plasma models, and discuss techniques for identifying basis invariants in their differential equation forms. Gauge transformations are one method for constructing new basis invariants. Additionally, in Chapter 2 we develop a new gauge transformation for construction of basis invariants, and include plasma model analogs for other applications of gauges. Gauge transformations raise the question of interpretation of the new quantity resulting from the transformation. In Chapter 3, we address this by extending the interpretation of fluid impulse and fluid impulse density to the plasma models, and demonstrate the global invariance of the fluid impulse in each case. In Chapter 4, we go on to generate new integral invariants for Hall and eMHD plasma models based on results from Chapter 2. We construct new topological invariants in Chapter 5 where we also provide an application of topological invariants to magnetic reconnection. Finally, in Chapter 6, we revisit fluid impulse density by making specific predictions for its behavior in three different models.

7.2 Discussion and Future Work

The use of techniques based on differential geometry to find fluid invariants speeds the process substantially. The use of invariants simplifies understanding of fluid behavior. Once an invariant is found, we know that it must continue to exist, regardless of the complexity of fluid motion. The techniques are general enough that they should be applicable to any new fluid model. In one sense, the more complex the fluid model, the more potentially successful the invariant approach. The diversity of invariants is limited by the variety of basis invariants. A more complex model may have more basis invariants and, thus, a greater richness of invariants overall.

Gauge transformations allow the construction of basis invariants in some situations. They are useful for other purposes as well. The gauge transformations discussed here have been primarily designed to maintain vorticity after the transformation. In models where the preservation of some other quantity would be a higher priority, a different set of gauge transformations may be available. Also, as mentioned in **section 2.10**, the versatility offered by the two gauge fields in plasma models may be worth further investigation.

It is interesting to note that the gauging procedure works in a qualitatively different manner for each of the plasma models and for hydrodynamics. In the hydrodynamic case, we gauge the equation of motion, while, in the MHD case, attempting to gauge the equation of motion is fruitless, and, thus, we gauge the magnetic field, instead. For Hall MHD, a combination of both gauges is required, while eMHD lacks a velocity evolution equation entirely. Much of this difference is due to the diverse ways the differing terms interact with the curl operator, and has consequences beyond identification of geometric invariants. For instance, each technique for finding circulation theorems for fluid flow seems to work in one model, with each technique unable to be extended to other models.

We have mentioned the use of topological invariants to study magnetic reconnection; however, the geometric invariants seem to offer promise, as well. If it were possible to express the frozen-in field in one model in terms of frozen-in field (or other invariants) in another model, this may provide a mechanism for tracking their behavior during transition between regimes.

We studied the behavior of the fluid impulse density, but what does this information tell us? Certainly, we can find now find the evolution of the vorticity via the curl. However, perhaps we would like to determine the velocity field, as well. This raises the question of the inversion of the gauge transformation. For incompressible models, the velocity is the divergence-free projection of the fluid impulse density, and can be computed. The fluid impulse in compressible models is more problematic and requires more study, although some results have been obtained by **[Shivamoggi (2009)]**.

Finally, the topological invariants seem to offer a lot of promise. For each system, we have a large number of integrals containing various amounts of terms related to frozen-in fields. It would seem that a variety of knot invariants other than the linking number may exist within. With each such translation, more of the tools of knot theory will become available for predicting fluid behavior.

A APPENDIX: DERIVATION OF MHD EQUATIONS

To better understand the nature and assumptions of the various plasma equations, let us sketch the derivation of the MHD equations. Consider the phase-space particle distribution function, F. If we could determine the evolution of F exactly, the fluid problem would be solved. If our system is Hamiltonian, then the distribution function evolves according to Louisville's equation (conservation of phase space volume)**[Fitzpatrick]**

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial F}{\partial x_i} + a_i \frac{\partial F}{\partial v_i} = 0 \tag{A.1}$$

or

$$\frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F + \vec{a} \cdot \nabla_{\vec{v}} F = 0 \, . \tag{A.2}$$

where $\vec{a}$ is acceleration and $\nabla_{\vec{v}}$ is the gradient with respect to the velocity variables only. Particles in the plasma will be interacting via the Lorenz force, so the acceleration is given by

$$\vec{a} = \frac{e}{m}(\vec{E} + \vec{v} \times \vec{B}) \, . \tag{A.3}$$

Unfortunately, solving this apparently simple equation for this formulation of plasma physics is neither possible nor particularly useful. The electromagnetic acceleration of each particle is affected by the behavior of every other particle, making the resulting system of equations (one set for each particle) intractable. Furthermore, this description of the plasma is in terms of a microstate, obfuscating more useful macroscopic properties which could be observed in a laboratory.

The solution is to average the distribution, F , over an ensemble, making the equations simultaneously more useful and more complicated. Such averaging gives

$$\begin{aligned}
&\left\langle \frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F + \vec{a} \cdot \nabla_{\vec{v}} F \right\rangle \\
&= \left\langle \frac{\partial F}{\partial t} \right\rangle + \left\langle \vec{v} \cdot \nabla F \right\rangle + \left\langle \vec{a} \cdot \nabla_{\vec{v}} F \right\rangle \\
&= \left\langle \frac{\partial F}{\partial t} \right\rangle + \left\langle \vec{v} \right\rangle \cdot \left\langle \nabla F \right\rangle + \left\langle \vec{a} \cdot \nabla_{\vec{v}} F \right\rangle \\
&= \frac{\partial \left\langle F \right\rangle}{\partial t} + \left\langle \vec{v} \right\rangle \cdot \nabla \left\langle F \right\rangle + \left\langle \vec{a} \cdot \nabla_{\vec{v}} F \right\rangle = 0
\end{aligned} , \tag{A.4}$$

where the third line uses the fact that the velocity of each particle is independent of the distribution. If we also had

$$\left\langle \vec{a} \cdot \nabla_{\vec{v}} F \right\rangle = \left\langle \vec{a} \right\rangle \cdot \nabla_{\vec{v}} \left\langle F \right\rangle , \tag{A.5}$$

then we could reconstruct the same simple equation which we had in the microscopic case for the more useful macroscopic case. Unfortunately, the acceleration, is, in general, dependent on the particle distribution and we must introduce a covariance term, called the collision operator, C, to cancel the dependency. This gives

$$\frac{\partial\langle F\rangle}{\partial t}+\langle\vec{v}\rangle\cdot\nabla\langle F\rangle+\langle\vec{a}\rangle\cdot\nabla_{\vec{v}}\langle F\rangle=C\,. \tag{A.6}$$

Finally, since the system is Hamiltonian, we have

$$\nabla\cdot\langle\vec{v}\rangle=\nabla_{\vec{v}}\cdot\langle\vec{a}\rangle=0\,, \tag{A.7}$$

and can write

$$\frac{\partial\langle F\rangle}{\partial t}+\nabla\cdot\left(\langle\vec{v}\rangle\langle F\rangle\right)+\nabla_{\vec{v}}\cdot\left(\langle\vec{a}\rangle\langle F\rangle\right)=C\,. \tag{A.8}$$

The above equation is in terms of phase space variables. To continue with the derivation of the fluid equations, we must put it in terms of the space coordinates by integrating out the velocity variables. In integrating over velocity, we will compute different velocity moments of the quantities in this equation. If all the velocity moments of the distribution function are known, then the distribution function can be determined. We need to know at least the zeroth and first moments in order to compute the charge density and current density which will be needed by Maxwell's equations in order to find the electric and magnetic fields.

Given that the zeroth moment of the collision operator is zero, the zeroth moment of the kinetic equation, (A.8) is

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\vec{v})=0\,, \tag{A.9}$$

conservation of particle number density, n. Since all particles of the same type are assumed to have the same mass, this is equivalent to conservation of mass. *For legibility, expectation value symbols will be omitted from this point forward.*

The first moment of the kinetic equation, after multiplication by mass, transformation to Lagrangian coordinates and identification of physically significant quantities leads to

$$\rho\frac{D\vec{v}}{Dt}+\nabla\vec{p}-en(\vec{E}+\vec{v}\times\vec{B})=\vec{F}\,. \tag{A.10}$$

Here, $\vec{p}$ is the pressure tensor and arises from the second moment of the distribution, and $\vec{F}$ is the collisional friction force arising from the first moment of the collision operator.

Due to the presence of the pressure tensor in the equation for the first moment, (A.10) , we must also consider the equation for the second moment. After treatment similar to that of the first moment we obtain

$$\frac{3}{2}\frac{Dp}{Dt}+\frac{3}{2}p\nabla\cdot\vec{v}+\vec{p}_{ij}\frac{\partial v_j}{\partial x_i}+\nabla\cdot\vec{q}=W\ . \tag{A.11}$$

Here, p is the scalar pressure: the trace of the pressure tensor, $\vec{q}$ is called the heat flux density and arises from the third moment of the distribution function, and W is the total energy change due to collisions, and comes from the second moment of the collision operator.

We now see that we have a problem. Due to the presence of the velocity factor in the second term of the kinetic equation, the equation for a moment of any order will always include a reference to the moment of the next order. In order to make the problem tractable, we need a method to limit the number of equations that we will have to solve. Which approach we take depends upon which physical circumstances we are trying to model, and which approximations we are willing to make. Fluid plasma models only apply when the length scales associated with variations in fluid properties are much longer than the mean free path of fluid particles. Additionally, MHD models specifically refer to circumstances where particle thermal velocity is on the order of fluid velocity. In these limits, the third moment terms disappear from the equation for the second moment along with other simplifications. What remains is

$$\begin{aligned}
&\frac{\partial n}{\partial t}+\nabla\cdot(n\vec{v})=0\\
&\frac{\partial(\rho\vec{v})}{\partial t}+\nabla\cdot\vec{p}-\delta^{-1}en(\vec{E}+\vec{v}\times\vec{B})=\vec{F}\ ,\\
&\frac{3}{2}\frac{Dp}{Dt}+\frac{5}{2}p\nabla\cdot\vec{v}=\delta^{-1}W
\end{aligned} \tag{A.12}$$

where δ is a small parameter and the extra $p\nabla\cdot\vec{v}$ comes from the fact that the trace of the pressure tensor is the only remaining contribution from $\vec{p}_{ij}\frac{\partial v_j}{\partial x_i}$. Now, if we separate terms by order we get

$$\begin{aligned}
&\frac{\partial n}{\partial t}+\nabla\cdot(n\vec{v})=0\\
&\frac{\partial(\rho\vec{v})}{\partial t}+\nabla\cdot\vec{p}=\vec{J}\times\vec{B}\\
&\frac{3}{2}\frac{Dp}{Dt}+\frac{5}{2}p\nabla\cdot\vec{v}=0 \quad .\\
&en(\vec{E}+\vec{v}\times\vec{B})=0\\
&W=0
\end{aligned} \tag{A.13}$$

The time evolution equation for the magnetic field can be found by taking the Maxwell-Faraday equation and using (A.13) to eliminate the electric field. This will give

$$\frac{\partial\vec{B}}{\partial t}=-\nabla\times\vec{E}=\nabla\times(\vec{v}\times\vec{B})\,. \tag{A.14}$$

For our applications, it will be useful to restate the above in a different form. By applying a vector calculus identity, we have

$$\frac{\partial\vec{B}}{\partial t}=\vec{v}(\nabla\cdot\vec{B})-\vec{B}(\nabla\cdot\vec{v})+(\vec{B}\cdot\nabla)\vec{v}-(\vec{v}\cdot\nabla)\vec{B}\,, \tag{A.15}$$

which can be rewritten as

$$\frac{D\vec{B}}{Dt} = -\vec{B}(\nabla \cdot \vec{v}) + (\vec{B} \cdot \nabla)\vec{v} \; . \qquad \text{(A.16)}$$

Now, the first of our plasma equations, when multiplied by the species mass, becomes mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad \text{(A.17)}$$

which can be rewritten as

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} + (\vec{v} \cdot \nabla)\rho = 0 \\ &\frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{v} \end{aligned} \; . \qquad \text{(A.18)}$$

Applying the material derivative to the combination $\frac{\vec{B}}{\rho}$ and utilizing the (A.16) and (A.18) gives

$$\frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right) = \frac{1}{\rho}\frac{D\vec{B}}{Dt} - \frac{\vec{B}}{\rho^2}\frac{D\rho}{Dt}$$

$$= -\frac{1}{\rho}\vec{B}(\nabla \cdot \vec{v}) + \frac{1}{\rho}(\vec{B}\cdot\nabla)\vec{v} + \frac{\vec{B}}{\rho^2}\rho\nabla\cdot\vec{v} \,. \qquad \text{(A.19)}$$

$$= \frac{1}{\rho}(\vec{B}\cdot\nabla)\vec{v}$$

Thus,

$$\frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right) = \left(\frac{\vec{B}}{\rho}\cdot\nabla\right)\vec{v} \,. \qquad \text{(A.20)}$$

In these derivations, we have ignored the fact that real plasmas have multiple particle species. The derivations for the MHD equations that govern electrons and ions are basically the same. Additionally, the approximations we have made ensure that the differences between ion and electron velocities are small, and can be replaced by a general fluid velocity that effectively approximates them both. This is where MHD and Hall MHD diverge. If there is a large difference between electron and ion masses, the system can still be modeled effectively by using a center of mass velocity. The ion velocity will be approximately the same as the center of mass velocity, but the electron velocity will go like

$$\vec{v}_e = \vec{v} - \frac{\vec{J}}{ne} \qquad \text{(A.21)}$$

where $\vec{J}$, again, is the plasma current. When using substituting this into (1.7) we get

$$\vec{E} + \vec{v} \times \vec{B} = \frac{\vec{J}}{ne} \times \vec{B}. \tag{A.22}$$

The term on the right is called the Hall term and the replacement of $\vec{E} + \vec{v} \times \vec{B} = 0$ with the above is the defining difference between MHD and Hall MHD. **[Lighthill]**

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