# DESIGN AND PRODUCTION OF CALCULUS ASSESSMENTS

by

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# ABSTRACT

The AP Calculus program at Green High School was not adequately preparing students for college level calculus as more than 80 percent of the students failed the AP exam. The students were concurrently receiving high marks on in-class assessments. Hence, the in-class assessments were not good indicators of students calculus understanding according to the AP standards. A needs assessment was conducted, focusing on the difference in high school and college calculus, how to assess calculus and where Green High School's assessments were falling short. More research was conducted to examine the college Board expectations of calculus learning. A content matrix was designed to measure how well an in-class assessment aligns with college and AP calculus expectations of calculus knowledge. From this, new assessments were created that meet the goals of the content matrix.

This dissertation is dedicated to the original Dr. Wenzel, who lives unflaggingly committed to his calling.

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# CHAPTER 1: INTRODUCTION

## Problem of Practice

Green High School (GHS) has an 85 percent pass rate, discounting math, for its advanced placement program (AP), while the AP calculus pass rate is just 19 percent. The overarching goal for the AP program is to provide a course that reaches the same learning goals as its college equivalent as demonstrated through the AP test score. The goal is for 100 percent of students to pass the AP test, thus there is a gap of 81 percent of students in the AP Calculus program, indicating that the class is not reaching its goal. The problem of practice this Dissertation of Practice will address is that Green High School's calculus program is not preparing students for college calculus success.

Green High School is a small private preparatory school in Clearwater, Florida. The school has a mission statement that emphasizes the importance of academic rigor and college preparation. Green High School has a waitlist and largely admits gifted students. The school has a thriving AP program, offering 14 different courses spanning six subject areas. The institution's self-proclaimed weakness, as demonstrated by end of the year standardized tests and AP scores, is mathematics. In fact, while the remainder of the school, discounting math, has an AP pass rate of 85 percent, well above the national average (which is different for each subject but stays roughly around 60 percent), the AP calculus exam has a pass rate significantly below average at just 19 percent (ACT, Inc., 2010).

The AP calculus program is relatively new to Green High School. In 2013, the school restructured its math program and added a pre-calculus and calculus track to better support students with STEM interests and abilities. Calculus is foundational to a STEM trajectory and studies link success in college level calculus with positive experience in a high school AP calculus course (Ubuz, 2011;

Laurent, 2009). When Green High School received the scores back from the AP test, indicating that students were not grasping the material at a college level, a gap analysis was conducted that focused on the in-class assessments. The analysis used quantitative and qualitative methods and examined the problem from a human resources, structural, symbolic, knowledge, and motivational framework. The thrust of the gap analysis was that the assessments indicated an emphasis on procedural knowledge, rather than conceptual, which has been a trend in most of Green High School's math courses. Thus, in finding a solution, the school's head of instructional effectiveness conducted a professional development meeting with the math department to demonstrate how to incorporate more conceptual knowledge-based questions on formal assessments. Further, the calculus teacher received summer training from the AP College Board regarding the content of the AP calculus exam. However, these minor interventions did not improve AP calculus scores in a statistically significant way. This is likely due to the infrequency of training and accountability regarding the requested transition from testing procedural knowledge to incorporating conceptual knowledge.

A study of the AP test scored at Green High School revealed that less than 1 in 5 students enrolled in the AP calculus program earned a score on the AP test that indicated they were ready for college level calculus (ACT, Inc., 2010). This is concerning because success on this AP exam is statistically linked to college success and direction (Bressoud, 2015). Indeed, taking an AP class is only proven to be beneficial to a student's college readiness if the student passes (ACT, Inc., 2010). Research demonstrated that students who tried but did not achieve success in AP Calculus went on to have lower grades in college calculus than those who never took high school calculus (Barnett, Sonnert & Sadler, 2014; Bressoud, 2015). Studies also showed that regardless of ability, perceived personal success was the paramount factor in students' continued interest and pursuit of math related majors and careers (Martin, Anderson, Bobis, Way, & Vellar, 2012; Pyzdrowski, Sun, Curtis, Miller, Winn, & Hensel, 2013; Worthley, Gloeckner, Kennedy, 2016). Thus, students who fail the AP calculus test, as 81 percent of Green High School students did are more likely to struggle with math in college and lose interest in pursuing math-related fields.

## Impact of the Problem

Entry-level math (most commonly college algebra for non-STEM majors, and calculus I for STEM majors) is often cited as the largest obstacle for college graduation. Studies indicate that more than half of high school graduates test as not being college ready in math (Zachry, Diamond & MDCR, 2015). This statistic is the highest of any subject area and is not limited to non-stem majors. One study examined top math performers in high school and found that 17 percent of those students who passed the AP calculus test, and were therefore deemed college ready by the AP board, still required remedial calculus in college (Stoumbakis, 2010).

Calculus is the first post-secondary course, setting a foundation for STEM majors. More and more students are taking pre-calculus and calculus in high school to prepare them for college and yet, even as this rate increases, the corresponding percentage of students graduating with STEM majors decreases (Stoumbakis, 2010). A recent study conducted at Middle Tennessee State University found that more than two-thirds of the students entering as STEM-majors had switched out of their original program by the end of the first year (Cheatham, Rowell, Nelson, Stephens, & Tenpenny, 2014). There are many reasons that students are dropping out of STEM majors, including loss of interest, lack of connection with faculty, and low grades (Stoumbakis, 2010). However, research demonstrates that the most common reasons all correlate to college preparedness, and poor performance in calculus appears as the largest factor in discouraging students from the STEM path (Cheatham, et al., 2014).

Nearly 30 percent of all students nationally who take college calculus fail or drop out of the course (Worthley et al., 2016). One of the reasons for this high failure/withdraw rate is that students are entering under-prepared. A study conducted at Ohio State University found that less than a third of the students who had completed five or more preparatory mathematics courses were ready for college calculus (Waits & Demana, 1988). High school and college instructors affirm that the expectations, setting, and style of the high school and college mathematics classroom differ greatly, creating a rough transition for many students (Ye, Kerr, Klymchuk, McHardy, Murphy, & Spencer, 2009).

One of the largest differences among high school and college calculus instructors concerns mastery of procedural verses conceptual mathematical knowledge (Stoumbakis, 2010; Zelkowski, 2011). Calculus professors, compared with high school calculus teachers, are more interested in students having a deep understanding of foundation concepts in calculus (Wade, Sonnert, Sadler, Hazari, & Watson, 2016). Indeed, Bressoud (2010) argues that this cognitive shift stems from the curriculum itself. College calculus covers less material than AP Calculus, which is the most frequent of high school calculus, but goes into much greater depth with limits and derivatives (Bressoud, 2010). The greater amount of material forces a more surface understanding and limits the cognitive demand to procedural knowledge at best (Bressoud, 2010). Additionally, college calculus tests were more likely to use different function notations and require students to include an explanation (Ryals, 2016).

## Historically Addressing the Problem

Several programs have been instituted at the collegiate level to ensure success in calculus and thus help retain STEM majors, including STEM Talent Expansion Program (STEP), which is funded by the National Science Foundation (Avila, 2013). While the problem may proceed from unpreparedness in high schools, there is little being done on a high school level to directly address it. In fact, a joint statement from the National Council of Teachers of Mathematics (NCTM) and Mathematical

Association of America (MAA) emphasizes that the issue of college calculus is a college issue, not a high school problem. The goal of high school math is to provide a foundation for students who can then strengthen their mathematical knowledge in whatever manner they see fit in college (NCTM, 2012). However, the number of high school calculus students has been steadily increasing by more than six percent as more and more high schools offer calculus each year (Stoumbakis, 2010). Thus, despite the official position, it would appear that high schools believe that offering calculus will give students an advantage in college.

Green High School restructured its math program to enable students to succeed in STEM majors in college, however, the low scores on the AP calculus test indicate that students are still not equipped for success in college calculus, the foundation of STEM courses (Bressoud, 2015). Statistically, students need to score a minimum grade of a three to make the high school calculus experience beneficial to college calculus (Barnett et al., 2014; Bressoud, 2015).

Indeed, studies of students who earned at least a three on their AP test demonstrated that these student go on to succeed in their respective calculus classes (either calculus I, or calculus II) at a higher rate than their college peers (Rosasco, 2014; Bressoud 2015). It is the goal of Green High School to reform their current calculus course (and any previous preparatory courses) so that all of the students enrolled in AP calculus earn at least a three on the AP test.

## Focusing the problem

Students are not succeeding in the course at an AP level and yet they are largely achieving high marks on in-class assessments. In fact, 73 percent of the students who earned at least a 90 percent average on the in-class formal assessments still failed to meet the AP benchmark and thusly are more likely to struggle with calculus in college (ACT, Inc., 2010; Barnett et al., 2014). The cumulative average score on formal assessments in the AP calculus class should be a good indicator of performance on the AP calculus test (Morgan & Ramist, 1998). In other words, the students who excel in the class should likewise excel on, or at least pass, the AP test. Conversely, if a student did not pass the AP test, there should have been some indication via his/her formal assessments in the class of learning gaps (Morgan & Ramist, 1998). Accordingly, there should be a correlation between the scores in the class assessments and the scores on the AP test. Students had a class test average of 88 percent. However, the class had an average AP test score of 1.667, where a one is the lowest possible score and three is the minimum score for passing. This average score on the AP exam indicated that students were not grasping calculus concepts on a college level (Morgan & Ramist, 1998). While 69 percent of the class received an 'A' average on the formal in-class assessments, less than 20 percent received college credit. This disparity indicates that the tests are not aligning with the AP standards.

To further investigate if the formal assessments for Green High School's AP calculus class were a good indicator of performance on the AP test a Fisher Exact Test was performed (see table 1.1). Due to the small number of students passing the exam, a chi-squared test could not be used. With the null hypothesis stating independence of variables, a strong p-value of .71 demanded that we not reject the null, (a p-value less than .05 is necessary for rejection). The test showed that passing the AP test was not related to receiving high marks on the formal assessments in-class. This means that receiving an 'A' on tests did not increase the chance of passing the AP Test.

**Table 1.1:** Fisher Exact Test

| | Average test grade of an A | Average test grade below an A | Totals |
|---|---|---|---|
| Passed the AP Test | 4 | 0 | 4 |
| Failed the AP Test | 11 | 7 | 18 |
| Totals | 15 | 7 | 22 |

Hence, the AP calculus program at Green High School is not aligning with the college standards for calculus. Success in Green High School's calculus program does not translate to success in college calculus. In fact, according to a survey that included the entire population of students who completed AP calculus at Green High School during the 2015-2016 school year, only 2 of the 22 students continued on to a STEM trajectory in college. Moreover, the survey revealed that the majority of students (over 68 percent) began their college experience with college algebra, the lowest level of college level math.

Green High School students' scores on the standardized AP test did not align with their cumulative in-class test averages. An analysis of the in-class assessments revealed that conceptual knowledge accounted for less than four percent of the points on formal assessments, with rote memory and procedural knowledge accounting for the other 96 percent. Students picked up on this with 16 of 22 students lamenting that they wish they better understood the concepts and how they related to one another. Moreover, the tests missed important concepts, did not require explanations and gave hints for which formulas to use, thus reducing the need for transfer (Mayer, 2011). These discrepancies between the in-class assessments at Green High School and the College Board's (as well as college calculus) standards matters as summative assessments are the driving force behind student learning (Raupach, Brown, Anders, Hasenfuss, & Harendza, 2013). Even when students are aware of the learning objectives, students prioritize studying according to what is being assessed on summative in-class tests rather than on the learning objectives (Anderson & Krathwohl, 2001).

At Green High School, AP teachers choose the curriculum and write their own tests. Green High School is satisfied with the textbook it currently utilizes, which does not have testing tools associated with it. With in-class tests being paramount in student learning, aligning the content of the tests with the concepts and cognitive demand of the college curriculum (as addressed through the AP standards) is essential for student success. In this way, the formal assessments scaffold the curriculum. The purpose of the project to address the problem of Green High School program not preparing its students for college success is to create formal assessments that align with college standards and help students achieve a deeper level of understanding of the AP concepts.

## Literature Review

Entry-level math (most commonly college algebra for non-STEM majors, and Calculus I for STEM majors) is often cited as the largest obstacle for college graduation. Studies indicate that more than half of high school graduates test as not being college ready in math (Zachry et al., 2015). This statistic is the highest of any subject area. Moreover, less than half of the students who enroll in remedial courses ever go on to complete college level work in that area (Crisp & Delgado, 2014). Students are not graduating high school with the skills necessary to succeed in college math to such a degree that the best predictor of college graduation is simply the completion of a single college level math course within the first two years of college (Schaffhauser, 2009). Thus, most of the research on preparing high school students for college level math is focused on the lowest levels of college math. However, for STEM majors, Calculus I serves as a gatekeeper to the program.

Nearly 30 percent of all students nationally who take college calculus fail or drop out of the course (Worthley et al., 2016). High school calculus and college calculus courses are often dissimilar in content, style, and cognitive demand, making the transition difficult for many high school students (Ye et al., 2009). This begs the question, what can high school math teachers do to better prepare their students for success in college Calculus? In order to answer this over-arching question, some sub-questions must be answered: Is prior experience with calculus beneficial and, if so, what classes best prepare students for success? Are there study habits or traits that teachers should be trying to instill in students in order to increase their success rate? How much of college calculus success can be attributed to natural ability? Is there really anything high school teachers can do to prepare their students for college level calculus?

## *Methods*

In order to answer these questions, I did a preliminary search through ERIC and other databases. The searches included key indices for terms like *college calculus* and *calculus* crossed with index terms for *high school calculus*, *predictors, AP calculus, statistics*, and *success*. Using this technique I arrived at 459 unique articles and dissertations. I then narrowed the field in two ways. First, I eliminated all articles/dissertations that grouped college Calculus in with other math classes (including Calculus II) and did not run separate statistics on Calculus I alone. Second, I eliminated all articles that focused solely on college curriculum or factors of success unique to the college that were independent of student characteristics or high school experience. After running the articles through these filters, the field dropped to 57 articles. In answering my research question it was important that the study focus on factors that can be influenced or changed. Regarding this, I eliminated studies that focused on gender, race, or other factors that are not malleable by the implementation of programs or curriculum at a high school level. This narrowed the scope of the literature to 18 articles or dissertations. After carefully reading the articles and cross-referencing the bibliographies, an additional article was found for a total of 19 articles.

There are studies attempting to correlate the use of calculators in high school with success in college. These have notably been left out of the literature review simply because the use of calculators in high school have not had a positive impact on the success of college calculus (Mao, White, Sadler & Sonnert, 2017). However, it bears noting that the unrestricted use of calculators in high school is correlated with low performance (Mao et al., 2017).

### *Findings*

After compiling the research, I sorted the findings into three categories: ability factors, effort/attitude and experience. The label *ability factors* stems from the designation that many studies used in explaining why the factors (like GPA and SAT scores) predict success. However, some studies viewed high school rank and GPA as a measure of persistence rather than ability. Interestingly, most of the studies that focused on these factors were investigating the accuracy of placing math students and ran calculus statistics as an aside. Further, some studies found that any one factor by itself was not a good predictor of calculus success, but combined with another, could predict the success rate with a statistically viable degree of accuracy. The largest, most thorough study (for our focus) found that the math ACT score was the best indicator of success (Messina, 2008). Overall, the findings showed statistically significant, but small, correlations between ability factors and success in college calculus.

The *effort/attitude* category covered students' study habits and posture toward math. The findings in this category have implications for high school math teachers at every level. Studies indicate that students who are confident and have a positive attitude toward math perform better in college calculus. More research would need to be done to ascertain if the confidence and attitude stems from ability and prior success and if, in turn, those are better indicators. However, the calculus specific research is in line with Bandura's (1977) theory of self-efficacy and subsequent studies which link self-efficacy to higher performance (Usher & Pajares, 2008). Perseverance was also a predictive factor for calculus success which is in line with Edge's (1984) interpretation that perseverance is the reason that high school ranking is found to be an indicator of college success. Finally, Barnett et al., (2014) found that the length of time students spent using good study habits in high school math was a strong predictor of college calculus success. The correlation between the length of time studying per night in high school was double that of the math SAT/ACT score in predicting college calculus success.

*Experience* was often cited in studies that crossed categories to be the best predictor of success in college calculus. However, studies varied on what defines calculus experience. It should be noted that AP Calculus was first introduced in 1990 and was altered for content again in 2004, with additional minor alteration occurring every couple years. Some studies indicated that experience was only valid if a pre-test based on calculus concepts could be passed. Other studies defined a success in a high school class as receiving a grade of an A. However, most studies argued that the more exposure to the content in high school, the better the student would preform in college. The exception being Wilhite, Windham, & Munday, (1998) who used high school experience as a bimodal discrete metric (1 = has taken, 0 = has not taken) in relation to other more fluid metrics like GPA and ACT score and found that experience was not a significant influencer of success in college calculus. Interestingly, most other studies used experience as a metric based on grade (or weighted grade) and found it to be a statistically significant indicator.

### *Implications*

While many studies seem to suggest that introducing students to calculus concepts in high school will help students succeed in college calculus, making the increase in high school calculus courses an appropriate measure, a closer look at the studies is necessary to interpret when high school calculus is an effective aid to college calculus understanding (Ferrini-Mundy & Gaudard, 1992; Fayowski, Hyndman, & MacMillan, 2009; Ubuz, 2011). Although multiple studies show that enrollment in high school calculus has a positive effect on college calculus grades, these studies are not unilateral in their conclusion that exposure to calculus increases college success rates (see figure 2). The studies that concluded that experience in high school calculus (and not necessarily success) all used experience as a bimodal discrete metric (Burton, 1989; Ferrini-Mundy, & Gaudard, 1992; Barnett et al., 2014). However, when high school experience in calculus is used a bimodal discrete metric, it is not a statistically significant variable when compared to other more fluid metrics like GPA and ACT score (Wilhite et al., 1998). Thus, increasing fundamental math knowledge, as tested on the SAT and ACT (which only tests through concepts found in Algebra 2), would be a better solution in ensuring calculus success than attending a calculus class.

Experience is only significant when the metric of high school calculus experience is based on a weighted grade, meaning that only students who found success in high school calculus are likely to succeed in college calculus. Indeed, after a long-term national study of college calculus, Bressoud (2015) concluded that high school calculus is only beneficial for students who receive a score of three or higher on the AP test. Thus, the increase in high school calculus offerings might not be the key to ensuring students success in college calculus. Therefore, Green High School is not meeting its mission by the current AP Calculus track in place. While AP is largely considered the best route to enable students for STEM careers in high school, the program must be implimented so that students are meeting the benchmark for success, namely, a score of three or higher on the AP Exam. The current tests do not align either in topic or cognitive modality to the AP curriculum.

Another indicator of college success that is formed through high school experience is the students' belief that they experienced a quality education in past math courses (Worthley et al., 2016). More considerably, effort channeled in the proper manner, was the largest indicator, after AP success, of college success (Barnett et al., 2014). High school calculus students who studied an hour a night in high school scored two letter grades higher in college calculus than their high school peers who did not study (Barnett et al., 2014). Unfortunately, the formal assessments at Green High School impacted students attitudes and beliefs concerning their education and effort. One student responded to why she did not study outside of class for the AP test, "I realized quickly on the first test that the class was designed to fit the class, not the AP test, so I just worked to succeed in the class." This student indicated that outside of homework for class, no additional effort was made. Another wrote, "I studied maybe 2 hours outside of class all year, I was doing well in-class so I thought I would be fine." These two students differed on how they viewed the tests but both were demotivated to put forth effort. Interestingly, when asked how much time they spent outside of class studying, nearly every student referenced the in-class formal assessments. Examinations clarify the expectations of what students should know and how they should know it, this in turn has direct implications on students' discipline in learning the material. Hence, altering the formal assessments will not only affect the content but also the students' effort and attitudes.

# CHAPTER 2: METHODOLOGY

The AP Calculus program at Green High School has high in-class test scores on average but low test scores on the AP exam. The tests are not consistent with the rigor of the College Board's AP program. In fact, only 27 percent of students who received an 'A' average (a cummulative average of 90 percent or more) on tests, likewise passed the AP test. This means that the AP calculus program at Green High School did not meet its goal of preparing students for success in college calculus since 73 percent of students who achieved an 'A' still failed the AP test. The tests were not a good indicator of student achievement as measured through the College Board's standards. Since passing the AP test is a significant indicator of success in college calculus courses, it is imperative that a high school program adopt the AP standards (Tyson, 2011).

The AP test is both a norm-referenced and criterion-referenced test. It is criterion-based in that it measures the depth to which students understand the standards outlined on the College Board syllabus. Points are likewise tabulated based on the accuracy with which the students can display their knowledge on these standards. However, it is also norm-based in that scores are assigned relative to the overall points of the population of students taking the test. The results of the AP tests allow us to gather information concerning how well students have met the standards of the curriculum and how well the AP program is doing compared to other schools (Fitzpatrick, Sanders & Worthen, 2011). Since the AP test is a criterion-referenced test that is statistically linked to college readiness, setting a goal for students to pass the AP test inherently focuses on increasing student learning of the content.

The assessment development project will have two stages. The first stage will be a formative needs assessment to determine what the formal assessments should encompass. After the needs have been determined, the second stage will be curriculum development where formal, summative assessments are created. Both phases will be included in this project designed to strengthen the AP Calculus program by creating new assessments that will guide the curriculum. The execution of the phases will incorporate a mixed methods design, utilizing both quantitative and qualitative data. Data sources include extant data concerning high school calculus, college calculus and AP Calculus standards and new data that will be collected using focus groups, interviews and formal assessments.

## Needs Assessent Evaluation Questions

As the project is multi-tiered, so are the evaluation questions. The data collected will correlate directly to the questions, as demonstrated in table 2.1.

**Table 2.1:** Questions and Data Collection for Needs Assessment

| **Questions** | **Possible Indicators** | **Data Collection** |
| --- | --- | --- |
| 1. How does the cognitive demand of college calculus differ from that of high school calculus? | • Description of the indicators of success<br>• Description of college calculus expectations<br>• Summary of interviews | • Literature Reviews<br>• Interview with college Calculus professors |
| 2. How do we assess calculus knowledge? | • Descriptors of types of questions<br>• Summary of interviews | • Literature Review<br>• Interview with college Calculus professors |
| 3. What areas of Green High School's Calculus assessments need improvement? | • Summary of focus group<br>• Summary of weaknesses on AP tests<br>• Summary of current tests | • Focus group with recent graduates who took Calculus<br>• Interview with GHS AP Calculus teacher<br>• Analysis of GHS current tests and AP results |

The needs assessment will rely heavily on extant data, collected through a literature review of what it means to learn calculus as well as a literature review comparing high school and college calculus. The information will then be coded and charted. Qualitative data will also be used to bring clarity to the literature reviews. Mixed methods of this nature are frequently employed in developmental stages of evaluation (Fitzpatrick, Sanders & Worthen, 2011). A focus group of Green High School graduates who have completed at least a semester's worth of college math will be conducted to expose any areas of need that are particular to Green High School's calculus program as it pertains to assessment. Focus groups are particularly acute in gathering data regarding the needs and wants for a program and thus are an effective source of information within a needs assessment (Fitzpatrick, Sanders & Worthen, 2011). While traditionally a focus group is composed of members that do not know one another, the small size of the school and the relative newness of the program make this unfeasible. Notably, students have already completed a survey detailing their perspective on the course and their relative success in it. These surveys will be used to form some of the focus group topics. In addition to the feedback from past program participants, college calculus professors and Green High School calculus teachers will be individually interviewed to clarify the differences in high school and college calculus pedagogy on assessments. Interviews are the preferred method since it allows for the interviewer to gain clarification when answers are vague or to probe further into particular statements (Fitzpatrick et al., 2011). While a focus group might also be effective, gathering college calculus professors from different schools would not be achievable within this study. This mixed methods approach allows for the triangulation of data, where the focus groups and interviews can contextualize and particularize the results of the literature reviews and analysis of previous exams (Fitzpatrick, Sanders & Worthen, 2011).

Due to the nature of a needs assessment, the standards and criteria for success is the completion of the deliverables, in particular, a content matrix that provides a measure for how well a calculus assessment aligns with the standards of the AP curriculum both in content and cognitive demand. This will be developed through the other deliverables, namely, the literature reviews, a summary of focus group, and a summary of the interviews. The information will be coded and charted. The charts will detail the differences in high school and college assessments and predictors of success in college calculus that stem from high school patterns relating to assessment. The portions of these predictors or differences that Green High School alumni particularly struggled with will be highlighted.

Curriculum Development Evaluation Questions

Using the data provided through the needs assessment and new data collected during this phase, as shown in table 2.2 below, the questions corresponding to the creation of the curriculum will be answered.

**Table 2.2:** Questions and Data Collection for Curriculum

| **Questions** | **Possible Indicators** | **Data Collection** |
|---|---|---|
| 4.What are the cognitive demands correlating to AP Calculus expectations? | • Description of AP emphasis, taxonomy and vocabulary | • Chart of past AP Calculus FRQs |
| 5.How well do the tests align with the objectives and cognitive demand of college Calculus and the AP program? | • Summary of feedback from college professors<br>• Summary of comparisons data relating to AP and college cognitive demand | • Interview with college professors<br>• Content Matrix chart that emphasizes cognitive demand<br>• Comparison chart with AP data |

In addition to the needs assessment, past AP calculus tests will be examined, and the free response sections will be coded and charted to note the cognitive demand, the objectives that are emphasized, and the variation in vocabulary. Criterion-based tests like the AP exam are not only integral in evaluating the overall success of the program but also in identifying particular areas of weakness (Fitzpatrick et al., 2011). Incorporating this data with the listed objectives on the College Board website, seven formal assessments will be created. These assessments will be charted on the content matrix developed through the needs assessment, outlining the topics covered and cognitive demand for each question. In this way, the test would have a reference table providing the topic covered and the taxonomy of cognitive demand for each topic. Each question will be analyzed according to the AP standards and an explanation provided for where the question fits on the content matrix. Further, the tests will be independently coded by a college calculus professor and an AP Calculus teacher. Interviews with the calculus instructors who coded the tests will be completed to discuss the tests and the assessments will be reworked according to feedback. As mentioned with the needs assessment portion, the nature of the curriculum design project demands that the criteria for success again be the deliverables, in this case, the completed assessments and corresponding matrices.

# CHAPTER 3: NEEDS ASSESSMENT

Students were succeeding on Green High School's formal assessments yet failing the AP test. The majority of these students went on to start college at college algebra, the lowest level of college math. Green High School did not prepare them for success in college calculus in line with its mission. A needs assessment was conducted to better understand how to create assessments that properly gauge student learning in calculus.

Before properly assessing students' knowledge of calculus, one must determine what it means to know calculus. Hiebert and Carpenter (1992), state, "Mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of the connections(p.67)." In other words, being able to compute a derivative does not equate to a full understanding of the concept. Relating derivatives to functions, picturing how derivatives effect graphs, perceiving contextually what a derivative means in a situation, inferring how a derivative can be used to solve real world issues (like optimization), and distinguishing how derivatives are similar (and different) from slope, all factor into the understanding of derivatives. It is only when one understands the relationships between ideas that they can connect and apply them.

It is helpful to have terms that classify knowledge, and the two most commonly used when referencing calculus understanding are conceptual and procedural knowledge. In fact, many cite the distinguishing difference in high school and college calculus as being related to the classification of knowledge, with high school emphasizing procedural and college emphasizing conceptual (Bressoud, 2010; Zelkowski, 2011; Ayebo, Ukkelberg & Assuah, 2017). Conceptual knowledge signifies an understanding of a mathematical idea, which requires not only a grasp of the definition but also how each part interrelates, and in a more generalized way, how it fits into the greater scheme of mathematics (Davis, 1985). Procedural knowledge is the sequence of actions that must be applied to compute a solution. While many debate the import of one type of knowledge over the other, both are necessary for a thorough understanding of a topic.

There is an academic trend toward a more conceptual view of calculus. The reformed calculus movement stressed that the purpose of calculus centers on its applicability (Chang, Cromley, & Tran, 2015). This movement shifted the direction of calculus to incorporate relational elements that aid in application. This shift toward a more conceptual view of calculus was reactionary to what was viewed as a purely procedural calculus course (Davis, 1986). The two types of knowledge are not at odds, as knowing how to approach a problem is only useful if one can proceed with that approach (Hiebert & Lefevre, 1986). Though both are necessary, when courses focus on imparting conceptual knowledge, students have a better understanding of concepts while not falling behind on procedural knowledge (Chappell & Killpatrick, 2003). Indeed when students grasp how the ideas of calculus relate to the procedures, they retain the information longer and apply it better (Barrett & Van Mol, 1986). Thus, in creating assessments designed to ascertain a students level of understanding, questions must be designed to judge a students' conceptual/relational knowledge of a topic.

## Content Matrix

This project will revise Porter's (2002) methodology of aligning math assessments with objectives to better fit the Calculus curriculum. Porter developed a two-dimensional content matrix that charts mathematical topics alongside the cognitive demand in order to form a common language for mathematical content. The topics are listed in the first column and the cognitive demand headlines the remaining columns. We will keep this format but enhance the language to better fit with research on understanding Calculus.

### *Cognitive Demand*

Concepts are by nature intangable ideas, and it is thus necessary to represent them in more tangable ways. The most common manner of representing upper level mathematical ideas are through numbers and occasionally with symbols. However, graphs, charts, and pictures often provide a more visual picture of a topic. Bremigan (2005), argues that, "visual representation is particularly important to learning the theories and applications of calculus." When conceptual knowledge is defined in terms of a topics relationship to its parts and to other topics, visualizing the web of connectedness is of the utmost import(Zimmermann, 1991). The use of multiple representation aids this organization and is linked to learning gains in conceptual knowledge (Hughes-Hallett, 1991).

The reform calculus movement outlined the import of the relationship between four modes of representing math: graphs, numbers, symbols and words (Bremigan, 2005). In agreement, the College Board focuses the free response section (worth half of the test) on students' ability to navigate these four representations of calculus ideas. Studies demonstrate that when students understand how the representations of a topic relate to one another, they not only perform better but continue to be more successful in higher level learning (Chang et al., 2015). The categories of cognitive demand for our matrix reference the representational (and thus relational) flexibility inherent to a question. The transition from representing an idea involving numbers to a graphical or tabular representation will be categorized as *External Representations*. *Application* questions assess a students ability to confer to and from a verbal representation. Understanding a concept through symbolic representation falls into the *Non-routine* classification. Finally, *Routine* problems comprise the numerical representation of concepts, which generally involve a procedural knowledge.

*External Representations*

External representations are vital to a conceptual knowledge of math. Hughes-Hallett (1991) states, "Students who are operating with few mental pictures are not really learning mathematics" (p.121). Studies have found that students' struggles in calculus often corresponds to a weak understanding of functions and how they relate graphically (Bremigan, 2005; Eisenberg & Dreyfus, 1991). While many students can graph, or recognize the graph of various functions, they struggle to identify the significance of the graph (Zimmermann, 1991). Even students with a competent algorithmic understanding of calculus frequently miss basic problems due to a lack of visual clarity in calculus (Eisenberg & Dreyfus, 1991). Graphical significance weighs heavily in calculus, as patterns in rates of change, optimization of functions, and net change of a function are readily available through this mode of visualization. Indeed, graphs are essential to solving a variety of Calculus problems (Chang et al., 2015). Sometimes students will come across problems that require extreme rigor in solving algebraically but a visual understanding simplifies the problem dramatically. Not only useful in solving problems, there are many complex proofs in calculus which students at this level might not be able to comprehend, however, a visual representation can provide a clear understanding of why the theorem works (Renz, 1986). While graphs are a common external representation that students should have a background in, tables are also important visual aids that help students link concrete information to complex tasks (Chang et al., 2015). Tables might appear more straightforward but students often struggle in extracting information from them, especially when the table records data in non-uniform increments (Boardman, 2009). Students' ability to read and alter external representations is directly correlated to success on the AP Calculus free response section (Bremigan, 2005). Wade, Sonnert, Sadler, and Hazari (2017) notably found that when high school instruction included a multiple representations construct, there was a negative correlation with college success. Graphical representation is a factor in this multiple representations category. However, a closer examination of the definitions of the construct reveals that the study is not at odds with a reformed (conceptual), calculus curriculum. In the study, when students reported that their teacher explained how to do things multiple ways and showed various procedures for solving problems, this was characterized under the multi-representational model. While using graphs and explaining ideas clearly is also in this category, (and surely nobody would argue that explaining ideas clearly has a negative effect on students), the largest weight of this linear regression model was the demonstration of different pathways to a solution. This multiple procedure approach to problems is not in line with the ethos of conceptual calculus, which seeks to connect the idea to models.

When students are asked to pull information from a graph, create or alter a graph, or otherwise transition from words, symbols, or numbers to a graph (or vice versa), the question will be coded in this category. Further, when students must create or alter a table, or otherwise make an informed decision about what is pertinent with a table, this too falls into the category of external representation. It should be noted that even when graphs are available, students ignore the graphs and opt for an algorithmic approach whenever possible (Eisenberg & Dreyfus, 1991). This means that if a function is accompanied by a graph and the solution to a problem can be done without the use/understanding of the graph, we will not code the problem under external representations. Further, if a chart provided requires no interpretation then it will also not be coded under external representation.

*Applications*

Using words to represent mathematical ideas and likewise pulling mathematical ideas from words is the crux of math application simply because words are the most common format in expressing ideas and problems in non-mathematical contexts. However, students often struggle to pull appropriate meaning from variables in context (White & Mitchelmore, 1996; Klymchuk, Zverkova, Gruenwald & Sauerbier, 2010). This difficulty points to a lack of deep understanding and is therefore not to be avoided. "Applications are a critical part of teaching and learning calculus precisely because they constitute one of the best places both to expose and to reinforce intuitive conceptual understanding (Rodi, 1986, p.123)."

Not only do word problems engage students in thinking conceptually, they also place the ideas of calculus in contexts that relate to other STEM fields. As Reed (1987) states, "It is the ability to do word problems that makes mathematics applicable" (p.30). Contextualizing problems can also enhance the depth of learning by relating calculus to known and practical knowledge (Klymchuk et al., 2010). For example, students have a preconceived knowledge of how position relates to time through speed. The rate of change as velocity is relatively easy for students to understand because it works within their framework and time increases at a constant rate (Jones, 2017). Using this familiar relationship, students have a backdrop to picture rate of change and other relationships related to the derivative (Marrongelle, 2004). In line with this, Czocher (2017) argues that when students can connect math to real world applications, their understanding of formulas is enhanced as they attribute meaning to a solution and relate it back to the equation.

Verbal communication is an important modality for mathematical concepts not only in providing contexts for students to decifer but also in allowing students to express their mathematical knowledge. Bushaw (1987) argues, "Justice simply cannot be done to some 'central ideas' by using only elementary mathematical symbolism or multiple choice responses. Writing is often the natural vehicle (p. 123)." This is in line with College Board's expectation of explanations and justifications on the free response sections. Multiple chief readers of the AP Calculus exam have expressed concern over the lack of students' ability to clearly express their knowledge (Boardman, 2011; Kokoska, 2012; Davis, 2016). While not requiring full sentences; clear, unambiguous, verbal communication is necessary to explain how equations relate to the context, how formulas relate to one another and how diagrams or graphs relate to a solution (AP Calculus Development Committee

& Chief Reader, 2011).

Just as with the multi-representational model, Wade et al., (2017) found that an application construct in high school calculus had a negative effect on college calculus success. The applications utilized in the high school classroom were associated with "motivational questions" intended to explain why students had to learn a topic. Interestingly, an emphasis on application was only a negative indicator for college students who did not succeed in high school calculus, as measured by their grade. So only students who performed poorly in their applications-based high school calculus course were less likely to succeed in college. Students who performed well in high school courses that included applications were statistically as likely to succeed in college as other successful high school students. Thus, application questions in a high school curriculum can not be viewed as having a negative impact on college success.

Questions that require students to take verbal communication and relate it to functional equations will be coded under applications. However, the mere presence of words does not necessitate that a question be categorized as an application. If a verbal statement directly addresses each component of an equation, thus eliminating the need to understand the context, the question will not be coded as application. However, by asking for interpretation of the answer within the context, this same scenario could again be an application question as it now requires students to relate the function or result to the verbal context. Questions that require a verbal explanation will be coded as application. However, grading of these questions must focus on the verbal communication or the question no longer requires the transfer to a verbal mode.

*Non-Routine Problems*

The last conceptual relationship deals with abstract data. In an abstract mode, relationships are less tied to a specific context but instead deal in generalities (Hiebert & Lefevre, 1986). Not only do abstract problems explore how and why formulas work, they enhance the understanding of variables. This is why White & Mitchelmore (1996) argue that, "A feature of all advanced mathematics is the need for abstract concepts (p. 80)."

However, the use of variables and symbols does not necessitate an abstract understanding. The presence of symbolic notation does not necessarily require students to relate a topic to anything beyond a procedure. For instance, "Find $f'(x)$ when, $y = f(x) = x$, includes symbolic notation, yet the symbolism isn't associated with anything abstract. Students need only compute a simple derivative to get the question correct and need not know anything about what a derivative means or its relationship to other ideas.

The categories title, *non-routine*, is named after Seldon, Seldon, Hauk & Mason's (1999) influential study in which they posture that non-routine questions require a mathematical flexibility not present in procedural based questions. In line with this, non-routine questions encompass more than just symbolism and abstraction. Instead, these questions require a conceptual knowledge in a number of ways. One manner is by connecting two topics that are not traditionally combined (Seldon et al., 1999). Students must understand the inner-workings of a topic to combine it with another in a coherent way that has not been modeled for them. This method relies on novelty and so a familiarity with the curriculum and instruction is necessary at times for its classification. Questions can also be non-routine when they require students to alter a known procedure in some way (Seldon et al., 1999). By incorporating a twist on a traditional procedure students must understand the intricacies of how/why the formula works in order to effectively change a component and achieve the desired result. Another method is to approach a question from an angle which does not allow solving from a procedural manner. This can be done in an abstract manner or by having students work in reverse through a known procedure (Seldon et al., 1999). Even though symbolism does not necessitate conceptual knowledge, questions that are characterized as non-routine, often use symbolic notation (and stay within this notation throughout the problem). This is not an exhaustive list, and questions that hit at the conceptual nature of a procedure are categorized as non-routine.

*Routine Problems*

The first three types of problems assess a students' conceptual understanding. Routine problems, labeled because of their frequency in math textbooks, refer to questions that gauge procedural and rote knowledge.

Within a conceptual calculus model, these questions are sometimes looked down on as mechanical and useless in the face of growing technology. Ralston (1987) lamented, "How ironic that there has been a steady, perhaps accelerating trend in recent years for college calculus to be dominated by the teaching of just those symbol manipulations which humans do poorly and computers do well(p.23)." Oft has it been theorized that the reason students are failing college calculus is because of the shallowness of the notational, procedure based calculus taught in high school (Bressoud, 2010; Carlson, Oehrtman, & Engelke, 2010; Wade et al., 2016). Why then would we want to include these routine problems on assessments?

There are a few reasons why I advocate the inclusion of routine problems on a test. First, rote questions that focus on precise definitions can promote conceptual understanding, as it is easier to memorize material that has meaning (Epp, 1986). Indeed, when a high school curriculum had an emphasis on exact definitions and precise vocabulary, students saw a larger degree of success in college level calculus (Wade et al., 2017).

Secondly, my interviews with college calculus professors were thematic in claiming incoming students were under-prepared in applying the algebraic manipulations to calculus computations. This is in line with multiple studies and that demonstrate an underdeveloped knowledge of basic algebra leads to both mechanical and conceptual calculus errors (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; White & Michelmore, 1996). While complex algebraic gymnastics can be found within a conceptual problem, it is sometimes helpful when including problems that delve deep into procedural knowledge, with tough and layered algebraic manipulations, to allow that brand of knowledge to be the focus. Further, one of the goals of formal assessments is to determine what students know and on which level they understand the content (Lyons, McIntosh, & Kysilka, 2003). Students who are confused about the translation of a topic from one representation to another might not be able to adequately express their procedural abilities due to a gap on the conceptual side. If the test only includes conceptual problems then identifying where the gap in knowledge occurs becomes difficult. A teacher might not be able to differentiate if students are struggling with translation or if there is an issue with the requisite knowledge dealing with strenuous algebraic application.

Finally, when there are multiple topics each with several subtopics, it can be difficult to design a test which includes all the material in a conceptual manner while also being accessible to a 50 minute time frame. By including some procedural questions, which typically take less time to solve, it becomes easier to balance the time with the demands of the content. While one might argue that not every topic need show up on the test, it is clear that students use tests as a model to condense and synthesize which material is of actual import (Raupach et al., 2013). If the material is significant enough to warrant a topic on the matrix, it should likewise be assessed formally.

Routine problems can aid in a conceptual understanding of calculus, strengthen algebraic knowledge and provide balance to an assessment. Moreover, it is the overuse of routine problems, not their very existence that proponents of a conceptual calculus are against (Hughes-Hallett, 1991). Therefore, routine questions, coded when a problem requires a procedure to be performed or asks students to relay rote knowledge, are an appropriate part of a college preparatory AP Calculus curriculum.

*Topics*

The College Board lists and categorizes all the essential knowledge for Calculus AB (College Board, 2016). The course is first divided into the three big ideas: limits, derivatives, and integrals. From there, the content has a nested structure. The largest subgroup is *Enduring Understanding*, which highlights the major themes into roughly four groups. Further, the topics are broken down into *learning objectives*. Finally, the essential knowledge necessary to complete the learning objective, retain an enduring understanding and grasp the big picture are listed. Mathematical facts in the *essential knowledge* division have a numerical/alphabetical listing that represents where it falls in the schema. For example, the essential knowledge that, "the derivative can be used to solve optimization problems, that is, finding a maximum or minimum of a function over a given value" is coded by 2.3C3. The beginning 2 directs us to the second big idea, derivatives, the first 3 indicates that is is the third topic of enduring understanding within derivatives, namely, "the derivative has multiple interpretations and applications including those that involve instantaneous rates of change." The letter 'C' denotes the learning objective and the following 3 references that it is the third piece of essential knowledge under the learning objective.

I have borrowed the organization of the College Board's topics but changed some of the language. The main topics correspond to the grouping of the learning objective. Rather than adopt the wording, with its cues on cognitive demand, a generalization of the essential knowledge heads the subtopics. Under the the overarching topic, the corresponding pieces of essential knowledge are listed. Due to the lengthy wording of many of these facts, the phrasing has been changed to synthesize the topic so it is readily understood on the chart. However, all the topics and subtopics are coded according to the College Board's content, which can be used to obtain the exact wording. For example, 2.1D has the learning objective "determine derivatives of higher order." Rather than focus on the learning taxonomy language "determine" the topic was paraphrased to "higher order derivatives". The two corresponding pieces of essential knowledge are 2.1D1 "differentiating $f'$ produces the second derivative $f''$, provided the derivative of $f'$ exists; repeating this process produces higher order derivatives of $f$," and 2.1D2, "Higher order derivatives are represented with a variety of notations, for $y = f(x)$ notations for the second derivative include $\frac{d^2y}{dx^2}$, $f''(x)$, and $y''$. Higher order derivatives can be denoted $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$". These two subtopics, denoted with a bullet point on the content matrix, also have shortened language, namely 2.1D1, "definition" and 2.1D2, "notational variety."

Some portions of essential knowledge (and learning objectives) have been omitted simply because it would be redundant to include them in the matrix. For instance, 2.1A4 states that "the derivative can be represented graphically, numerically, analytically and verbally." This part of essential knowledge is intended to cue teachers into the cognitive demand concerning the topic, which is covered on the columns of our matrix. An explanation will accompany any and all topics that have been omitted. Finally, the content matrix for each test will focus on the topics relevant to the unit and section being tested and as such, the order of the topics might change. For instance, even though 1.1C3 "L'Hopital's rule", is found before 1.2A1, "Continuity at a point", the former topic does not show up until Unit 2, where the latter is included in Unit 1.

# CHAPTER 4: CURRICULUM DESIGN

Green High School's AP Calculus curriculum is not aligning with the AP standards in a way that is reflected through formal assessment. The in-class assessments are not reliable indicators of success or failure on the standardized examination. While students who receive an 'A' on class assessments should indicate that the student will likewise receive college credit, 73 percent of these high achievers failed the AP test in 2016. A needs assessment revealed that students focused on procedural knowledge because of how the assessments were structured. However, a full understanding of calculus requires not only a procedural competency but also a relational understanding of how topics intertwine.

The content matrix provides a taxonomy of how all the essential topics for understanding calculus are tested. In designing new tests for Green High School's curriculum, the aim is for every topic to be included in at least two categories of cognitive demand on the content matrix. This will ensure that student's grasp of a topic is such that they can move flexibly from one representational mode to another. Additionally, each subtopic should be assessed in some manner. Since students prioritize information that is formally tested, each subtopic should appear on an exam (Anderson & Krathwohl, 2000). Finally, at least 75 percent of the questions should be conceptual in nature or have a conceptual component. By directing students' priorities toward conceptual learning, statistically students will perform better overall on assessments (Chappell & Killpatrick, 2003).

In addition to meeting the goals of the content matrix, the assessments should enable students to succeed on the AP AB Calculus exam or at least expose areas where students are falling short of AP expectations. The AP test that capstones the course is divided into two equally weighted portions. The multiple choice section and the free response section (FRQ). The free response section has six questions (two where students may use calculators) each with three or four parts. On these questions, students are expected to show all supporting work using proper notation; calculator notation is not accepted even on questions where calculators are permitted (College Board, 2016). Moreover, specific notation is required. When students use language like "it," "the function," or "the graph," to reference their logic, they do not receive credit for explanations of correct mathematical ideas because their communication is too vague (Davis, 2016). As the former chief reader Boardman (2008) wrote, "Students need practice communicating mathematics through the use of clear, unambiguous, and mathematically precise language (p.5)." In-class assessments should likewise promote specific language in communication of mathematical ideas. Questions throughout all assessments will also be graded in a manner similar to the AP free response section by discounting unsupported answers (see appendix A). The expectation, as outlined on the directions, should be that all answers arrived at through functional notation must be accompanied by verbal support which demonstrates the logic behind the formulas. Special emphasis should be given to the specificity of language used in these explanations. While graphs and figures can be utilized within a justification, sign charts will not be looked at. By holding students accountable to the standards of the College Board throughout the year, students will gain fluency in expressing their knowledge.

The free response section from the last ten years of AP tests (2007-2016) have been coded on the content matrix (see appendix I). Additionally, questions have been analyzed for vocabulary and algebraic rigor. When topics prioritized on the free response section correspond to the topics on a section test, the assessment will include questions similar in vocabulary, algebraic intensity and mode of the free response section. By mirroring the style and rigor of the AP assessments, students' become familiar with the format and expectations of the AP exam. On student surveys, multiple students reflected that they were confused, or uncertain about the type of questions asked on the free response section before taking the exam. These students often felt that the in-class assessments were dissimilar and wished to see more compatibility.

Not all topics are present on the free response section, though every topic and subtopic is essential for a thorough understanding of calculus. Moreover, the absence of a topic on the free response section does not indicate that only a procedural knowledge of the topic is necessary. The multiple choice section frequently includes questions of a conceptual nature. However, gauging the depth and degree to which students must understand these topics is difficult. As such, we will consult studies that suggest common errors and areas that students struggle within these topics. By assessing common learning mistakes, errors and learning gaps will be mitigated before the AP exam.

The new formal assessments for Green High School's calculus curriculum will stress conceptual understanding and relational fluency by meeting targets laid out on the content matrix. The tests will typify the standards of the College Board by echoing the styles and themes of the free response section and assessing common student errors. The tests will be graded according to the College Board's guidelines, familiarizing students with communication expectations. This will ensure that students finding success on the assessments are understanding calculus in a way that leads to success in college calculus. Further, the tests will act as a benchmark for student learning. Misunderstandings on the assessments will reveal gaps that need to be addressed before the AP exam.

## Unit 1 Section 1: Limites

### *Content*

The first test covers one of the three big ideas of calculus, limits. In understanding limits, a formal $\epsilon - \delta$ definition is not part of the curriculum. Instead, students are exposed to an intuitive informal understanding of limits as predictors of functional behavior in 1.1A. As an overview, 1.1A introduces limit notation (1.1A1) and the idea of one sided limits (1.1A2). 1.1A3 mentions that sometimes limits do not exist, which is explicated on in 1.2A and thusly 1.1A3 is not charted but instead incorporated into the topics under 1.2A. Students experience difficulties with the meaning of limits. Studies suggest that students are unclear how the x-values being approached relate to the function's value (Nagle,Tracy,Adams & Scutella, 2016). In particular, students have misconceptions concerning (1) if the limit or the function is approaching a value, and (2) if the limit of a function is a value or is just near a value (Nagle et al., 2016).

Topic 1.1C covers the rules for computing limits algebraically. Typical errors relating to this deal with students not recognizing the need to manipulate a function. Instead it is assumed that the limit does not exist whenever the function does not exist at the point the limit approaches (Parameswaran, 2007; El-khateeb, 2015). Sometimes students also mistakenly view limits purely in terms of algebraic evaluation and have little understanding of what the results mean either algebraically or graphically (Ferrini-Mundy & Graham, 1991). 1.1D leans into this error by highlighting some ways that limits relate to functions. Finally, 1.2A deals with limits relationship to continuity, honing in on where and when limits do not exist. Students have difficulties when the function exists at a point but the limit of the function does not, or exists but is not equal to the function at that point (Parameswaran, 2007). The existence of limits, or lack thereof (1.2A), is a major topic explored in college calculus (college calculus professors, personal communications, February-April, 2018). However, the AP free response section does not cover this material in a meaningful way. Over the last ten years, only three questions have covered limits directly. Both of these questions assessed students understanding of the relationship of limits and continuity, asking for students to interpret a graphical representation or a piece-wise function.

The topics for the first unit cover nearly all of the essential knowledge for limits, with the exception of L'Hopitals Rule (1.1C), limits and graphs (1.1B) and continuity as a condition for theorems (1.2B). L'Hopitals Rule is excluded because it requires the computation of derivatives, which is a skill learned in unit two. 1.1B addresses the fact that limits can be portrayed graphically, which is covered through the external representation category for cognitive demand on the content matrix. 1.2B has been reserved for section three as it is not possible for students to relate continuity to the theorems of calculus before they have been exposed to any such theorems.

### *Previous Test*

**Table 4.1:** Content Matrix for GHS Section 1 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 1.1A Definition/Existence of limits | | | | |
| • 1.1A1 Notation | 2,4 | | | |
| • 1.1A2 Concept of limits | 2, 3, 12, 15 | 9 | | |
| 1.1C Algebraic Rules for Limits | | | | |
| • 1.1C1 Sums and Products | | 11 | | |
| • 1.1C2 Algebraic manipulation | 8, 16, 18, 20 | | | |
| 1.1D Limits Relate to Functions | | | | |
| • 1.1D1 Asymptotes | 17 | | | |
| • 1.1D2 Magnitude, End Behavior | 6, 7, | | | |
| 1.2A Limits relationship to continuity | | | | |
| • 1.2A1 Continuity at a point | 4, 5, 13, 14 | 10 | | |
| • 1.2A3 Types of Discontinuities | 19 | | | |

Assessments are important because students are motivated by what they believe will be on the test (Rosasco, 2014). In one study, when students were introduced to a conceptual view of limits, they still believed the test would ask only the procedural-based questions (Hardy, 2009). It is therefore imperative for assessments to reflect a conceptual knowledge in order to redirect students' focus. The previous Green High School section one assessment focused primarily on routine questions. Though it fell short of the goal of assessing every topic in at least two areas of cognitive demand, the test did incorporate some conceptual-based questions related to an external representation. Questions nine through 11 require students to interpret a graph. While question 13 referenced an external representation by asking what must be true about a graph, the answer choices can all be understood in an analytic, non-graphical manner. Studies show that even when students are prompted toward a visual representation, they gravitate toward a purely analytic understanding of calculus (Eisenberg, Dreyfus, 1991). Thus, question 13 was categorized as routine.

Question 19 asked students to explain the differences in a removable and a non-removable discontinuity. The intent was for students to express their conceptual understanding of types of discontinuity in verbal form (Brown, personal communication, June 16, 2017). However, in practice, students received full credit for algebraic (or geometric) examples without verbal communication to demonstrate the logic behind the computations. Verbal explanations that referenced the computational difference alone, without reference to the implications for the function were also accepted for full credit. While some students did use a verbal form of communication, they demonstrated a lack of conceptual understanding of the difference in types of discontinuity and still receiving full credit. This was credited to the lack of clarity in directions on the test (Brown, personal communication, June 16, 2017).

Relying on procedures and graphs, there was little variation in the manner of which students were tested. Moreover, the typical areas of weakness were not assessed in a meaningful way. For instance, where students are known to struggle by not resorting to algebraic manipulation (Parameswaran, 2007; El-khateeb, 2015), the instructions specifically told students to "use algebra to evaluate" or to "find each limit algebraically." These cues remove the area of difficulty and give students a false sense of mastery. Students were also never asked to assign meaning to any of their results.

## *Assessment Analysis*

### *Topic 1.1A: Definition and Existence OF Limits*

On the new section one assessment, the first main topic, namely, the definition of limits, is approached from several perspectives. In the first question, students are given a chart and must create the appropriate limit equation. Pulling information from an external representation, students should recognize that $x$ is approaching negative four from both sides and simultaneously the function $f(x)$ is getting close to three. They must then transfer this information to notational form. There are no hints within the question as to what type of equation should be attained. While traditionally students are not asked to go from a chart to create an equation, the chart and equation interact so often in the curriculum that this is not coded as a non-routine problem. Still within this topic, the third question requires students to understand the dynamic relationship of limits through an external representation in graphical form. Approaching from one side alone ensures that students understand the notation of $x \to 1^+$ on an physical level. Question 11 also assesses students' knowledge of the definition but this time in a contextual environment. Within the word problem the revenue is represented in functional notation, however, students must still pull the limit equation from the verbal statement. Additionally, the function is in terms of $t$ and not $x$, which requires a more acute understanding of variable, which students struggle with (White Mitchelmore, 1996). Moreover, the contextualization of limits is particularly difficult for students (Ferrini-Mundy Graham, 1991). Hence, this application-based question addresses several of the common student errors related to limits.

*Topic 1.1C: Algebraic Rules for Limits*

When approaching the topic of algebraic rules for limits (1.1C), there were several typical student errors to address. The justification of results is therefore a critical portion of these problems. Questions two, six and eight assess the algebraic rules for limits, incorporating areas where students typically faulter.

The first subtopic, the sums and products of limits is addressed through questions six. Encountering a limit of two distinct functions, students demonstrate knowledge of this topic by breaking up the limit appropriately and evaluating the pieces separately. Within this, aquiring the limit requires a graphical understanding. There is a cusp and hole where students are evaluating the point, adding to the difficulty of utilizing the graphical representation (Ferrini-Mundy & Graham, 1991). The other portion to be computed requires a requisite knowledge of trigonometry.

Manipulating equations is the second subtopic, addressed through questions two and eight. Beginning with a routine problem, question two provides only the limit function and expects that students will recognize the need to compute additional algebraic manipulations without any cues through directions. If students merely plug five into the equation they would assume the limit does not exists, a common student error (Parameswaran, 2007; El-khateeb, 2015). Adding to the complexity, students are asked to interpret the result as it relates to the continuity of the equation. Difficulty might arrise as the limit exists while the function has a discontinuity at the point (Parameswaran, 2007). Students must verbally expound on what is happening to the function at this point, placing aspects of this problem within the application sphere of the content matrix. Since most of the student common errors intersect with this topic, question eight also focuses on algebraic manipulations, though it approaches the subtopic in an entirely different manner. While traditionally students are asked to find the limit of a polynomial as $x$ approaches a value, students must work backwards in question eight. Reversing the process involves the incorporation of several ideas related to limits. Continuity at a point is addressed since the limit is said to exist even though substitution renders the denominator zero. Algebraic manipulations must be performed in an abstract manner in order to remove the discontinuity, (thus a recognition of the removable point) and set equal to what the limit approaches. After this portion is solved, substitutions and distributions still need to be correctly performed. This question sets the tone not only in requiring a deep conceptual knowledge of limits but also in demanding a sophisticated procedural knowledge of quadratics.

*Topic 1.1D: Limits Relate to Functions*

Topic 1.1D conveys how limits relate to functions through asymptotes and end behavior. Questions five, nine and 12 focus on this topic, while evaluating knowledge in different manners. Question five has students note that the limit does not exist graphically and then explain what happens to the function as it approaches an asymptote. No points should be awarded for explanations that don't display an understanding of the discontinuity as it relates to limits. For example, "the graph shows an asymptote" is not a justification of how students know the function is discontinuous. Also related to a graphical understanding, question nine presents students with a verbal description of a function with several characteristics given through limits. Students must translate this verbal description into a graph of the function. While limits are frequently derived from a graph, students are not typically required to transfer information into graphs. This requires students to understand dynamically what is happening through the algebra well enough to create their own external representation of the data. This again ensures that students have a multi-representational view of limits (Ferrini-Mundy & Graham, 1991). While questions five and nine focus on an external representation, question 12 approaches the topic in a non-routine manner. Students are given a mildly abstract rational function as well as the location of horizontal and vertical asymptotes. The emphasis is on constructing the appropriate limit equations that enable the variables to be solved.

Question 11, discussed under 1.1A has aspects that require an understanding of end behavior. Students must recognize that the verbal cue, "what will they make ultimately" translates the limit as time approaches infinity.

*Topic 1.2A: Limits Relationship to Continuity*

The final topic, 1.2A, the relationship between limits and continuity received a large coverage on the assessment due to its relative importance. The most straightforward question, number four, asks students to pull information from an external representation and explain the results. While there could be confusion because the function has a value at $h(1)$, the graph clearly displays a jump discontinuity. The explanation should use clear language concerning the behavior of the function on either side of the point function. The verbal statements that students make are important and should be graded accordingly. It is the function, not the limit, that approaches a value on either side. Language in describing the graph helps identify students understanding of the concept of limits (Nagle et al., 2016). While using the same graph as question four, question seven is set up in a different manner. Students are given an equation and asked where on the graph the equation is true. By replacing the number that x approaches with $c$, thus having a variable approach another variable, the functional notation becomes somewhat abstract. Adding to the abstraction is the fact that the function $h(x)$ is not defined numerically and set equal to the non-numerical value $h(c)$. Students are working in reverse, searching the graph for properties that meet the requirements of the equation. Notably, this assess a concept that students typically find difficult, namely locating where the function exists at a point but does not equal the limit of the function at that point (Parameswaran, 2007). The last question that focuses on this topic, question 10, engages students in a functions continuity at a particular point. Students are given a piece-wise function with variables representing quantities in both portions of the function. Students must work in reverse to find variables for with the function is continuous. Questions similar to ten were used in two studies to test the "visual" and "non-routine" knowledge of college calculus students. In both studies students performed poorly, with the first citing a lack of visual interpretation of continuity and the second citing the unusual manner the question is posed in (Eisenberg & Dreyfus, 1991; Seldon et al., 1999). The AP free response section also included a similar question in 2003, with students largely earning zero of the possible three points (Riddle, 2003). Additionally, questions two and eight are coded under continuity at a point due to the presence of the removable discontinuity in these problems.

**Table 4.2:** Content Matrix of Section 1 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 1.1A Definition/Existence of limits | | | | |
| • 1.1A1 Notation | 2,3 | 1 | 9 | |
| • 1.1A2 Concept of limits | | 1,3, | 11 | |
| 1.1C Algebraic Rules for Limits | | | | |
| • 1.1C1 Sums and Products | 6 | 6 | | |
| • 1.1C2 Algebraic manipulation | 8 | | | 2 |
| 1.1D Limits Relate to Functions | | | | |
| • 1.1D1 Asymptotes | | 5 | 5 | 12 |
| • 1.1D2 Magnitude, End Behavior | | 9 | 9,11 | 9 |
| 1.2A Limits relationship to continuity | | | | |
| • 1.2A1 Continuity at a point | 2, 8 | | 8 | 2,10 |
| • 1.2A3 Types of Discontinuities | | 4, 7, 8 | 4, 8 | 7 |

*Summary*

The new test meets the demands of the content matrix. Not only is each topic covered in at least two ways on the cognitive demand but each topic is covered in at least two conceptual ways. Every subtopic has been assessed in a conceptual manner and 100 percent of the questions have a conceptual component. Thus the goals of the content matrix have been met.

In addition to being on target with the cognitive demand categories, there is a variety in the nature of the relationships. External representations include both chart and graphs. Students must go from a chart to functional notation, use a graph to evaluate an equation and create a graph from a verbal description of limits. Applications include words problems where a limit must be derived from a verbal statement and word problems where the limit is given through a verbal statement and must be put into an equation. Students must also solve abstract equations in a non-traditional manner.

Not only was there variation within the relationships of limits, but also within the notations utilized on the assessment. Limits are expressed through charts, with standard notation (including one sided limits), in direct verbal form, and in a contextualized verbal form.

The section one assessment engages all the main issues where students err according to the studies addressed in the content section. Without prompting, students must decide when and how to manipulate functions. Students are required to justify and interpret their results. Further, a requisite knowledge of trigonometric functions and polynomials adds to the degree of difficulty in the abstract applications of questions three and six.

## Unit 2 Section 2: Derivatives

### *Content*

Unit 2 focuses on the concept of the derivative. The curriculum has broken this unit into two portions: derivatives (section 2) and application of derivatives (section 3). The first portion includes the following topics: 2.1A, definition of derivative, 2.1B, estimation of derivatives, 2.1C, rules for calculating derivatives, 2.1D, higher order derivatives, 2.2B, relationship between continuity and derivative and 2.3B, derivative as tangent line.

In 2.1A, the definition of derivative is in terms of limits, lending itself to a conceptual understanding of how limits and derivatives relate to one another. While students often struggle in articulating this relationship, a clear understanding of this topic is correlated with higher performance on future topics like graphic derivatives and conceptualizing the derivative at a point (Vincent, 2016). Though the limit definition has not been formally required on the AP free response section in the last ten years, it is related to the estimation of derivatives, which is tested once or twice each year on the free response section. The two ideas are so closely related, that many students have trouble distinguishing them and interchange their respective formulas (Vincent, 2016).

Students find better success with the ideas of 2.1C, the rules for calculating derivatives. This is perhaps because of the procedural nature of this concept (Jones, 2017). However, some students still make algebraic mistakes and have difficulty when questions require them to use a rule in combination with another rule (Jojo, Maharaj & Brijlall, 2013). Further, students can be confused when confronted with functions that require a change in form before common rules can be applied (Ferrini-Mundy & Graham, 1991). Finally, errors are frequent when these algebraic rules are presented in an abstract manner (White & Mitchelmore, 1996). Approximately every other year, the AP free response section will include both routine and non-routine questions regarding these ideas. The emphasis is generally in presenting them in an abstract form. Even when not addressed directly, these topics are requisite knowledge for numerous other free response questions.

The derivative as tangent line (2.3B) has been tested every year for the last ten years by the College Board. This concept is used in tandem with several other topics on the AP exam and often requires an interpretation of the approximation. This topic is frequent because it is foundational to interpreting graphs and estimating functions. While many students can successfully set up an equation for the tangent line at a point, they don't always know what it being estimated by the line (Vincent, 2016). Student also can encounter confusion when it comes to the graphical representation or interpretation of the tangent line (Vincent, 2016).

L'Hopital's rule (1.3C) can now be introduced as students are able to compute derivatives. L'Hopital's rule applies only to limits that are otherwise indeterminate. The AP FRQ section has not directly included this topic in the last 15 years. When taking the limit of a function that is a fraction, students are generally successful in applying the theorem (Rodi, 1986). Sometimes students even apply the rule when the condition of the limit being indeterminate is not sufficiently met (Rodi, 1986). Anectdotal evidence suggest students do not understand the crux of the problem with these limits, namely that the quantities are both growing very large (or small) but at different rates (Ash, Ash, & VanValkenburg, 1986).

*Previous Test*

**Table 4.3:** Content Matrix of GHS Section 2 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 1.3C3 L'Hopital's Rule | | | | |
| 2.1A Definition of derivative | | | | |
| • 2.1A2 Limit definition at a point | | | | |
| • 2.1A3 Limit definition of function | | | | |
| 2.1B Estimation of derivatives | | | | |
| 2.1C Rules for calculating derivatives | | | | |
| • 2.1C2 Solving directly | 5, 7 | | | |
| • 2.1C3 Product/Quotient Rule | 2, 10, 12 | | | |
| • 2.1C4 Chain Rule | 9, 11 | | | |
| • 2.1C5 Implicit Differentiation | 8, 13 | | | |
| • 2.1C6 Inverse Function | | | | |
| 2.1D Higher order derivatives | | | | |
| • 2.1D1 Definition | 1, 11, 12 | | | |
| • 2.1D2 Notational variety | 1, 11 | | | |
| 2.2B Continuity and derivatives | | | | |
| • 2.2B1 Points not differentiable | | | | |
| • 2.2B2 Differentiable functions | | | | |
| 2.3B Derivative as tangent line | | | | |
| • 2.3B1 derivative as slope | 3, 4, 12 | | | |
| • 2.3B2 tangent line as approximation | | | | |

Green High School tested the students on these topics in a purely routine manner. While question 10 aimed for a conceptual understanding by applying the problem to the rate of change in area of a triangle, every relationship represented through words was also explicated in a corresponding notational manner, eliminating the need to transfer from a verbal to a functional form. Through the hints at the end of the question, the relationships were specified further and students were prompted even in the type of calculation to use when taking the derivative, rendering the question purely procedural in nature. In addition to the gaps in cognitive demand, the conceptual matrix reveals that integral topics were also not tested (see table 4.3). Over half of the topics/subtopics in this section are not being formally assessed, which signals to students that these unassessed topics lack significance (Raupach et al., 2013). This means that statistically students' are not prioritizing half of the content in this section that the College Board deems necessary for understanding calculus. Notably, some of the gaps correspond to topics of particular emphasis' by the College Board, like the estimation of derivatives (which account for an average of 1.8 points per test) and tangent lines as approximations (.8 points per test). Topics like 2.1C are also on the AP test in a conceptual manner accounting for an average of 1.1 points.

*Assessment Analysis*

*Topic 2.1A: Definition of Derivative*

The section two assessment approached the definition of derivative (2.1A) in a routine and applied manner. The first question specifies the manner in which students should take the derivative so there is no translation involved in students choosing the correct method. However, students do need to know the definition and how to use it for computations, hence a procedural knowledge which places this question in the routine section. I employed the notation $f'(x)$ but also clued students in through the directions that the reference was for a derivative. Students must explain what the result means in terms of the graph. By not mentioning the tangent (or slope) by name, but rather asking them what the derivative means, there are no verbal clues hinting at the answer. This application is placed under definition of derivative but since this application also relates the slope of the tangent line, it is likewise coded under topic 2.3B.

Inspired by question 37 on the multiple choice section of the 1993 AP Calculus AB test, question 12 demands an intricate knowledge of how each parts inter-relates in the limit definition of derivative. Students must wade through potential alternate definitions to see which accomplish the goal of finding the slope over an infinitely small region. There are minute differences in some of the definitions making the translation challenging. Further, students are known to confuse the derivative function with other similar functions, like the estimate of derivatives (Vincent, 2016). The results must be accompanied by a verbal explanation that is specific in describing each limit in terms of slope.

*Topic 2.1B: Estimation of Derivatives*

Estimating derivatives (topic 2.1B) appears on multiple AP free response sections accompanied by tables. This test takes two different approaches in the estimation of derivatives, though both rely on tabular information. Question eight incorporates a verbal, contextual problem. The accompanying chart maps hours (recorded in non-uniform increments) and smartphones in stock at a store. This problem requires only a base level procedural knowledge of the derivative as an estimation. However, the added layer of pulling information from a chart and then explaining the result in context means that students must transfer data into the algebraic notation and then impart the meaning through words. A similar set up appeared on the 2012 AP exam. The results showed that while students could largely compute the estimation, they often misinterpreted the meaning and used incorrect units (Kokoska, 2012).

Question six, while not contextual, involves layers of transfer. Students should know how to estimate a derivative using a table. They should likewise know that a second derivative is simply a derivative of a derivative. However, these two concepts are not generally combined. Combining them requires that students fully understand the estimation equation and how to alter the equation to accommodate a second derivative. By requiring a novel modification to a known procedure, this falls into the non-routine category.

*Topic 2.1C: Rules for Calculation*

The rules for calculating derivatives splinter into several subtopics all of which are essential procedures for calculating derivatives of functions. Questions 3, 4, 5, 7 and 10 cover these subtopics in a variety of manners.

Question three focuses on the procedure of calculating derivatives. The notation is switched to the $\frac{dy}{dx}$ format with $y$ being replaced by the function and with the irregular placement of the squares that is required for higher order derivatives. This symbolic representation is purposefully more difficult for students to recognize. Similar notation on the 2015 AP exam proved difficult for students, with many unable to recognize that a second derivative was necessary (Kokoska, 2015). Additionally, each derivative requires a different computational rule. For the first derivative the chain rule is applied twice. However, the second derivative requires both the product rule and chain rule. By combining the rules, this question assesses students on an area that statistically leads to error (Jojo et al., 2013).

The fourth question also delves into a common misconception related to calculating derivatives by testing students on the rules in an abstract manner (White & Mitchelmore, 1996). This question asks students to compute a derivative of a composite function, composed of two functions, neither of which is expressed algebraically. In order to arrive at a correct answer, students must first use abstract functions to perform the appropriate derivative. Next, the information needs to be extracted from the chart or graph in the proper order. This question is non-routine due to the abstract aspects. While the setup is slightly different, this question resembles question six on the 2016 AP exam, where nearly one-third of students received no points and struggled to compute the derivative of the abstract composite function (Davis, 2016).

Similar to question four, question five approaches the computation of derivatives from an abstract perspective that requires the calculations of unknown functions. With this problem in particular, the derivative must be found in the abstract sense before plugging in data as there is an implicit chain rule inherent to $h(3x)$ that is missed if substitution occurs to early. This problem also requires the use of the quotient rule, an aspect of computing derivatives that has not been previously required on the test. As with question four, once the abstract derivative is arrived at, information from the table and graph must be used to solve. By bringing together abstract functions with multiple computational rules, this questions assesses two common difficulties for students (White & Mitchelmore, 1996; Jojo et al., 2013). The free response section of the 2014 AP exam involved a similar set up where the quotient rule must be applied to an abstract function. Students were relatively successful though some struggled with the algebra, particularly with using parenthesis when squaring the denominator (Kokoska, 2014).

Inverse functions are assessed through question seven, which again uses the abstract form to gauge understanding. Part D of question three on the 2007 AP free response section had a comparable setup, with students taking the derivative of an inverse through a chart. This portion of the question was widely missed, with students using a reciprical instead of an inverse or not attempting the question at all (Diefenderfer, 2007). A deep understanding of functions is necessary to take the derivative of the inverse in a generalized form before applying the appropriate information from the chart.

Question 10 addresses the final subtopic of calculating derivatives, implicit differentiation. This rule is used in tandom with the chain rule, again evaluating a typical area of student confusion (Jojo et al., 2013). This question will be further analyzed under the topic of continuity and derivatives (2.3B).

*Topic 2.1D: Higher Order Derivatives*

Topic 2.1D, higher order derivatives, is assessed through two problems, questions three and six. Question three concurrently tests the computational rules for derivatives while question six is discussed under the estimation of derivatives. Question three is routine in nature but question six requires the second derivative to be combined with the estimation formula (non-routine) and then interpreted from a chart (external representation). These questions each use a different notation to express the second derivative so that students develop a notational fluency.

*Topic 2.2B: Continuity and Derivatives*

The relationship between continuity and derivatives is addressed in a non-routine and a graphical manner. Question 10 is modeled after question six on the 2015 AP exam. Typically students are asked to find the slope of a tangent line at a particular point. Here, and on the 2015 exam, students are asked to find points where a line tangent to the function is vertical. Approaching from a different angle, students must take the derivative and find where it is not continuous. Many AP test-takers faltered on this step, with a large percentage not attempting the problem and others setting the numerator equal to zero (Kokoska, 2015). After conquering this hurtle, students must again understand the nature of tangent lines to make appropriate substitutions. Rich in algebraic manipulations, this question still hinges on a conceptual understanding of both continuity and tangent lines. Also addressing continuity, question 11 approaches the topic from a visual angle.

Students must generate a graphical example of a continuous function that is not differentiable (at least at a point) and verbally reflect on the relationship of continuity and derivatives.

*Topic 2.3B: Derivative as Tangent*

The derivative represents the slope of the tangent line at each point on the function. The AP free response section often asks for the tangent line or an approximation using a tangent line in connection with differential equations, the fundamental theorem of calculus, abstract functions, and verbal problems. Although many of the ideas the tangent line is used in tandom with on the AP test have yet to be taught, this test incorporates the tangent line within a variety of contexts.

Question nine provides a verbal background from which a tangent line approximation must be considered. Within the question, the abstract function $C$ is said to model the amount of corn syrup in a vat after $t$ minutes. However, the given differential equation $\frac{dC}{dt}$ has no contextual meaning given with the problem. Students must use this equation and the initial condition, given verbally in the first sentence, to create a tangent line. Since the line must be evaluated at a time of 2 minutes and 30 seconds, a simple conversion must be used before substituting into the equation. The answer should be given with appropriate units, indicating that the student understands the result in terms of the context. The contextualization of the problem adds to the difficulty as students struggle with the meaning of a tangent line approximation (Vincent, 2016). This question is riddled with translation from symbols to words and words to symbols, hence its placement under application. The 2007 and the 2011 AP tests likewise asked students for tangent line approximations in verbal contexts. Students did very poorly in 2011 with over half the test takers not receiving any points (Boardman, 2011). In 2007, only one third received no credit but many students who computed the approximation could not adequately explain their result (Diefenderfer, 2007).

Discussed extensively under other topics, questions four and five and 10 all include aspects that judge students conceptual understanding of the derivative as a tangent line. Questions four and five require students to find the slope of the tangent line of a function at a point, utilizing it as the derivative within an equation. While these might seem somewhat straightforward studies show that students struggle with the graphical interpretation of tangent lines (Vincent, 2016). Question 10 asks students where on a function the tangent lines are vertical, pressing students to proceed with tangent lines in multiple unusual ways. Similarly, the 2004 and 2012 AP exams asked students to find places where the tangent line was horizontal, with the 2004 test placing it in the parallel context of implicit differentiation.

*Topic 1.3C3: L'Hopital's Rule*

This section's assessment also includes L'Hopital's Rule, which explores the idea of limits while also incorporating derivatives. Question two requires the recognition and operation of L'Hopital's rule. Whenever specific theorems or rules of this nature are used, it must be accompanied by a justification demonstrating that the conditions for the rule or theorem have been met. Thus, this question inherently falls within the spectrum of application as students verbalize their logic. A knowledge of trigonometric functions, together with the chain rule, adds to the complexity of this problem. Question 13 ensures that students are not only familiar with the procedure of L'hopital's rule but also the conditions for which it applies. Here students must use specific language to explain why the fraction is not in an indeterminate form.

**Table 4.4:** Content Matrix of Section 2 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 1.3C3 L'Hopital's Rule | 2 | | 2, 13 | |
| 2.1A Definition of derivative | | | | |
| • 2.1A2 Limit definition at a point | 1, 12 | | 1 | |
| • 2.1A3 Limit definition of function | 12 | | 12 | |
| 2.1B Estimation of derivatives | 8 | 6, 8 | 8 | 6 |
| 2.1C Rules for calculation | | | | |
| • 2.1C2 Solving directly | 3 | | | |
| • 2.1C3 Product/Quotient Rule | 3 | | | 5 |
| • 2.1C4 Chain Rule | 3,10 | 4,5 | | 4,5 |
| • 2.1C5 Implicit Differentiation | 10 | | | |
| • 2.1C6 Inverse Function | | | | 7 |
| 2.1D Higher order derivatives | | | | |
| • 2.1D1 Definition | 3 | 6 | | 6 |
| • 2.1D2 Notational variety | 3,6 | | | |
| 2.2B Continuity and derivative | | | | |
| • 2.2B1 Points not differentiable | | 11 | 11 | 10 |
| • 2.2B2 Differentiable functions | 11 | | | |
| 2.3B Derivative as tangent line | | | | |
| • 2.3B1 Derivative as slope | | 4,5 | 1,9 | 10 |
| • 2.3B2 Tangent line approximation | | | 9 | |

*Summary*

The new test not only covered a wider breadth of topics but also moved the cognitive demand toward a more conceptual understanding. The goal of having every major topic covered by a minimum of two levels of cognitive demand was met, with the majority of topics being charted on three or more categories (see table 4.4). Additionally, each subtopic was covered at least once while the length of the test remained reasonable for a 50 minute class meeting. Only two questions (one and three) do not have a conceptual component, meaning that over 75 percent of the questions have a relational element.

A routine problem was added for the definition of derivative because the application question did not require a procedural knowledge of how derivatives are related to limits. Additionally, a routine problem was introduced to check the procedural knowledge of derivatives using a combination of rules. These two questions covered topics which were not tested in a computational manner on other areas of the test. Further, these questions were purposefully difficult from an algebraic standpoint. The test is strategically set up to first require procedural knowledge in order to boost student's confidence and refresh student's memory before moving into more conceptual problems that require transfer.

Within the external representation category, students must interpret information from charts and graphs. There are two charts, one that employs functional notation and the other that conveys contextual knowledge. In addition to delineating data from existing graphs, students must sketch a graph of a function that meets a particular set of requirements described in verbal form. Application questions include word problems and questions in which students must give a verbal response. There are several non-routine problems that involve students abstracting information concerning algebraic rules for limits. The test also employed a diversity of language and notation to depict the functions and ask the appropriate questions, assuring that students recognize derivatives in any format. Even with the diversity of conceptual problems, the computations do not lack rigor. A prerequisite knowledge of inverse, trigonometric and exponential functions undergirds multiple questions.

Areas that are known to be difficult for students were all addressed. Students needed a clear concept of the definition that can be distinguished from other similar formulas for question 12. The computational rules for calculating the derivative were mixed, involved a change in form and were also computed in an abstract manner. Tangent lines needed to be properly evaluated in context and understood graphically.

## Unit 2 Section 3: Applications of Differentiation

### *Content*

Section 3 is aptly named as it focuses on the topics that naturally lend themselves to a conceptual understanding. The topics focused on in this chapter are 2.2A derivatives relationship to a graph of a function, 2.3C derivatives as related rates 2.3E derivatives relationship to differential equations, and 2.4A the Mean Value Theorem. Necessarily, 2.2A requires the translation to and from graphical form. Encompassed within 2.3C is the relationship between derivatives and rectilinear motion and derivatives as a way to optimize functions. To properly assess knowledge of this nature, the recognition of the meaning and purpose of a derivative as well as how the derivative relates to a problem is necessary - again leading to questions that fall into the conceptual categories. However, the manner in which the topics are described makes it difficult to confer subtopics in 2.3C to require a graphical translation. This is because an optimization problem, which clearly falls into 2.3C in the context of a graph now becomes a minimum/maximum problem which falls under the topic of 2.A. Conversely, most applications outside of a graph for 2.2A is better charted under 2.3C. The final topic (2.4A) deals with the Mean Value Theorem which should be understood conceptually but also requires a procedural knowledge in application. This is the first section to introduce a major theorem and so topic 1.2B which deals with understanding continuity as a condition for theorems has been saved until this section.

In 2.2A, the emphasized relationships between functions and graphs not only deal with the derivative as the slope of a tangent line at a point on the function but take a broader stance in introducing the derivative function as a measure of the (mostly continuous) rate of change. This portrays vital information about whether the original function is increasing or decreasing. Further the idea of the second derivative function relating to the original in terms of concavity and relating to the first derivative in terms of the rate of change increasing or decreasing are integral to understanding functions graphically. The derivatives relationship to a graph features prominently on the free response section of the AP test. Over the last ten years, this concept appeared an average of two times on every AP test. Students are generally given the graph of a function or the graph of the derivative of a function and asked to "find values" of the function and derivative. They are asked to find critical points, local or absolute minimums and maximums, and also to find infection points. Other prompts ask for intervals where the function is increasing/decreasing, concave up or down, and to interpret what is happening at certain points. Further, students are sometimes asked to make composite functions with the graph/graphs given. About a quarter of the questions ask for an interpretation of what the answer means in the context of a situation. The graphs in general are sometimes, but not always, accompanied by an algebraic expression of the function, or the functions derivative. Notably, students have not been asked to sketch a graph since 2005, leaving speculation that this skill is no longer required on the AP test. However, there was similar speculation concerning the integration of trig inverse functions, which received at least a five year hiatus from the test, only to reappear in recent years.

There are a number of common conceptual issues that students face with the relationships of 2.2A.

On a basic level, studies found confusion between how the first and second derivative relate to one another (Jones, 2018). Students who could identify that the first derivative represented an increase/decrease and the second derivative concavity, had trouble comprehending that a function could be getting larger while the rate it increased was getting smaller (Jones, 2018). Students also experienced trouble relating the points of inflection they found graphically to the context of the situation the graph represented (Jones,2018). Further, students sometimes could not interpret graphs without an equation being present, demonstrating a reliance on algebraic expressions (Asiala et al., 1997). Finally, when the derivative was non-linear, students dramatically failed at being able to graph the derivative function (Teuscher & Reys, 2012; Vincent, 2016).

2.3C covers three important aspects of derivatives use. The first, a derivatives relationship to motion, or kinematic contexts of derivatives, is popular not only because of its association to physics (a course required for most STEM careers) but also because of the intrinsic relatability to students. The concept of speed and how position relates to time, is one that students have an early notion of. Moreover, relationships of this nature are readily interpreted because it is always position, not time, that affects the magnitude of the expression (i.e. time changes at a fixed rate). Many students use the understood physical relationship involved in these kinematic contexts to strengthen their concept of derivatives and graphs (Marrongelle, 2004). These question are a favorite of the AP free response sections, appearing as the framework for a question 13 of the last 15 years. Students are asked to find: where something changes direction, when something reaches its highest point, when something is speeding up or slowing down, the average velocity, the average acceleration, the average rate of change of velocity, the rate that the distance is changing and to interpret these in light of other objects in motion.

Motion can be a gateway for students to understand derivatives as rates of change in general. In these related rates, most often students are asked to relate how something changed over time, but students might also be asked to examine how the volume of an object changes based on the height. Appearing at least once, on average, each year on the AP free response section, related rates questions require a deep understanding not only of the derivative but also of the situation. Students are asked questions related to quantities, estimation, and the rate at which things change. Further, they are asked to interpret these rates in context with properties of the first and second derivative. Identifying the way the rates are related to one another in non-kinematic settings is often difficult for students (Jones, 2017). Indeed, it appears that the difficulty in these problems is interpreting the context. The more clues students are given as to the notational interpretation of the context, the more they are able to find correct solutions (Martin, 2000). However, even when students can associate the rate of change with the derivative and find a correct answer, some are still confused about what the answer means in context (Jones, 2017).

Though listed separately by the College Board, optimization is an example of a type of question that can be asked within the context of related rates. The free response section on the AP test frequently employs questions that test students knowledge of this relationship, including them around 70 percent of the time. As with the larger category of word problems, the success of solving optimization problems hinges on the ability to construct an appropriate formula (LaRue & Infante, 2015). Students are confident with the required procedure, though they can't explain why the procedure works (LaRue & Infante, 2015). The inability to identify the appropriate pieces of information to use in an optimization formula is likely linked to the lack of understanding in how the formula works.

The final topics 2.4, the Mean Value Theorem, and 1.2B, continuity condition for theorems appear often on the AP exams. Though named *the Mean Value Theorem*, topic 2.4 also covers the Intermediate Value theorem and Rolle's theorem. The Mean Value Theorem, or the Intermediate Value Theorem, appeared on roughly 60 percent of the last 10 AP exams. It is asked in connection to an estimation of a derivative nearly every time. While sometimes the theorem is named, most of the time prompts like, "explain why there must be a value $c$," on an interval or "is there a time $t$," on an interval, cue students to employ the theorem. Whenever these theorems appear, topic 1.2B is also being assessed as theorems can never be employed without demonstrating that the conditions for the theorem have been met (Boardman, 2011).

### *Previous Test*

**Table 4.5:** Content Matrix of GHS Section 3 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 1.2B Continuity condition for theorem | 7 | | | |
| 2.2A Derivatives relationship to a graph | 2, 5, 9, 12, 14 | 11 | | |
| 2.3C Derivatives as Related Rates | | | | |
| • 2.3C1 Motion | | | | |
| • 2.3C2 Related Rates | | | | |
| • 2.3C3 Optimization | 4 | | | |
| 2.4A Mean Value Theorem | 1, 15 | 6 | | |

While this section may lend itself to application, the former Green High School section three assessment did not require any. The one word problem, question four, denoted the perimeter and the area, then went further in reminding student they must use both of these functions to find the result. One could easily cover up all the words and still compute a correct solution, making transfer from verbal statement irrelevant. The only conceptual category of cognitive demand covered on the test was the derivatives relationship to external models, in this case, graphs. However, it is worth noting that even when graphical interpretation was tested, areas that students statistically find more difficult (as summarized above), are avoided. Only a shallow knowledge of how the derivatives relate to the graph is necessary. For example, when the graph of a function was given in question five, no calculus was needed to answer either portion of the question. Students could merely look at the graph and find where local maximums and where the graph is concave up. Even if students chose to use a calculus approach, they could easily avoid the graph altogether and use only the function, which was given in notational form. In fact, when supplied with a graph and notational form, students repeatedly ignore the graph in favor of an algorithmic route (Eisenberg & Dreyfus, 1991). Again in question nine, the graph accompanies the notational form of the function and so all calculations (and justifications) can be done without reference to the graph. Even when students are asked to sketch a graph in question 11, the function to be graphed is given and a graphing calculator is permitted, making the relationship of the derivative to the function completely irrelevant. The students can simply type the function into their graphing calculator and sketch the resulting image. Thus, in every case where the content matrix labels a question as demonstrating students ability to relate functions to a graph, an understanding of the graph is not actually necessary.

There were only six topics/subtopics in this section but one-third were missing from the assessment. Markedly, the neglected topics are applications that are popular within the AP free response section. In the last few years an average of nine points on each AP free response section corresponded to topics in this section alone that students never engaged with on a formal level.

*Assessment Analysis*

*Topic 2.2A Derivatives Relationship to a Graph*

Represented multiple times throughout the assessment, the derivatives relationship to a graph is essential in understanding calculus. Questions seven and nine directly address the topic while questions five and six integrate the topic within the solution process.

Rather than approach the topic from a graphical nature, question seven uses an abstract function to test if students understand how and why the procedures work. Typically students are asked to find the local maximum or minimum of a function. Here, students are given the maximum and must work backwards to get the function. Taking an abstract derivative (which requires an understanding of how different variables correspond to the variable of differentiation), students then must substitute the x-value of the maximum into the expression and set it equal to zero. An additional equation is needed, where students substitute x-value of the maximum into the original and set it equal to the value of the local maximum. After solving the system of equations, students finally have a functional notation they can use to find the local minimum. The accompanying justification should reference the second derivative to demonstrate that the second critical point is a minimum rather than another local maximum.

Question nine provides test takers with six graphs, three of unique functions and three of the corresponding derivatives. Students must not only pair the derivative with its function but also identify which graph is the original and which is the derivative. Notably, none of the chosen functions have linear derivatives, which are more easily recognized by students (Teuscher & Reys, 2012; Vincent, 2016). Similarly, the functions appear only graphically and not in notational form (Asiala et al., 1997). Students must rely on their graphical understanding of extrema and critical points to correctly match the graphs. The accompanying justification ensures that students have a correct and thorough conception of the topic rather than just pairing graphs they felt looked similar.

The fifth question emphasizes the derivative as it relates to motion but has components that demand a clear understanding of the derivatives relationship to a graph. Students must relate optimization to the absolute maximum and thusly check a half-closed interval. Verifying the endpoints is a skill that falls squarely in 2.2A since it requires the clarification between local and absolute extrema. In order to check the boundary, students must have a firm grip on the context, recognizing that at t=0, the car is at its starting point and no distance has been covered. Finally, students must use the graph of the derivative to find and classify the local extrema on the interval. This portion of the question can lead to error since the function is never represented in functional notation but only given through the graph (Asiala et al., 1997). The process and solution should be justified in verbal form. Prior AP tests with similar tasks demonstrate that students have trouble explicating global critical points and instead revert to local language (Boardman, 2009; Kokoska, 2013). Within this problem the maximum value must be evaluated in verbal form and through external representation.

Question six, like question five, stresses the derivatives relationship to motion more than the derivatives relationship to a graph. However, embedded within the process of solving is a problem related to the interaction between the first and second derivative. Not only is this relationship tricky for students in general, but this question relates a situation of particular error where the derivative get larger but the rate at which it increases gets smaller (Jones, 2018). Students must look for areas where the sign of the first derivative is opposite the sign of the second derivative. This must be done through the graph alone, adding to the degree of difficulty for students (Asiala et al., 1997).

*Topic 2.3C: Derivatives as Related Rates*

Heavily emphasized on the AP free response section, derivatives are utilized as a rate of change in a variety of manners. 2.3C1, demonstrates how derivatives are utilized in kinematic settings. Questions five and six both assess this relationship. Since students have a more natural awareness of the context of motion, these questions incorporated subtopics that students tend to error on in understanding (see 2.2A above). Question five gives students a graph of the velocity of a car and asks them to interpret when the car is furthest from its starting point. Steeped in relationships, this question first requires students to take a verbal statement concerning velocity and position and recognize the inherent derivative relationship. Next, students should realize that the verbal cue, "furthest," implies an optimization. Bringing these concepts together, the correct interpretation of the graph is necessary to arrive at the proper result. To further ensure a proper understanding of the graph's relationship in context, the question requests verbal justification and interpretation. Question six likewise asks for students to expound on their answer. Hinging on an acute understanding of the interplay between velocity and acceleration, this question mirrors the language and objective used on portions of questions from five separate AP exams within the last 10 years. Students typically miss problems of this nature or do not correctly justify the answer by specifically comparing the velocity (or first derivative) and the acceleration (second derivative) (Boardman, 2008; Kokoska, 2012; Davis, 2016).

Derivatives as related rates in non-kinematic contexts (2.3C2) are assessed on questions three and eight. The third question, while a word problem, provides students with limited verbal clues. For example, though the rate of sand filling the cone is mentioned, the question never references this as indicating a change in volume, requiring students to make that leap. More information is given contextually than is needed to answer the problem, again ensuring that students can identify which pieces of information are relevant in the desired equation. By leaving out functional notation, limiting verbal denotations and providing additional information, students must master the most difficult portion of contextual problems (Martin, 2000). Solving this problem requires multiple substitutions (some involving fractions), implicit differentiation and calculations involving $\pi$. Students statistically struggle when implicit differentiation is introduced within contextual problems and also tend to make errors with units when formulas for volume involve *pi* (Jones, 2017; Dorko & Speer, 2015). The 2002 and the 2008 AP exams asked students related rates questions corresponding to the volume of a cone in like manner. In these situations, students experienced difficulty in setting up the appropriate equations and in the differentiation (Boardman, 2008).

Again testing students ability to solve related rates problems, question eight requires students to visualize (or draw) a triangle and connect the rates of change in the base and height to the area. Setting the rates of change of the height and base to be direct opposites might lead some students to believe that the area does not change. However, through implicit differentiation students should recognize that the area depends not only on the rates of change but also on the height and base at a given point. Thus, the area only remains unchanged when the base and the height are equal (which is not stipulated on the multiple choice selections). Here, students must not only solve the equation but interpret the results in terms of the context. In doing so, students should set the rate of change greater than zero to find when it is increasing. By forcing students to place the results back into context, this problem addresses the common issue students face in not being able to interpret their results (Jones, 2017). This question was modified from the 1998 multiple choice question 90. Though the question remains in a multiple choice format, it requires students to have a verbal, external and functional knowledge of related rates.

The final subtopic of 1.3C is optimization. We have seen that question five is in fact an optimization problem, requiring students to find the zeros graphically. While question five utilizes the skills of optimization, question one is solely focused on the subtopic. Here the farmer's predicament is described verbally. Students must create a mental (or preferably draw a physical) diagram of the situation in order to set up the proper equations for both perimeter and area. This step can be difficult since neither equation is given functionally within the problem (LaRue & Infante, 2015). These equations must be manipulated, solving the perimeter for a single variable and substituting it into the area equation before any calculus can be performed. After solving, students must answer the question using the appropriate units.

*Topic 2.4A: Mean Value Theorem*

Topic 2.4A deals with the Mean Value Theorem which also encompasses the Intermediate Value Theorem and Rolle's Theorem. This assessment, in accordance with the AP emphasis, focuses on the Mean Value Theorem. Question two begins with a simple understanding/application of estimating a derivative. However, using this estimation to assess the validity of Mark's claim requires a graphical understanding both of what the resulting estimation means and what the derivative means in terms of the graph. These two concepts must be balanced with an understanding of both the results of the mean value theorem and the theorem's necessary conditions. Since the equation is never given for the function, the Mean Value Theorem and its conditions must be interpreted solely through an external representation. The 2011 AP exam set up a similar graph and explicitly asked why the Mean Value Theorem does not apply on the interval. Most of the students could not articulate why the theorem didn't hold (Boardman, 2011).

Question four likewise does not provide any functional notations. A chart is provided with the relevant functions being defined situationally and in verbal form. Never is it stated that $R(t)$ is the derivative of $W(t)$, students must pick up on key words like "rate" and reference the setting to make this jump. The chart also has an unnecessary row so that students must understand the context in order to pull the applicable data. Similar to the 2013 AP exam, students are asked if there is a time when the rate at which water is pumped is 4.5 liters/minute. Checking that the average rate of change on the interval is 4.5 liters/minute, the Mean Value Theorem must be employed. In order to receive full credit, students must demonstrate through the justification portion that the conditions for the Mean Value Theorem have been met.

*Topic 1.2B: Continuity Condition for Theorems*

The introduction of theorems naturally lends itself to the topic of continuity conditions for theorems. The two questions where the Mean Value Theorem is assessed both necessarily address this topic. Question two hones in on the conditions, having students identify through an external representation that although the function is continuous, it is not differentiable at a point. In question four, students must verify that the conditions hold, making the jump from twice differentiable to continuous. In both of these questions students must express the relationship of the functions to these conditions verbally.

**Table 4.6:** Content Matrix of Section 3 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 1.2B Continuity condition for theorems | 4 | 2, 4 | 2, 4 | |
| 2.2A Derivatives relationship to a graph | | 5, 6, 9 | 5, 6, 9 | 7 |
| 2.3C Derivatives as Related Rates | | | | |
| • 2.3C1 Motion | | 5,6 | 5,6 | |
| • 2.3C2 Related Rates | 8 | 8 | 3, 8 | |
| • 2.3C3 Optimization | | 1, 5 | 1 | |
| 2.4A Mean Value Theorem | 4 | 2, 4 | 2, 4 | |

*Summary*

The section three topics all correspond to three areas of cognitive demand. There are fewer questions, allowing this test to fit comfortably within the time frame of Green High School classes. All subtopics are included and every question has at least one conceptual aspect being assessed. Problems are diversified even within the categories of cognitive demand. Students are given charts and graphs to pull pertinent information from, thus transferring from external representations into functional (or verbal) notation. Additionally, students must construct diagrams (at least mentally) moving toward external representations. The majority of questions are given in verbal form, where unique circumstances inform the derivative relationships. Likewise, students must also explain answers or contextualize results. There is also a non-routine questions which takes a more abstract approach towards derivatives graphical nature.

The assessment further utilized an assortment of notations and vocabulary to hit at the ideas related to derivative. Phrases like, "as much area as possible," "average rate of change," "how fast the height is changing," "furthest," "slowing down," and "local minimum" are all prompts for how to apply calculus. Prerequisite geometric knowledge involving perimeter, area and volume are tested alongside an understanding of multiple families of functions. Rates of change not only depend on time but also on other changing quantities. Common student errors were assessed, sometimes through multiple questions. Finally the themes of the free response questions were present with appropriate rigor and language.

## Unit 3 Section 4: Anti-Derivatives

### *Content*

Unit three is broken into four sections: anti-derivatives, logarithmic, exponential, and transcendental functions, differential equations and applications of integration. The first section of unit three, anti-derivatives deals primarily with how to calculate anti-derivatives. The five main topics covered in this introduction to Anti-Derivatives are: 3.2A integral definition using Riemann Sum, 3.2B Approximations using a Riemann Sum, 3.2C Properties of Integrals as they relate to a graph, 3.3A Relationship of definite integrals to functions, 3.3B Properties of integration.

3.2A and 3.2B present integration in terms of limits of Riemann Sums. Prior to calculus, many students are often not familiar with summation notation. However, it is in taking the limit of the sum where students typically falter conceptually (Orton, 1983). As with word problems, students also encountered issues setting up the appropriate equation, specifically in partitioning the base of the region (Sealey, 2014; Orton 1983). The College Board has not directly tested the limit relationship of integration, but the free response section incorporates an approximation based on a Riemann sum 9 out of 10 years. These estimations are accompanied by charts or graphs and asked in conjunction with both averages and net change. Most of the time, the estimation must be interpreted in terms of the context.

Similar to the definition of integrals, 3.2C views the integral as area under a curve. This topic appears directly on the AP free response section around 70 percent of the time. Used with piece-wise functions or undefined graphs, these problems are often incorporated into application problems or are used in tandem with the fundamental theorem of calculus. Within this framework, students struggle to understand negative area (Kouropatov & Dreyfus, 2013) and likewise the antiderivative of the absolute value of a function (Serhan, 2015; Radmehr & Drake, 2016).

3.3A acquaints students with the fundamental theorem of calculus, specifically the second portion which brings together derivatives with integration. Unsurprisingly, the free response section of the test is riddled with problems that utilize this topic. It is found with graphs and functional notation, in the abstract form accompanied by charts, and in the midst of verbal application problems. While only directly tested once each year, it is required at least on a procedural level within multiple other problems. In execution, errors are frequent when the bounds of integration include variables rather than constants (Radmehr & Drake, 2016). On a conceptual level, students have difficulty explaining what the fundamental theorem means verbally or in terms of a graph (Radmehr & Drake, 2016).

There are similarities in 3.2C and 3.3B; the two topics both cover finding the integral and the properties of integration. 3.2C views these from the relationship of the integral to area under a curve where 3.3B views these in terms of functional notation. Hence, any attempt at testing 3.3B that requires the function to be thought of or translated into a graph would then be classified at 3.2C. Very rarely, (only once in 10 years) does the free response section directly test these computational skills.

Notably, some topics and subtopics have been removed when translating the AP Essential Knowledge into topics for the content matrix. The entire section of 3.1A focuses on a definition of anti-derivative by way of working backward from differentiation and notes that the same rules of differentiation apply (but in reverse). While this is an important concept, it is covered thoroughly in both the properties of integration (3.3B) and in the fundamental theorem of calculus (3.3A). Further, several subtopics deal with the translation of knowledge (3.2A3, 3.2B1,3.3A), and were also omitted.

*Previous Test*

**Table 4.7:** Content Matrix of GHS Section 4 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 3.2A Definition | | | | |
| • 3.2A1 Riemann Sum notation | | | | |
| • 3.2A2 Limit of Riemann Sum | | | | |
| 3.2B Approximations | | 7, 8 | | |
| 3.2C Properties from graph | | | | |
| • 3.2C1 Evaluate using Area | | B | | |
| • 3.2C2 Add/subtract Integrals | | | | |
| • 3.2C3 Discontinuities | | | | |
| 3.3A Integral and Functions | | | | |
| • 3.3A1 Integral as function | | | | |
| • 3.3A2 Fundamental theorem | | | | |
| 3.3B Computing of integrals | | | | |
| • 3.3B1 Vocabulary | 1 | | | |
| • 3.3B2 Definite | 1, 2, 5, 9, 12 | | | |
| • 3.3B3 Indefinite | 3, 4, 6, 13 | | | |
| • 3.3B5 Algebraic manipulations | 3, 4, 5, 6, 13 | | | |

The content matrix of Green High School's assessment for section four reveals a primarily routine expectation of students' learning. Eleven of the 13 questions asked students to compute a definite or indefinite integral, though always using the same notation. Markedly, even with the procedural based problems, hints are given to aid computations, suggesting students consider "C," use a u-substitution and giving the structure of the anti-derivative.

Two questions ventured into the conceptual sphere of cognitive demand by adding charts from which data must be pulled strategically for Riemann estimations. The bonus question, denoted by a B in table 4.7, required students to compute an integral from a graph that involved negative area. This questions was optional and frequently missed. Apart from the bonus, the graphical interpretation of integrals was missing from the test, even with regard to Riemann sums. Application and non-routine problems were also missing from the assessment.

While the computation of integrals was heavily covered and approximations also made a dual appearance, the majority of topics were not included.

### *Assessment Analysis*

#### *Topic 3.2A: Definition*

The final unit introduces the idea of anti-derivatives. This concept is explored through the use of Riemann sums, which are new to students. This first topic specifically assesses students ability to recognize and use summation notation and the limit definition of integrals. Question one relies on a generalized non-particular function and asks students to rank the results (which cannot actually be computed) of several types of approximations. The right, left and midpoint Riemann sum are all represented in summation notation. Students need to identify the type of approximation the summation references and also determine if the estimation is an over-estimation or under-estimation of the functions area. This requires students to picture, or create, a graphical representation of the situation. Further, the comparative magnitude of error in the estimation must be considered. Finally, these estimations must be considered with respect to the actual integral. This skill has proved difficult on past AP tests (Davis, 2016). Since the problem does not involve any computations but rather asks students to consider the theoretical ranking of areas, it is both abstract and novel, falling squarely within the range of non-routine.

The sixth question gives an exact Riemann sum in standard notation and asks students to convert it into the more familiar integral notation. While seemingly straightforward, this question addresses the two areas of difficulty regarding the topic. First, students must understand how the bounds of integration relate to the product within the Riemann sum (Orton, 1983). Further, students must relate the limit to the summation in order to arrive at the appropriate expression (Orton, 1983). After setting up the integral, students must apply a right Riemann sum using the function. This requires the appropriate partioning of the base and the calculation of the function's height without the aid of a chart.

*Topic 3.2B: Approximation*

Traditionally, approximations on the AP free response sections provide a chart that students are asked to interpret. In this vein, question 12 provides a real world scenario and an accompanying chart. Instead of labeling $t$ in hours past, the chart gives certain times of the day in non-uniform increments. Students must make sense of how the situation relates to the Riemann sum, particularly how the base should be partitioned to set up the appropriate product. This is a typical area of weakness, thus the layers of context and uneven timetable make the set-up particularly challenging (Sealey, 2014; Orton, 1983). Similar problems on past AP exams reinforce the idea that students struggle in setting up and executing Riemann sums (Boardman, 2010; Kokoska,2013).

These difficulties are likewise present in questions six, which was discussed under 3.2A. The first question also requires a generalized understanding of each type of Riemann sum and the resulting approximation (see 3.2A).

*Topic 3.2C: Properties of Integrals from Graphs*

Topic 3.2C, graphical properties of integration, has two areas of concern for students: a confused conception of negative area (Kouropatov & Dreyfus, 2013), and comparably, the computation integrals that involve that absolute value of a function (Serhan, 2015; Radmehr & Drake, 2016). Question two references the first of these by providing a piece-wise continuous graph of $g(x)$ that has both positive and negative area. Further, a function $f(x)$ is defined to the the integral from zero to x of $g(x)$. Students are then asked to find $f(6)$, $f'(4)$, and $f''(3)$. With the first of these, students must find the area under the curve from zero to six, which has regions with negative area. This question captures the ethos of multiple questions on the AP free response section where common errors include not associating the integral with area, misunderstanding the bounds and misunderstanding negative area (Boardman, 2011; Kokoska, 2012; Kokoska, 2014).

Question five directly concerns the second major student misconception by asking for the integral of the absolute value of $x + 1$. The function is markedly simple, since the analysis is more complex. A sketch of the graph or at least the mental image is neccessary for proper evaluation. The absolute value is not differentiable at all points and so this function must be broken into two-pieces around the non-differentiable point. To break this up students must apply properties of integration. Understanding the graph helps dramatically in setting up the two new integrals. Since the absolute value translates the function over the x-axis to make everything positive, the areas where the graph would have been negative must be adjusted for by negating the function. Each integral can then be computed and the results combined arithmetically. A similar question was used in Eisenberg and

Dreyfus'(1991) study to test students' visual understanding of calculus. Students largely failed to take the graph into consideration and therefore were unable to compute the integral correctly (Eisenberg,& Dreyfus, 1991).

Approaching this topic in a non-traditional manner, question three asks students to compute a bounded integral of an abstract function. Rather than provide a graphical, notational or verbal depiction of the function, only the upper and lower bounds for the function are given on the appropriate interval. Students are given a range of potential values for an unknown function and asked the greatest possible value of the integral. By relating the integral to the area of the function, students should recognize that the boundaries provide a base of two. Further, in order to maximize the area, students should choose the largest value of the function to use in the computation.

Question 10 is modeled after Kouropatov and Dreyfus's (2013) study concerning high school calculus' students ability to solve non-routine questions about integration. Even with the use of their textbooks and calculators, only 42 percent of students answered the question correctly (Kouropatov &Dreyfus, 2013). While the computations are elementary, the visual representations with abstract variables demand that students have a strong connection between the integral as area under the curve and how it relates to the calculations of integration. The problem is made even more complex with the addition of the jump discontinuity.

*Topic 3.3A: Integrals and Functions*

The second part of the fundamental theorem of calculus is assimilated into multiple areas of the AP test. The section four assessment addresses it on all spectrums of cognitive demand, using a variety of formats.

Question seven envelops several relational aspects of 3.3A. Garner (2008) designed this question to relate integration to multiple previous topics (Garner, personal communication, April 30, 2018). Parts A and B deal with continuity and differentiability as they relate to integrals. However, parts D, E, F, and G all assimilate the topic of the fundamental theorem of calculus with the topic of the derivative's relationship to a graph. This important connection often appears on the free response section of the AP test. Even though this is a popular combination of ideas, students still have trouble assimilating the topics (Radmehr & Drake, 2016). Part D of this questions resembles part D of the third question on the 2012 AP test, which was heavily missed as students had trouble using the fundamental theorem of calculus in combination with an additional derivative (Kokoska, 2012).

Denoting the fundamental theorem of calculus in a non-traditional notation, question 11 presents a trigonometric function that students can not integrate computationally. There are three further twists to this routine problem. First, the lower bound is not zero, but $\pi$, requiring evaluation. Second, the upper bound is a variable, which often presents a hurdle to students (Radmehr & Drake, 2016). Finally, the upper bound is a function, requiring the chain rule in evaluation. Every aspect of the fundamental theorem of calculus is altered to be more rigorous, exposing any potential misconceptions regarding the formula/procedure.

Wrapping up this topic, questions two and nine, treated under 3.2C and 3.3B respectively, assess the graphical and verbal implications of the fundamental theorem of calculus. Notably, the verbal interpretation has proved particularly difficult for students on past AP examinations (Kokoska, 2012).

*Topic 3.3B: Computing Integrals*

Unlike 3.2C, the computation of integrals in 3.3B does not rely on a visual representation but rather focuses on the algebraic procedures. The fourth question provides several generalized integration equations with various bounds. Though the abstract integration equations only cover $f(x)$ and $g(x)$, students are asked to find the integral of $f(x+1) - g(x)$. This should be broken into two integrals before $f(x+1)$ can be manipulated with a u-substitution to mirror $f(x)$. Since the function is not given, the substitution must be carried over into the bounds of integration. Since no computations are actually performed and all equations remain in an abstract form, this question is classified as non-routine.

Alternatively, question eight is purely procedural. Students might be inclined to use a u-substitution or break the numerator into two separate fractions over $2x - 1$, both techniques lead to dead ends. Instead, the function needs to be simplified by factoring the numerator, which is the difference of two squares. After canceling the denominator the anti-derivative is straightforward, however students must integrate over a region with a removable discontinuity.

Adding meaning to the integral, question nine provides a verbal backdrop of a realistic event. The function $c(t)$ is defined verbally as a rate of change and also given numerically. Students are told to find the anti-derivative and must use a u-substitution to evaluate the indefinite integral. Question four on the 2012 AP tests requires a similar integral computation. Since the function maps the number of cars washed per hour, the anti-derivative of the derivative (the fundamental theorem of calculus) gives back the number of washed cars. Students should comment on the meaning of C in the indefinite integral as the initial amount of washed cars.

**Table 4.8:** Content Matrix of Section 4 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 3.2A Definition | | | | |
| • 3.2A1 Reimann Sum notation | 6 | 1 | | |
| • 3.2A2 Limit of Reimann Sum | 6 | | | 1 |
| 3.2B Approximations | 6 | 12 | 12 | 1 |
| 3.2C Properties from graph | | | | |
| • 3.2C1 Evaluate using Area | | 2, 10 | | 3 |
| • 3.2C2 Add/subtract Integrals | 5 | 5, 10 | | 10 |
| • 3.2C3 Discontinuities | 5, 8 | 10 | | 10 |
| 3.3A Integral and Functions | | | | |
| • 3.3A1 Integral can be a function | | 10 | | 7,10 |
| • 3.3A2 Fundamental theorem | 11 | 2 | 9 | 7, 11 |
| 3.3B Computing of integrals | | | | |
| • 3.3B1 Vocabulary | 2, 9 | | | |
| • 3.3B2 Definite | 8 | | | 4 |
| • 3.3B3 Indefinite | 9 | | 9 | |
| • 3.3B5 Algebraic manipulations | 8 | | | 4 |

*Summary*

Table 5.8 displays the variety with which each topic was addressed. Two of the topics reached every spectrum of the cognitive demand and the remaining three topics were covered through 3 of the 4 areas of cognitive demand. External representations included continuous and discontinous graphs as well as a chart. Further, a graphical/visual understanding was sometimes necessary but not provided, requiring students to sketch the image. Two questions unfolded a real world scenario from which the equation must be derived and then re-contextualized. Since the topics were largely procedural in nature, abstracting was utilized multiple times to identify how well students understand the procedures. Strictly routine problems, which only account for 18 percent of the questions, incorporated layers of complexity through the notation, requisite functional knowledge, and algebraic rigor.

In line with the College Board's goals, there is a variety of notation and vocabulary throughout the assessment. Areas where students typically falter were dispersed amidst the questions to safeguard against common misconceptions. Finally, past topics were brought in and related to current ones. This keeps students from compartmentalizing topics and promotes a fluency in content in line with the AP exam's format.

## Unit 3 Section 5: Logarithmic, Exponential and Transcendental Functions

### *Content*

This chapter does not introduce any new topics, rather it hones in on a procedural aspect of previous topics. Logarithmic, exponential and inverse trigonometric functions are particularly difficult to differentiate and/or integrate. Performing these operations is listed on the content matrix under the topics of 2.1C2 and 3.3B. Since computing logarithmic, exponential and inverse trigonometric functions are a purely mechanical aspect of the topics, applications problem would feature this topic as an aside or interpretation of the results. To present it in graphical form would not necessarily aid in the solving of integrals of derivatives directly. Providing a chart would also not accurately represent the procedure required. Hence this subset of the topic(s) are difficult to apply in a meaningful way to other forms. The nature of this aspect of the topic is so narrow that it often stays within the routine division on the matrix in order to appropriately address it.

The free response section of the AP test frequently includes logarithmic, exponential and transcendental functions for which the derivative or integral must be computed. The integral of complex trigonometric, exponential or some combination thereof is generally set in the context of volume and students are permitted a calculator to solve. Without the use of calculators the integrals consist of basic trigonometric functions and exponential functions of base $e$. These integrals are used within motion problems or within separable differential equations. The derivatives involving logarithmic, exponential and transcendental functions are sometimes found in the midst of piece-wise continuous functions and computed in partial fulfillment of min/max, inflection problems, tangent line approximation, or some other application. Common errors are found in associating the integral with the correct technique for solving (Green & Ricca, 2015)

Rather than create a test of questions that are routine in nature, concepts from previous chapter will be incorporated that require computations in line with the theme of the section. This mirrors the manner in which the AP free response section incorporates the themes of this section. Since the curriculum has not yet introduced applications of integrals, some routine integration problems will be incorporated into the test. In order to better view the assessments emphasis on the cognitive matrix, review topics will be included for this section. Whenever these concepts are required in a rote or procedural manner, they will be left off the content matrix. However, when the topics from previous section are being tested in a conceptual manner they will be added to the bottom of the matrix, differentiated from the focus of the test.

*Previous Test*

**Table 4.9:** Content Matrix of GHS Section 5 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 2.1C Rules for calculating derivatives | | | | |
| • 2.1C2 Solving directly | 1, 2, 3, 4, 5, 6, 12, 13, 14 | | | |
| 3.3B Computing of integrals | | | | |
| • 3.3B2 Definite | | | | |
| • 3.3B3 Indefinite | 7, 8, 9, 10, 11 | | | |

On the previous assessment for section five, every question asked students to perform a procedure involving the derivative or integral (see table 4.9). The first 12 questions were straightforward derivatives and integrals found in standard form aimed to hone in on students' computational ability to deal with transcendental functions. However, formulas for the integrals of inverse trigonometric functions are provided (and by implication the formulas for derivatives of inverse trigonometric functions). Meaning that a third of the content being tested was given in a form that minimized even the procedural knowledge involved.

Moreover, a calculator was allowed on the entire test. Every derivative could be computed through the calculator, making even the computational skills for the first six questions irrelevent. Further, there are down-loadable (and AP-approved) calculator programs that can perform indefinite integrals. While a survey suggested that students did not have additional programs, it is still worth noting that an AP-approved calculator could render this test nothing more than an exercise in plugging in numbers to a calculator. This was not the intent of Ms. Brown, who left definite integrals off the test to prevent students from being calculator reliant (Ms. Brown, personal communication, June 16, 2017). While definite integrals can be evaluated by the calculator, evaluating them by hand requires the extra step of plugging in values to transcendental functions. This step can be tricky for students as it requires knowledge of the functions domain and range. Further, when the AP test allows calculators on questions, it is the definite integral that students generally need to compute.

The twelfth question requires implicit differentiation which is a change from the first six questions. The last two questions sought to have some applicability by requiring students to find the equation a tangent line, however, these were asked in a routine manner.

Overall, the test stayed within the routine division on the content matrix, had little variety in how questions were approached and was consistent in the notation. The use of the calculator and the inclusion of formulas eased the difficulty of the tasks being assessed. The definite integral, 1 of the 3 subtopics, was purposefully omitted from the test, leaving the assessment's coverage inadequate.

*Assessment Analysis*

*Topic 2.1C2: Solving Derivatives Directly*

The section five assessment does not allow calculators for the majority of questions involving the derivative. Students' ability to calculate these is the main thrust of this topic and the AP test rarely includes derivatives of logarithmic, exponential, trigonometric functions on the calculator portion of the free response section. However, question nine allows a calculator and has a second derivative involved. The topic received ample coverage on the assessments as it is assessed on questions 1, 2, 6, 8 and 9.

Set within a real-world application, the first question denotes the logarithmic function $v(t)$ explicitly. The verbal instructions are "find the acceleration," from which the students should infer that a derivative is needed. Arriving at a function for acceleration that is discontinuous when $t$ or $lnt$ equals zero, students must recognize that the domain of the acceleration function corresponds to the domain of the velocity function, which is defined only on the open interval (1,4). Since the acceleration is continuous on this interval, the only interval it is defined for, the function is continuous. More than a simple yes or no, students must express their understanding of the continuity of the resulting logarithmic function in verbal form.

The second question is also placed within the familiar context of motion. However, rather than relating position with velocity and acceleration, as is typical, the position is juxtaposed with the gas mileage. With a focus on the manipulation of related rates, this problem demands that students understand the meaning of each function in context and in notational form. Students must find how many gallons per hour the car uses at a certain time. This should be translated into the functional form $\frac{dg}{dt}$. Relating this to $\frac{dg}{dx}$, reveals a need to find $\frac{dx}{dt}$, which is the derivative of the position function. Taking this derivative requires the computation of a base five exponential function that includes a function within the exponent. The resulting function must be multiplied by a logarithmic function and evaluated at a point. Though there is no prompt directly within the question, students should justify each step as is specified on the directions for each in-class assessment.

Approaching in a non-routine manner, question six relies on novelty in incorporating two familiar concepts. Relating derivatives to tangent lines, test takers are asked where the tangent line of two logarithmic functions are parallel. This is novel because tangent lines are traditionally evaluated at a point or set equal to a slope, while here the slope of the tangent lines are set equal to the slope of another tangent line. The solution involves the differentiation of two logarithmic functions, one involving the quotient rule and the other requiring a change in form, which students find difficult even when not found inside a logarithm (Ferrini-Mundy & Graham, 1991). Students must then set these derivatives equal and solve for $x$. While they will have two possible solutions, only one falls within the domain of the functions so the other must be eliminated. This question approaches the slope of tangent lines in a nontraditional manner. It also forces students to consider the domain of the functions they are manipulating rather than just relying on the algebraic solution.

Modelled after the fifth question on the 2012 AP free response section, the assessment's eighth question involves the graphical interpretation of the second derivative of an exponential function. Students must first compute the derivative on a non-base $e$ exponential function which has been multiplied by a simple linear function. Then students must compare the algebraic results of the second derivative with a graph of the original function (for which the notation is not given nor solvable). Within the verbal comparison of the two function, conclusions should be drawn concerning the inflection points of the graph.

Question nine involves the computation of the derivative of a piece-wise discontinuous, trigonometric function. While this portion is routine, the initial conditions for the function are altered and students are asked how this affects the derivative. A clear understanding of how integrals and derivatives relate to graphs informs the result that initial conditions do not effect the slope of a function.

*Topic 3.3B: Computing Integrals*

At this point in the school year, students have practiced derivatives for months while integration is still a new technique. Questions 3, 4, 5, 7 and 9 all address integration involving special functions. Question nine presents a more complex integration of these transcendental functions but students are afforded a calculator. The AP free response section allows calculators for the first two questions and integrating trigonometric, logarithmic or exponential functions is present within these nearly every year.

On the non-calculator portion, question three gives students the acceleration and requests the velocity. While students have not formally learned the relationship with integration and motion, they are aware that the acceleration is the derivative of velocity. Students also understand the relationship between derivatives and anti-derivatives. Combining these ideas, students should integrate the inverse trigonometric function. Since there are no bounds of integration the constant C should be present in the solution. Further, students should comment on the domain of the velocity function.

Using a variable within the bounds of integration, question four places an unusual twist on integration. Students must compute an inverse trigonometric function and evaluate it to solve the missing boundary condition. Relying on novelty, this question requires an acute knowledge of variables, as the traditional variable $x$ is evaluated not at a constant, as is usual but at the variable $a$. A rich knowledge of trigonometric inverse functions and their relationship to the unit circle is also essential in arriving at a correct solution.

The non-calculator portion of this topic is wrapped up with two routine problems, five and seven. Question five is purposefully similar in form to a trigonometric inverse function, which students are clued into given the section. However, this question is actually only solvable through a u-substitution. Though strictly routine, difficulty is added through the manipulation of logarithmic functions. Question seven provides a different notation, asking for students to evaluate an indefinite integral with an initial condition. This was inspired by the 2012 AP exam where students needed to find the integral of $sin(\frac{\pi}{6}x)$ and struggled with the computation (Kokoska, 2012).

The final question allows the use of a calculator. Indeed, a portion of the piece-wise function can not be integrated without a calculator. Inspired by the sixth question of the 2011 AP exam's free response section, the second derivative and the integral must be evaluated. With similar questions on past AP exams, students have had issues with the bounds of integration and remembering to incorporate the initial condition (Kokoska,2015). The areas where students struggle are notably not eased by the use of a calculator. However, the incorporation of the calculator tests students ability to evaluate complex functions in a manner consistent with AP standards. In comparing two identical functions with different initial conditions, the question requires an awareness of how the constant of integration affects the function.

**Table 4.10:** Content Matrix of Section 5 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 2.1C Rules for calculating derivatives | | | | |
| • 2.1C2 Solving directly | 1, 2, 6, 8, 9 | 8, 9 | 1, 2, 9 | 6 |
| 3.3B Computing of integrals | | | | |
| • 3.3B2 Definite | 5, 9 | | 9 | 4 |
| • 3.3B3 Indefinite | 3, 7 | | 3 | |
| Review topics | | | | |
| • 2.2A Derivatives and graphs | | 8, 9 | | |
| • 2.3C1 Motion | | | 1, 3 | |
| • 2.3C2 Related Rates | | | 2 | |

*Summary*

The section five assessment tests each of the topics on at least three categories of cognitive demand (see table 4.10). While the content demands that each problem have a routine aspect, only 2 of the 9 questions remain purely routine. Notably, these two questions are modelled after previous AP free response exams problems where students had difficulties. Only one question related the material to an external representation directly, though an understanding of the graph is necessary for an additional question. There was a breadth of variety in the application section, where students related derivatives and integrals to motion, gas mileage and tangent lines. Explication in verbal form is also required for multiple questions. Non-routine problems were incorporated through novel applications and abstract boundaries.

The assessment incorporated past topics concerning the concavity, relationship to motion, related rates, tangent lines and continuity of functions. There is variety in the types of functions with the incorporation of logarithms, exponents of different bases, trigonometric, trigonometric inverses, polynomials, rational, piece-wise and combinations of the aforementioned functions. The themes of prior AP exams are present and typical student errors incorporated.

## Unit 3 Section 6: Differential Equations

### *Content*

Unit three section six presents the following topics related to differential equations: 2.3E verifying solutions of differential equations, 2.3F slope fields, 3.5A separable differential equations and 3.5B exponential growth and decay functions.

Slope fields were added to the AP curriculum in 2004 and since then have been featured as a free response question roughly 40 percent of the time, with students either being given a graph to interpret or drawing in the slopes on a blank graph. One study of students in a differential equations course found that students had trouble mapping solution curves from the slope field (Habre, 2000). On a conceptual level, not all students recognized that slope fields could impart specific information about particular solutions (Habre, 2000; Hughes-Hallett, 1991).

Topic 3.5A features the solving of separable differential equations both general and particular, with initial conditions. These questions are featured on roughly 70 percent of free response sections, involving relatively straightforward functions to be solved without a calculator. Students have only learned one method for solving differential equations, namely, separating the variables. However, students still struggle in performing the procedure (Boardamn, 2010; Kokoska, 2012). Even when students have mastered the analytic exercises, many have trouble communicating what the solutions mean (Habre, 2000). There was also a lack of notational fluency related to the understanding of variables with these problems (Raychaudhuri, 2008).

Exponential growth and decay functions (3.5B) are a type of separable differential equation. The difference in application is that these problems can describe a situation and allow the test-taker to set up the appropriate differential equation rather than providing the equation in notational form. These have not been featured on the free response section in the last ten years. Consistent with studies on applications, students have difficulty in recognizing when the data is exponential in form (Ellis, Ozgur, Kulow, Dogan & Amidon, 2016).

*Previous Test*

**Table 4.11:** Content Matrix of GHS Section 6 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 2.2E Solution of DEs | | | | |
| 2.2F Slope Field | | 2, 6, 10 | | |
| 3.5A Separable Differential Equations | | | | |
| • 3.5A1 Initial values | 10 | | 7, 9 | |
| • 3.5A2 Separation of variables | 3, 4, 5, 9, 10 | | | |
| • 3.5A3 Domain restrictions | | | | |
| • 3.5A4 General vs particular soln | 1, 3, 4, 5, 9 | | | |
| 3.5B Exponential Growth and Decay | 7, 8 | | | |

The section six assessment at Green High School is similar to the free response section of the AP exams in the emphasis of topics. Questions seven through 10 were explicitly intended to mimic the College Board's assessments, as Ms. Brown felt her students were ready for the rigor (Brown, personal communication, June 16, 2017). Questions seven and nine require students to interpret the initial condition from the context. However, the heavy verbal cues, "there was initially 10,000 cubic feet," and "initially there were 5 gallons," make this transfer straightforward. Question ten is similar to the sixth question on the free response section of the 2005 exam. However, the function utilized in Ms. Brown's assessment ($\frac{dy}{dx} = \frac{xy}{2}$ is more simple than the functions employed on the AP section: $\frac{dy}{dx} = \frac{y-1}{x^2}$, $\frac{dy}{dx} = (3 - y)cosx$, $\frac{dy}{dx} = \frac{y}{(3-y)x}$, $\frac{dy}{dx} = -\frac{2x}{y}$. Students at Green High School were also afforded the opportunity to use their calculator to solve questions seven through ten, where every differential equation is on the non-calculator portion on the AP exams.

Even though the test resembles the general focus of the AP free response section, there are topics missing on this formal assessment that the College Board believes necessary for students understanding of calculus. Questions covering 2.2E and 3.5A could be part of the multiple choice section and have appeared (though infrequently) on the free response section. Not only are two subtopics left off the test, but the goal of assessing every main topic on two levels of cognitive demand was not met.

*Assessment Analysis*

*Topic 2.2E: Solution of Differential Equations*

Recognizing and justifying that the solution of a differential equation is itself a function, requires both an understanding of the differential equation and a fluency with functions from an algebraic standpoint. The first question addresses this in a routine manner. A non-separable differential equation is used so that students can not solve the equation to verify and instead must substitute the appropriate forms of the trigonometric solution into the equation. This involves taking the second derivative of trigonometric function with an implicit chain rule. The justification should include a statement acknowledging that the particular solution satisfies the equation for all values of x.

The eighth question also requires students to verify a particular solution to a non-separable differential equation. This question reflects a portion of the fourth question on the 2015 AP Calculus AB free response section by giving one of the two equations in abstract form and asking students to solve for the variables. Students did not do well on this portion of the AP question and many fixated on solving the non-separable differential equation (Kokoska, 2015). The addition of the generalized formula marks this as a non-routine problem.

*Topic 2.2F: Slope Fields*

Slope fields are external representations of differential equations. As such, questions concerning slope fields naturally fall into this category of cognitive demand. This assessment approaches slope fields in three different manners on questions two, four and six.

Question two gives a portion of a slope field where students should sketch a slope field for a particular solution. The sixth question of the 2014 AP test includes a similar set up where the common error was not using the initial condition for the sketch (Kokoska, 2014). In this question, the solution curve will be used to approximate the value of the function when $x$ is two. While there is no prompt for explication, the directions stipulate that all answers must be justified. Hence, a verbal statement explaining the estimation should accompany the result.

Set within the backdrop of a population growth problem, students must identify which slope field could model the context in question four. There is no function or differential equation given in notational form (though students should have solved for this in question three), so the relative shape of the solution curves should be used for correct identification.

The final slope field question (number six), does not provide a slope field nor does it ask students to graph one. Rather, it provides the differential equation and asks questions concerning the general shape of the slope field. Similar to the 2004 AP test, this question requires students to sketch or mentally envision the field and find areas where the slopes are positive. Here students must provide verbal reasoning throughout the process of finding the domain and locating where the slope field has positive slopes.

*Topic 3.5A: Separable Differental Equations*

Students at this level only know how to solve one type of differential equation, namely separable differential equations. Still, test-takers on the AP exam often do not know how to separate the variables or make computational errors when required to solve (Boardamn, 2010; Kokoska, 2012). Questions 3, 5, 6 and 7 all relate to this topic, with all but question six requiring students to solve the differential equation. The sixth question, discussed under 2.2F, relates to this topic as it requires explication on the domain of a differential equation.

Question three does not give a differential equation but rather a real world problem that students can use to formulate a differential equation. Similar to a free response problem on the 1993 exam, very limited functional notation is used in describing the situation. This means that students have to analyze the sea lion proportions, connecting the term "rate" with a derivative to arrive at an appropriate differential equation. Then students then can perform a separation of variables and integrate both sides. After simplifying, the initial condition must be applied and the appropriate variables solved for. It is important when showing the steps that lead to the result, that students use the absolute value within the natural log function. Leaving out the absolute value, even in an intermediery step, is a common area where students lose points on the AP exam (Boardman, 2008).

Another contextual problem, number five, models a real world concentration application. Since these are not as accessible as exponential growth and decay, the differential equation is given using functional notation within the problem. The equation itself is similar to the equations used on the 2011 and 2012 exams, where students had issues separating the variables (Boardman, 2011; Kokoska, 2012). The prompt, "find the concentration of chlorine after an hour," tells students to solve the differential equation particular to the initial condition and evaluate it at a time $t = 60$, since $t$ is in minutes. The typical variables of $x$ and $y$ were replaces with $C$ and $t$, requiring an intuitive knowledge of variables. The result must be in correct units, indicating an understanding of what the result means in context.

Question seven presents an equation similar to that of the 2013 exam in that the $y$ variable is found within an exponential. On the exam students were, for the most part, able to separate the variables but had trouble computing the anti-derivative of $\frac{1}{e^y}$ (Kokoska, 2013). Question seven presents the differential using the $y'$ notation, rather than the more standard $\frac{dy}{dx}$, which confused students in a study (Raychaudhuri, 2008). This is the only question which asks for a generalized rather than a particular solution and as such students should incorporate the constant of integration.

*Topic 3.5B: Exponential Growth and Decay*

Setting up and solving differential equations which result in the common exponential equation related to $y = Ce^{rt}$ is a subset of separable differential equations. Question three, discussed under 3.5a, gives students a proportional word problem and has them set up an equation. Students must incorporate $k$, the constant of proportionality, before separation of variables. The result is the familiar exponential equation. Plugging in the initial value, students can solve for the constant C, but should leave $k$ in the final equation. While students need not immediately recognize this as an exponential problem, they must include the appropriate proportions to arrive at the correct form of the equation. The corresponding slope field question, number four, requires an intuitive understanding of the graphical representation of exponential growth functions.

**Table 4.12:** Content Matrix of Section 6 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 2.2E Solution of DEs | 1 | | 1 | 8 |
| 2.2F Slope Field | 6 | 2, 4, 6 | 6 | |
| 3.5A Separable Differential Equations | | | | |
| • 3.5A1 Initial values | 3 | | 3, 5 | |
| • 3.5A2 Separation of variables | 3, 7 | | 5 | |
| • 3.5A3 Domain restrictions | 6 | 6 | | |
| • 3.5A4 General vs particular soln | 3, 7 | 2, 4 | 3, 5 | |
| 3.5B Exponential Growth and Decay | | 4 | 3 | |

*Summary*

Section six covers topics with a narrow set of applications, as students only understand a small portions of differential equations. However, the assessment incorporates variety in how the topics are tested. Each of the four topics, and corresponding subtopics are represented on at least two areas of cognitive demand (see table 4.12). Only 1 of the 7 questions had no conceptual components. External representations are graphical in nature, with students manipulating a graph, judging the appropriateness of a graph and picturing a graph in order to explain areas with positive slope. Questions are given in verbal or functional form, with a variety in the notations used. The differential equations include standard set-ups that solve to exponential functions, functions that evaluate to logarithmic functions on both sides of the equation and functions that involve exponentials before integration. Further, students manipulated solutions within non-separable differential equations. A non-routine question was incorporated with the use of an abstract differential equation.

The questions were intentional in incorporating vocabulary, equations and calculator use in line with the expectations of the College Board as demonstrated through prior AP free response sections. Additionally, frequent student errors were addressed through the prompts, inclusion of non-separable differential equations, and algebraic rigor of separable differential equations.

## Unit 3 Section 7:Applications of Integration

### *Content*

The final section covers the remaining topics, namely 3.4A interpreting the integral as the net change on an interval, 3.4B viewing the integral relationship as an average value of a function, 3.4C the integral's relationship to motion, and 3.4D finding the area or volume of regions using integration.

Translating the integral as the net rate of a change (3.4A) and the average value (3.4B) requires interpretation of verbal statements into functional notation. These contextual problems are tested an average of just over two times each year on the free response section of the AP test. When the average is utilized on AP tests, the questions often provides the formula and ask students interpret the results. The average is used most often in conjunction with tabular information and requires a particular Riemann sum for computations. Similar to word problems involving derivatives, the degree of students success is related to their ability to decifer what the variables in the problem mean in relation to the formula (Klymchuk et al., 2010).

As mentioned in section three, kinematic settings are popular on AP exams, appearing on 13 of the last 15 exams. Twelve of the 13 appearances require the explicit knowledge of how integration relates to motion. The integral's relationship to motion (3.4C) is not solely relegated to working backward from the derivatives relationship to motion. Rather, there is a new distinction when integrating the velocity, namely the difference between distance and displacement. This distinction appears on about 70 percent of the free response sections with students being asked to find position, find the total distance, or explain the meaning of the definite integral of velocity, or the definite integral of the absolute value of velocity. Relatedly students might be asked, when something changes direction or to find the speed given the velocity. In finding the total distance from a velocity function, students must deal with the difficult task of computing integrals of absolute values (Radmehr  Drake, 2016).

Related to the Riemann sum, 3.4D demonstrates how integrals can be used to find the area of regions and volume of solids. Questions of this nature appear every year on the AP test without much variation in the vocabulary or cognitive demand (external representations). Within the last 15 years, 14 tests asked students to set up or find the area between two graphs. Twelve of the 15 asked students to find the volume of a region given a cross sectional area. Eleven had students setting up or finding the volume of a solid of revolution using the disk method. Finally, students were asked once for the volume of a solid of revolution with a solid center, once for the rate at which the differences in height of two functions changed, and twice for the set-up involved when a line divides a region in half. With all of these tasks, students are occasionally asked to simply set-up, but not evaluate the integral. This is most likely because the set-up requires a precise graphical interpretation of how the solid is generated. In line with this, the majority of student errors are not computational but rather in setting up the appropriate formula (Bremingan, 2005).

*Previous Test*

**Table 4.13:** Content Matrix of GHS Section 7 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 3.4A Integral as the net change | | | | |
| 3.4B Integrals as average value | | | | |
| 3.4C Integral relationship to motion | | | | |
| 3.4D Area/Volume of regions | | | | |
| • 3.4D1 Area of region | | 3, 4, 6, 7, 8 | | |
| • 3.4D2 Volume | | 1, 2, 5, 8 | | |

While the former test focused on conceptual knowledge, specifically the area and volume of regions using graphical representations, it lacked variety in both cognitive demand and topics. Students were never formally exposed to a verbal representation for integration. This means that students never had to apply their knowledge of integrals to any non-graphical contextual situation. Unfortunately, this skill is heavily underscored by the College Board. More than one fifth of the total points for the free response section have been dedicated to topics 3.4A,B and C over the last ten years.

## *Assessment Analysis*

### *Topic 3.4A: Integral as Net Change*

The first application of integration requires the interpretation of the results as the net change. Questions five and six address this within the backdrop of a bakery. Rather than just one function of a rate of change to be visualized in the context, question five juxtaposes two rates of change corresponding to the cookies at the bakery during the same time interval. The net amount of cookies can be thought of as the net cookies made minus the net cookies sold at a certain time. The net as a difference of two rates appears on the 2010 and the 2016 AP exam. One of these net changes requires the integration of an exponential function while the other a piece-wise non-continuous function. In addition to computing these, students must explain their reasoning verbally.

Question six relies on the same context but adds a twist. Replacing the piece-wise function with a linear expression for the cookies sold, students must find the time the bakery should begin selling cookies. Approaching from a new angle students must first decide the net cookies of the day. Next students should create an integral to represent the cookies sold, adding a variable to the lower bound, and set it equal to the amount of cookies made. This must be evaluated and the variable solved for. The result must then be interpreted within the context, rounding up so that the cookies don't run out and then establishing this answer in terms of the time of day. Again, students must give a reason for the steps they take in solving. The bakery situation used for both questions gives the functional notion for the rate at which cookies are being produced and sold, but gives no further explications, meaning that students must used verbal cues to decide which type of equation to set up and the bounds for integration. Keeping the notation minimal and not explicating the calculus components, these questions require multiple levels of interpretation. This is statistically the most difficult portion for students (Klymchuk et al., 2010).

*Topic:3.4B Integral as Average Value*

Students previously learned how to compute the average rate of change for a function on an interval and now through integration can calculate the average value of the function on an interval. The distinction between the two is often difficult for students. In fact, students repeatedly fail to understand when to use the average value formula and what it means (Boardman, 2010; Boardman 2011; Kokoska,2013; Kokoska,2014). Questions four, eight and nine allow students to convey their understanding on the topic.

Question four mirrors portions of the 2005, 2009 and 2013 AP free response in providing the framework of the integral equation for the average value of the function and asking students to interpret their results. Question four specifies to use a right Riemann sum that requires students to partition the base. Students are asked to set-up but not solve, since most of the computational errors on previous AP tests involve the set-up (Diefenderfer, 2007; Boardman, 2009). From a conceptual standpoint, students must use the description of the chart and relate the information to the symbolic notation of an integral average in order to correctly interpret what they have solved for. Traditionally, students have struggled to interpret the meaning, so much so that many do not even use the word average (Diefenderfer, 2007; Kokoska, 2013).

Question eight also incorporates an external representation, this time in the form of a graph. Previously students were given the formula and needed to know it was the average, here the question gives no formulas but asks for the average speed. First, students must take the verbal cue, "average," and recognize the appropriate equation. Students have struggled with this in similar AP problems (Kokoska, 2014). Yet, the traditional equation generates the average velocity not the average speed so appropriate alterations to the equation must be made. Solving requires the computation of an integral of absolute value which is traditionally difficult for students as they struggle with the concept of negative area (Kouropatov & Dreyfus, 2013). Correct units should be used and appropriate verbal explication included.

Question nine rounds out the assessment of averages. Modelled after question three of the 2015 AP test, this question requests two separate averages, one involving the integral notation and the other the derivative. Students often confuse these as they struggle with relating the results of averages to the function (Boardman, 2011; Kokoska, 2015). The numbers on the chart were purposefully simple since the thrust is on the interpretation of both averages.

*Topic 3.4C Relationship of Integral to Motion*

As mentioned above, motion problems have an added component once integrals are introduced, namely the difference between distance and displacement. Not only must students understand this difference in terms of the formulas but in question seven, the difference must be understood in terms of the physical realities of Brian's trip to the beach. The word *distance* was purposefully used to describe the length between Brian's house and the beach so that students must fully grasp the nuance between the two quantities and not just look for a "key word" within the question. Even when students distinguish the difference between distance and displacement, solving for the respective quantities requires relating the difference in formulas with a graphical representation of the trip. The AP exam frequently requires students to distinguish between distance and displacement yet on every occasion many students struggle to do so (Boardman, 2009; Boardman, 2011; Kokoska, 2012; Davis, 2016). Relatedly, question eight requires students to differentiate between speed and velocity, interpreting this again from the graphical representation of the function.

*Topic 3.4D: Area and Volume of a Region*

The first three questions address students' understanding of the way area and volume can be found through integration. While there is only one method of generating an area between graphs, there are two methods for generating solids, solids of revolution and solids with cross sectional areas. The three questions on the assessment correlate to these three distinct techniques.

Question one describes a solid that is formed by a cross section of semi-circles. This shape was specifically chosen because the 2012 AP exam noted that students did not know the formula for finding the area of semi-circles (Kokoska, 2012). While not all students will need to sketch a graph of this problem, a spacial understanding of the situation is necessary in correctly identifying the changing radius of the base of the semi-circles and the bounds of integration. Since minute differences in the positioning of the region equate to modifications in the formula, students must have a visual awareness of the region as it relates to the integral equation.

Rich in algebraic manipulations, question two could be set up oriented horizontally (which is a more simplified way), or vertically (in two portions). Having decided on a direction, students must find the bounds of integration and recognize the appropriate variable to solve for. Further, the equations need to transformed to isolate the chosen variable for integration. Finally, the region must be defined in terms of the distance between the appropriate functions for every sub-region necessary. A successful set-up signifies a comprehension of the relationship between notational calculus, the concept of area and the graph. Students are not asked to solve, as the set-up is traditionally the more difficult task (Bremingan, 2005).

Building on question two, the volume of a solid formed by a region that is being revolved around a line requires the integral of the circle created through revolution, $\pi r^2$. This is made more difficult when the solid has empty space within it; students must recognize this and account for it within the radius. A purely procedural knowledge of how to interact with the formulas for solids of revolution is not sufficient since the format of the formula itself is enjoined with the visualization of the graph. Picturing the circumrotation and the solid generated enables a proper formula from which to solve. Similar questions appear on nearly every AP free response section.

**Table 4.14:** Content Matrix of Section 7 Test

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 3.4A Integral as the net change | | | 5, 6 | 6 |
| 3.4B Integrals as average value | 8 | 4, 12 | 4 | |
| 3.4C Integral relationship to motion | | 7, 8 | 7, 8 | |
| 3.4D Area/Volume of regions | | | | |
| • 3.4D1 Area of region | | 2 | | |
| • 3.4D2 Volume | 1, 3 | 1, 3 | | |

*Summary*

Comprised of only nine questions, the section seven assessment meets the goals laid out through the content matrix (see table 4.14). Each of the four topics fall under two or three divisions of cognitive demand. External representation include both graphs and charts. Questions also describe functions with the expectation of students creating a graph or diagram to visualize the problem. Every question has a verbal aspect where the language is purposeful both in what it gives and what it leaves out. Abstract elements were present, requiring students to make generalizations and think through the learned processes. Every question had a conceptual portion while students were still challenged computationally.

Since these topics are prevalant on the free response section of the AP test, special attention was placed on the language and notations within the word problems so as to reflect the common themes. Moreover, the provision of graphs, or lack thereof, and the detail within the external representations are consistent with prior exams. This is also represented in the functions. Piece-wise discontinuous functions occasionally appear within integration applications as well as functions with exponential components. The main difference in functions is that many problems related to the volume of solids of revolution on the AP free response section have logarithmic and exponential components which can not be solved by hand and require a calculator. Although this is not mirrored on the section seven exam, questions involving these particular functions were assessed with a calculator on the section five assessment. Further, volumes of solids are not always included within the calculator section and when they are not, the equations are markedly similar to the ones found in this assessment.

Questions were also designed with common student errors in mind. Students were asked to set up integral equations related to area and volume (Bremingan, 2005), distinguish distance from displacement and find the corresponding integral of an absolute value function (Bremingan, 2005), and distinguish the appropriate equations given limited verbal cues (Klymchuk et al., 2010).

## Synthesis

Green High School's assessments were not accurate predictors of students' knowledge of calculus as judged through the AP exam. An analysis of the exam revealed multiple topics, many of which were prominent on the free response section of the AP Calculus exam, were never included on formal assessments (see appendix H). In fact, on the 2016 AP exam, which students at Green High School failed, students had not been formally tested on portions of questions on the free response section that conservatively totalled 26 of the 54 available points. That is, 48 percent of the points on this section covered material that was new (or at least not emphasized) to students at Green High School. Additionally, the cognitive demand of the materials were not in line with the College Board's curriculum, which stressed the visual representation and application of calculus. As such, new assessments were designed that better aligned with the goals of the calculus program.

The new assessments addressed every topic and subtopic within the scope of the College Board curriculum. Each topic is addressed on multiple categories of cognitive demand, with an emphasis on external representations and applications. The questions reflect the vocabulary, variation, emphasis, and rigor of the free response section of past AP exams. Further, common student errors have been identified and incorporated in each section. Calculus professors and AP Calculus teachers have reviewed each assessment and found them in line with the character and rigor of the college level calculus sequence.

# CHAPTER 5: CONCLUSION

## Findings

Green High School seeks to implement an AP program that equips students with the knowledge necessary to succeed in college. The school has a goal of a 100 percent pass rate for the AP tests. Currently, 85 percent of the students at Green High School pass the AP exams in general, though in AP Calculus only 19 percent of students passed the exam. The calculus program has high test scores on in-class assessments that are disproportionate to the number of students who passed the AP test. Thus, the in-class tests were not a good indicator of students learning at the AP level. In order to help students succeed in AP Calculus, Green High School wants to restructure the in-class assessments so that they correspond with the intensity of the AP goals. To gauge the effectiveness of the assessments, it is expected that 100 percent of students who merit an 'A' should likewise receive college credit by way of passing the AP test. This is important as students who pass the AP test statistically perform better in the calculus sequence in college.

To help Green High School meets its goal, a needs assessment was conducted that focused on the difference between high school and college calculus, what it means to effectively assess calculus and where Green High School was falling short. A literature review was conducted to give an outline of the differences in high school and college level calculus while interviews with college professors stressed how these differences impact the classroom and outcomes at the college level. The largest difference of concern is the procedural emphasis of high school calculus compared to the conceptual structure of college calculus programs (Ryals, 2016; Wade et al., 2016). This difference is of great import as understanding math requires a knowledge of how topics relate to one another and the ability to transfer ideas from one topic to another (Hiebert & Carpenter, 1992). For students to have a deep understanding of calculus, it is necessary that they not only can perform procedures but also comprehend the way in which the topics inter-relate and relate to the broader picture of math and science. As such, a content matrix was created that list topics down the first column and cognitive demand across the leading row. The cognitive demand is not broken down by a hierarchical structure following Bloom's taxonomy, but rather identifies the major modes of relating mathematical concepts. The goal then, for assessing calculus knowledge, is for questions to span the sections of cognitive demand, requiring students to have a procedural and conceptual knowledge of the calculus idea. By charting the tests used at Green High School, areas of weakness were revealed. Green High School did not test students on a large number of topics and emphasized a procedural knowledge in the vast majority of questions. This led to poor performances on the AP test.

After establishing a content matrix that reveals the manner in which a topic is covered, it was important to establish the AP expectations and compare them with the content matrix. The previous AP Free Response sections were charted on the matrix to reveal patterns. Having established that the College Board, similar to college calculus programs, focuses on a conceptual knowledge of calculus, the goal that at least 75 percent of questions on every test have a conceptual component was added. Other goals, like every topic being covered in at least two categories of cognitive demand and every subtopic covered in at least one way, help ensure that each test that meets the objectives of the content matrix will also satisfy the expectations of both the AP program and college calculus programs. Having answered all the evaluation questions, additional research was done to create the new assessments for Green High School to use. These assessments were read by, and altered accordingly, AP calculus teachers and college calculus professors. The new in-class assessments met every goal established in the content matrix.

## Classroom Implications

The import of this work extends beyond the classroom of Green High School. The content matrix can be used to evaluate in-class assessments in every AP calculus classroom to ensure that students are understanding concepts at a deep level in line with the college board's standards. The content matrix synthesizes layers of research on what it means to know calculus as well as how calculus is assessed through the AP program, resulting in an easy tool for classroom teachers to use. By changing the topics, the content matrix could be used for multiple math courses to ensure that students have both a conceptual and procedural knowledge of the course.

In redesigning the tests, it is important to note that an assessment is not, in and of itself, effective or ineffective. To be sure, examinations which do not align with the set standards of a course are ineffective. However, even a well-designed test can be ineffectual if the actual course content being taught on a daily basis does not also meet those same standards. Effective tests, by nature, reference the class time in key principals and ideas (Lyons et al., 2003). Students should not be caught off guard with the format, content, or emphasis of the in-class exams. Students should be used to providing explication and reasoning using specific language regarding answers on daily assignments. Even when employing non-routine questions, students should be deeply familiar with the concepts and how to generalize to an abstract form. This has strong implications for the curriculum outside of the assessments. Indeed it was the very aim of this project to re-align the day-to-day classroom activities by setting an appropriate benchmark through the assessments.

Since the tests are modelled after the College Board standards, it is important to also examine how the College Board assesses successful understanding. The AP exam is divided into two sections, the multiple choice and the free response section, which are both weighted equally. The free response section is composed of six questions that are each worth nine points for a total of 54 points. A survey of released data which analyzed several recent AP Calculus AB results suggested that in order for students to receive a passing score they need to answer about 35 percent of the questions correctly. That is, of the possible 108 points, in an average year somewhere between 29 to 41 points is the minimum to earn a passing score of three. Assessments should be graded with the rigor of the AP standards. However, the College Board's expectations for students' achievement level has implications for the scaling of graded tests.

One of the main purposes of an examination is to determine which concepts have been mastered and which need additional development (Lyons et al., 2003). Time should be taken to give explicit and detailed feedback on the test itself. Pinpointing for students where language was too vague, where units were incorrect and where calculations went awry. When used properly, the graded test is an effective learning tool for students (Lyons et al., 2003). After returning the tests to students, class time should be devoted to going over each question, outlining not only the solutions but also the appropriate language for justifications. Intervention could then be taken by having students work questions similar to ones they missed (potentially for additional points toward the exam).

## Limitations

All projects have limitations. Certainly, this dissertation addresses a particular need at Green High School and is therefore not designed to be generalizable. The results of this study do not judge whether the assessments succeeded in guiding the program in such a way that students are successful in college calculus. An additional long-term study with a control group will therefore be necessary to gauge the impact. Another potential criticism of the project is that it does not pay particular attention to the parents as stakeholders. Green High School partners with parents and values their input. Certainly, parental insights will be sought at a later time when the curriculum is being implimented at the classroom level.

## Future Studies

Green High School will continue in its evaluation of the AP Calculus program and more generally the math program. This continuation will be led by the head of instructional effectiveness, Ms. White. Ms. Brown will review the assessments and attend another course provided by the AP Summer Institute. Following this, Ms. Brown will meet with the head of instructional effectiveness to map out a plan for curriculum changes. During the school year, Ms. White will frequently observe the AP Calculus classroom to ensure that the instruction echos the standards modeled through the assessments. She will additionally monitor students' progress by examining the summarized results of each test and assist Ms. Brown with measures of intervention when necessary. While ideally, a control group could be set up to establish the predictive nature of these assessments and AP success, this is beyond the scope of Green High School's goals.

Future studies outside of Green High School might gather the assessments from several well established successful AP Calculus programs and look for patterns within the content matrix. These patterns might depict more precise goals for where future assessments should fall on the content matrix.

# APPENDIX A: DIRECTIONS FOR ASSESSMENTS

## Directions for AP Calculus assessments

Show all your work for every question and provide sufficient explanations even when not directly prompted within a question. Correct results without supporting work or justifications will not receive full credit. You will be judged according to your methods and the demonstration of your logic, in addition to the accuracy of your answers. Clearly label every diagram, graph, or figure you draw or use for justifications. When theorems are used, clearly indicate the conditions for which the theorems apply.

Write in pencil. Cross out any work that you do not want to be graded. There is no need to simplify answers, but should you simplify, and do so incorrectly, credit will not be granted. No calculator will be allowed unless otherwise specified.

When calculators are used, unless otherwise specified, answers should be correct within three decimal places. Questions answered with the aid of calculators still require justifications using non-calculator terms.

# APPENDIX B: UNIT 1 ASSESSMENTS

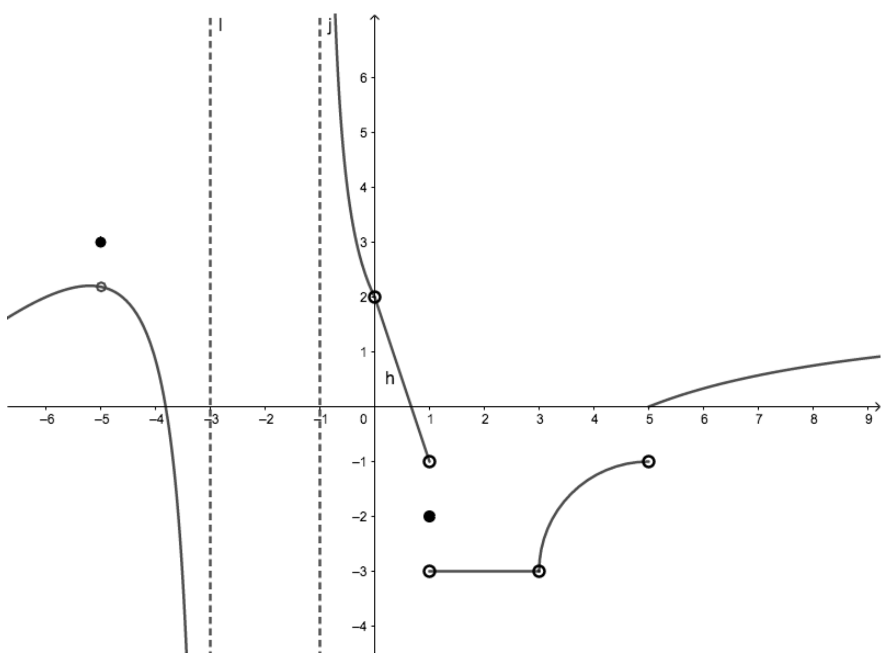
## Section 1

1. Write an equation that summarizes the following information.

| $x$ | -4.05 | -4.01 | -4.005 | -3.995 | -3.99 | -3.95 |
|---|---|---|---|---|---|---|
| $f(x)$ | 3.125 | 3.098 | 3.021 | 2.991 | 2.891 | 2.822 |

2. Find $\lim_{x\to 5} \frac{6x-30}{x^2-25}$. Is the function continuous at $x = 5$? Explain.

Let the function $g(x) = \frac{cos\pi x}{x}$. The graph of $h(x)$ is given below. Use the functions $g(x)$ and $h(x)$ to answer questions 3-7. If the limit does not exist, briefly explain how you know and classify the discontinuity.




3. $\lim_{x\to 1^+} h(x) =$

4. $\lim_{x\to 1} h(x) =$

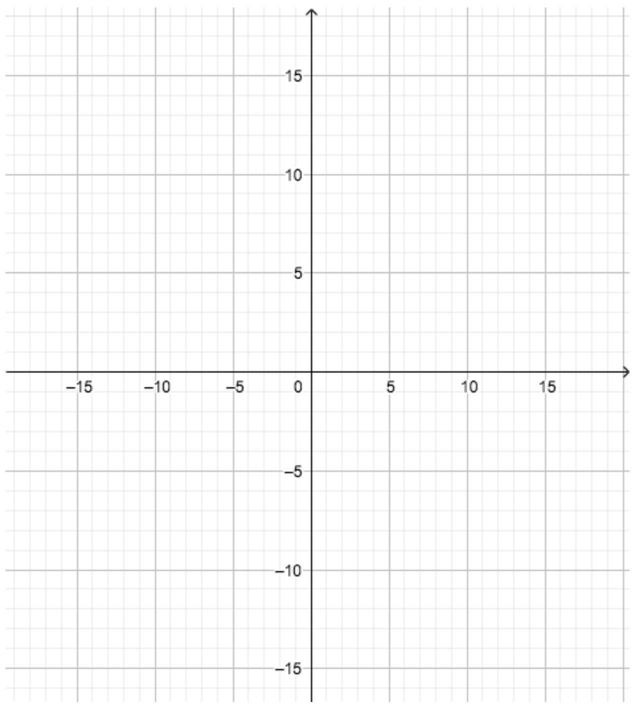
5. $\lim_{x\to -1^+} h(x) =$

6. $\lim_{x\to 3}(h(x)+g(x)) =$

7. Find all values of c, where both $h(c)$ and $\lim_{x\to c} h(x)$ exist but $\lim_{x\to c} h(x) \neq h(c)$.

8. Find the polynomial $2x^2 + bx + c$, such that $\lim_{x\to -2} \frac{2x^2+bx+c}{x^2+x-2} = \frac{7}{3}$.

9. A certain function on the coordinate axis has a limit of one as $x$ approaches infinity and a limit of negative two as $x$ approaches negative infinity. Additionally, the limit as $x$ approaches two is three. Sketch a potential drawing of the function on the coordinate axis below.




10. For what values of a and b is the function $f(x)$ continuous, where a and b are integers?

$$f(x)=\begin{cases} \text{a - x} & x \leq 1 \\ \frac{x}{b}+3 & x > 1 \end{cases}$$

11. Suppose that the revenue (in millions of dollars) that a certain band makes while touring can

be modeled by the function $r(t) = \frac{82t^2-3t+7}{2t^2+6t}$ where $t$ represents the number of months the band is on tour. If the band continues to tour, what will they make ultimately?

12. If the graph of $\frac{ax^2+b}{x^2+c}$ has a horizontal asymptote at $y = 3$ and a vertical asymptote at $x = 4$, find $a - c$.

# APPENDIX C: UNIT 2 ASSESSMENTS

## Section 2

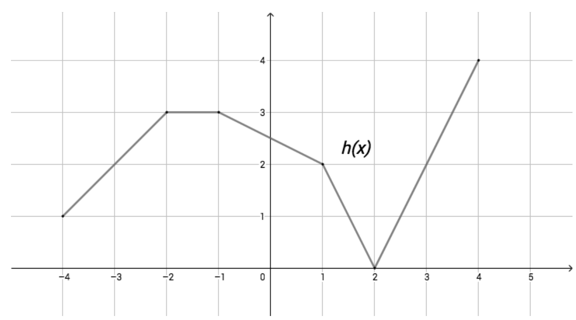
1. Let $f(x) = x^2 - 2x + 3$. Use the definition of derivative to find $f'(x)$ and evaluate it at $x = -3$. Explain what your solution means in terms of the graph of $f(x)$.

2. Find $\lim\limits_{x\to 0} \frac{3sin(3x)}{2x}$

3. Find $\frac{d^2}{dx}(3sin^2(2x)$.

For Questions 4 - 6, use the following functions.

| x | $f(x)$ | $f'(x)$ |
|---|---|---|
| -3 | 0 | -2 |
| -2 | 5 | 3 |
| -1 | 8 | 4 |
| 0 | 6 | -1 |
| 1 | 5 | -1 |
| 2 | 3 | -3 |




Let $f$ be a differentiable function. The table above gives values of $f$ and its derivative $f'$ at selected values of $x$.

Let $h$ be a function whose graph is shown above.

4. Let $g(x) = h(f(x))$. Find $g'(-3)$.

5. Let $k(x) = \frac{h(3x)}{f(x)}$. Find $\frac{dk}{dx}$ at$x = -1$.

6. Estimate $f''(-\frac{1}{2})$

7. Let $p(x) = f^{-1}(x)$. Find $p'(0)$.

| $t$ (hours) | 0 | 1 | 2 | 4 | 7 | 8 |
|---|---|---|---|---|---|---|
| S(t) (smartphones) | 1000 | 800 | 724 | 610 | 310 | 0 |

8. As part of a cyber Monday special, an electronics store started selling smartphones online at midnight and sold out 8 hours later. The function S(t) represents the number of smartphones the store had in stock at a time $t$, where t represents the hours after the sale began. Values of S(t) at various times $t$ are shown in the table above. Use the data in the table above to estimate $S'(3)$. Explain what the result means in context.

9. At Willy Wonka's chocolate factory a certain vat contains 150 liters of corn syrup. More corn syrup is then pumped into the vat. The function $C$ models the amount of corn syrup in the vat after $t$ minutes. The umpa lumpas believe that $C$ satisfies the differential equation $\frac{dC}{dt} = \frac{1}{10}(C - 50)$ for as long as the corn syrup is being pumped. Use the line tangent to the graph of $C$ at t = 0 to approximate the amount of corn syrup in the vat after the pump has run for 2 minutes and 30 seconds. Make sure to use appropriate units.

10. Let $(x + y)^2 + y^2 = 8$. Find all points on the curve that have a vertical tangent line.

11. Sketch the graph of a function f(x) that is continuous but not differentiable. Write a sentence(s) explaining why the function is continuous but not differentiable. Is it possible for the function to be differentiable but not continuous at a point?

12. Suppose $f(x)$ is differentiable at the point $x = 2$. Which of the following must be true? Justify.

A) $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$

B) $\lim_{h\to 0} \frac{f(2-h)-f(2)}{h}$

C) $\lim_{a\to 2} \frac{f(a+h)-f(a-h)}{h}$

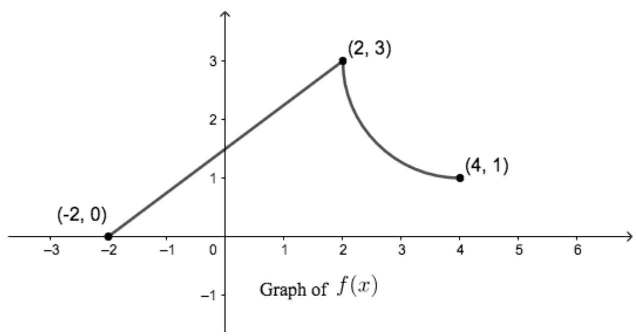
D) $\lim_{x\to 2} \frac{f(a+h)-f(a)}{h}$

E) $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

13.Can you use L'hopital's rule to compute $\lim_{x \to 3} \frac{cos(\pi x)}{x-3}$? If so, evaluate the limit, if not, explain.

Section 3

1. A farmer wants to construct a large rectangular fence. He wants to subdivide the enclosed area into 3 smaller, parallel rectangles using the same fencing. His goal is to enclose as much area as possible. If the farmer has 1200 feet of fencing available, what are the optimal dimensions of the larger rectangular fence?




2. Find the average rate of change of $f$ from $-2 \leq x \leq 4$ using the graph above. Mark believes that there does not exist a value $c$ such that $f'(c)$ is equal to the average rate of change. Is Mark correct? If so, does this contradict the Mean Value Theorem? Explain.

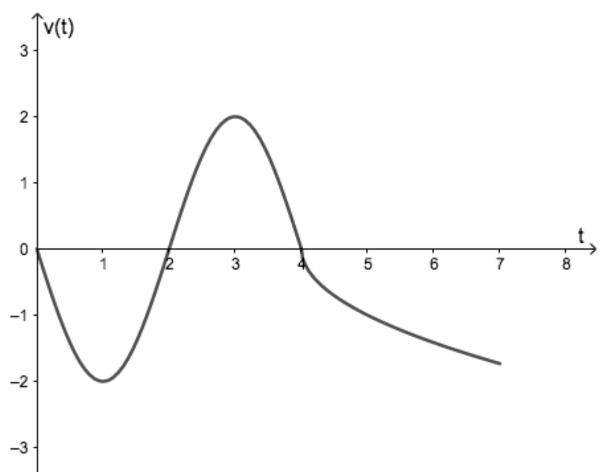
3. An hourglass consists of two right cones. Sand falls through the top cone into the bottom cone, filling the cone at a rate of 1.2 $cm^3/min$. The hourglass is constructed such that height of the sand is always equal to one-fourth of the diameter of the base in the bottom portion of the hourglass. Find how fast the height is changing when the height is 3 cm. (Volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

4. The amount of water in liters contained in a cistern at time $t$ is modeled by the twice differentiable function $W(t)$. The rate at which the water is pumped out of the cistern at time $t$ is expressed by $R(t)$, where $R(t)$ is in liters per minute. The table below shows $W(t)$ and $R(t)$ at various times.

| $t$ minutes | 0 | 10 | 30 | 40 |
|---|---|---|---|---|
| $W(t)$ | 0 | 82 | 58 | 103 |
| $R(t)$ | 3 | 3.2 | 2.1 | 3 |

Is there a time between 30 and 40 minutes where the rate at which the water is pumped out of the cistern is 4.5 liters/minute? Justify your answer.

Use the following graph and explanation to answer questions 5-6.




A taxi picks up a passenger at the airport and drives on a straight road for $0 \leq t \leq 7$ where t is in minutes. The taxi is moving with the velocity modeled in the graph above.

5. At what time on the interval $0 \leq t < 3$ is the taxi furthest from the airport? Explain your answer.

6. On what intervals is the car slowing down? Explain.

7. The function $f(x) = x^3 + bx^2 + cx$ has a local maximum of 9 at $x = -3$. Find the local minimum.

8. The base of a triangle is decreasing at a rate of 2 centimeters per minute while its height is increasing at a rate of 2 centimeters per minute. Which of the following is true about the area of

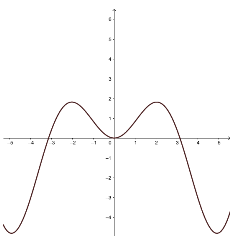
the triangle? Show the work that leads to your answer.

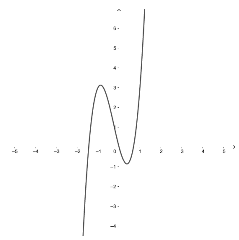
A) The area is always increasing    B) The area is always decreasing

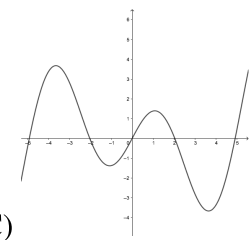
C) The area is increasing only when b<h    D) The area is increasing only when h>b

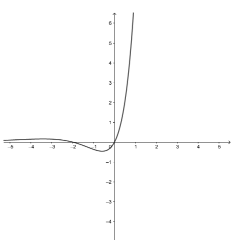
E) The area remains constant

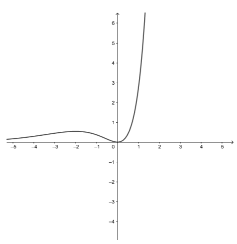
9. The six graphs below represent three functions and their corresponding derivatives. Match the function to its derivative and briefly justify your choice. Make sure to specify which graph is the original function and which is the derivative.

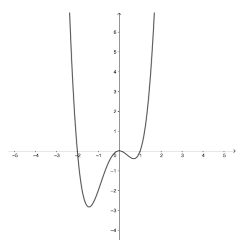
A) 

B) 

C) 

D) 

E) 

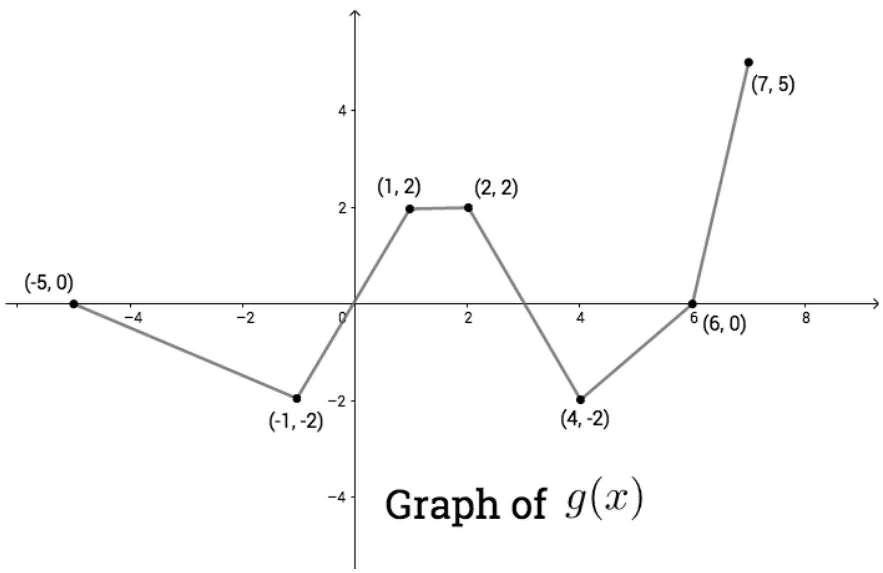
F) 

# APPENDIX D: UNIT 3 ASSESSMENTS

Section 4

1. Let $f(x)$ be a continuous, positive, increasing function for all $x \geq 0$. Rank the following in increasing order:

A) $\int_0^5 f(x)dx$ B) $\sum_{i=1}^{5} f(x_i)$ C) $\sum_{i=0}^{4} f(x_i)$ D) $\sum_{i=1}^{5} \frac{f(x_i)+f(x_{i-1})}{2}$




2. Let $f(x) = \int_0^x g(x)dx$ and let $g(x)$ be the function shown in the graph above. Find $f(6)$, $f'(4)$ and $f''(3)$.

3. Suppose that $h$ is a continuous function on the interval [1,3] and that $2 \leq h(x) \leq 6$. What is the greatest possible value of $\int_1^3 h(x)dx$?

Use the following to answer question 4.

$\int_0^3 f(x)dx = 2$ $\int_0^{-3} f(x)dx = -7$ $\int_5^3 f(x)dx = -1$ $\int_{-4}^2 g(x)dx = 3$

4. Find $\int_{-4}^2 (f(x+1) - g(x))dx$.

5. Find $\int_{-3}^{2} |x+1| dx$.

6. Express $\lim_{max \Delta x_i \to 0} 9 \sum_{i=1}^{\infty} x_i^2 \Delta x_i$ where $\Delta x_i = \frac{2-0}{n}$ as an integral expression. Then, use a right Riemann sum with three sub-intervals to estimate it.

7. Suppose that $f$ has a positive derivative for all $x$ and that $f(1) = 0$. Which of the following statements must be true of the function $g(x) = \int_0^x f(t)dt$. Justify your answers.

A) $g$ is a differentiable function of $x$

B) $g$ is a continuous function of $x$

C) The graph of $g$ has a horizontal tangent at $x = 1$

D) $g$ has a local maximum at $x = 1$

E) $g$ has a local minimum at $x = 1$

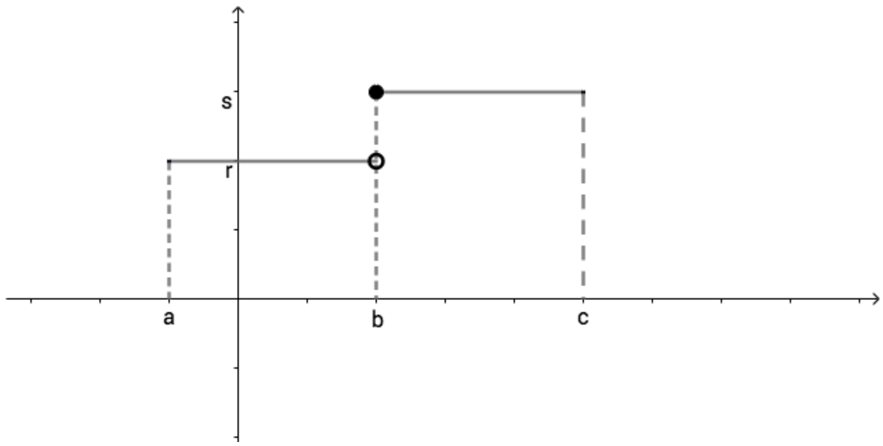
F) The graph of $g$ has an inflection point at $x = 1$

G) The graph of $\frac{dg}{dx}$ crosses the $x$-axis at $x = 1$

8. Find $\int_0^5 \frac{4x^2-1}{2x-1} dx$.

9. Find the anti-derivative of $c(t)$, where $c(t)$ models the number of cars per hour that were washed at a certain car wash and $c(t) = 4t^3 \sqrt{2t^4 + 7} dt$. Explain the result in terms of the context.

10. Set up an integral expression that represents the area under the curve in the following graph. Then evaluate the expression.




11. Evaluate $\frac{d}{dx}\int_{pi}^{x^2} sin(t^2)dt$.

12. During a hurricane a weather company recorded the rate at which rain fell during various times of the day. The following chart relays their finding where the rainfall is recorded in inches per hour.

| time | 8:00 am | 9:30am | 11:30am | 1:00 pm | 2:00pm | 4:00pm |
|---|---|---|---|---|---|---|
| $r(t)$ | 1.2 | 1 | 3 | 1.8 | .7 | .2 |

Estimate the amount of rain that fell from 8:00 am to 2:00 pm using a left Riemann sum.

## Section 5

1. A rocket is launched straight up into the air. The rocket's velocity was recorded to be $v(t) = ln(lnt)$ on the interval $1 < t < 4$. Find a function for the acceleration of the rocket, is this function continuous? Explain.

2. Suppose that a car drives on a straight road in one direction such that the position of the car, at a given time, $t$, is given by the function $x(t) = 5^{t^2-1}$, where t is in hours. The number of gallons of gas used by the car to travel $x$ miles is given by the function $g(x) = \frac{2}{3xln5}$ gallons per mile. How many gallons per hour is the car getting when t = 2, indicate the units.

3. The acceleration of a particle along the x-axis is given by $a(t) = \frac{1}{t(lnt)^3}$. Find the velocity.

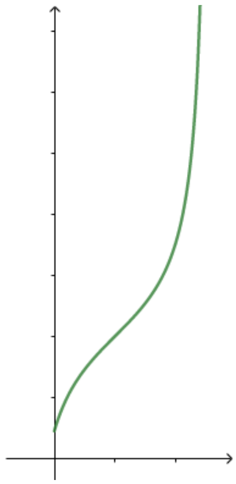
4. Find the value of $a$, $a$>1, for which $\int_1^a \frac{1}{1+x^2}dx = \frac{\pi}{12}$.

5. Evaluate $\int_0^1 \frac{x}{x^2+1}dx$.

6. Is there a common value of $x$ for which the tangent lines of $g(x) = ln(\frac{3}{x^2})$ and $h(x) = ln(-\frac{x+1}{x})$ are parallel to one another?

7. Find the function $f$ that goes through the point (0,0) and has a derivative, $f'(x) = sin(\frac{\pi}{3}x) - cos(\frac{\pi}{3}x)$.

8. A function $f$ has a derivative $f'(x) = x6^x$. Find the second derivative and use it to explain why

the graph of below can not model the function $f$. 

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

You may use a calculator on this portion of the test

9. Suppose the piece-wise twice-differentiable function $f(x)$ has a derivative as represented below and $f(1) = 5$.

$$f'(x) = \begin{cases} \frac{1}{2}e^{x^2} & -4 \leq x \leq 1 \\ x^3 cos(x) + 3 & 1 < x \leq 7 \end{cases}$$

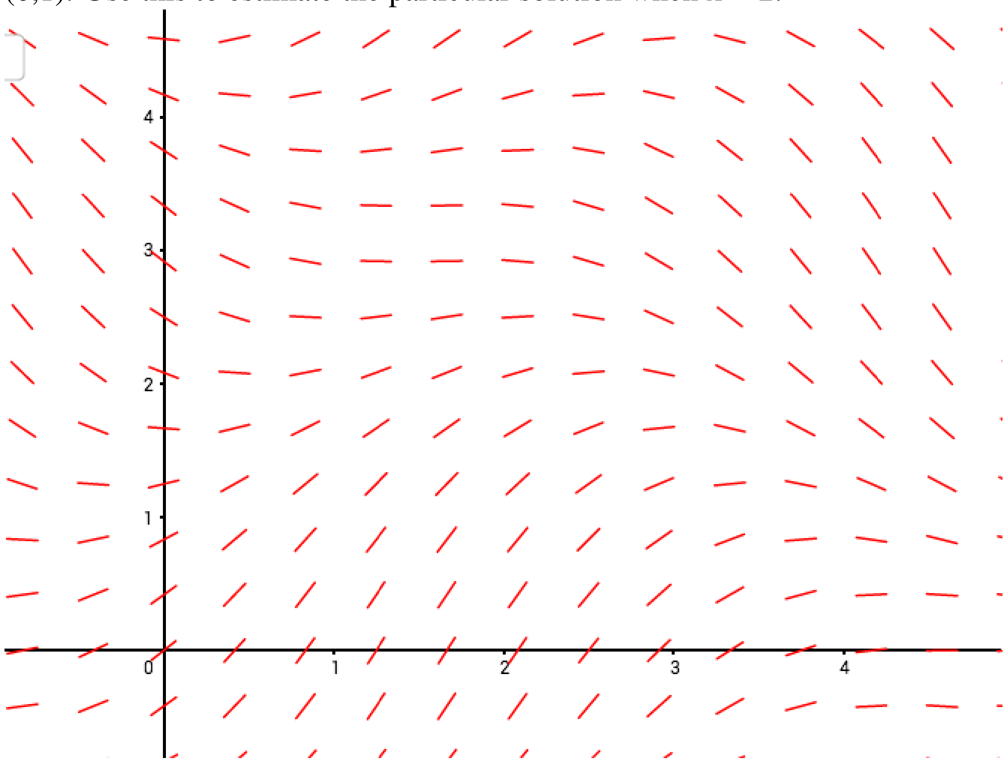
A) Find $f''(5)$. Suppose that a piece-wise twice-differentiable function $g(x)$ has the same derivative as $f(x)$ but has the condition that $g(1) = -100$, compare $f''(5)$ to $g''(5)$.

B) Find $f(5)$. Suppose that a piece-wise twice-differentiable function $g(x)$ has the same derivative as $f(x)$ but has the condition that $g(1) = -100$, compare $f(5)$ to $g(5)$.

Section 6

1.Is $y = sin(2x)$ a solution of the differential equation $y'' + 4y = 0$? Justify.

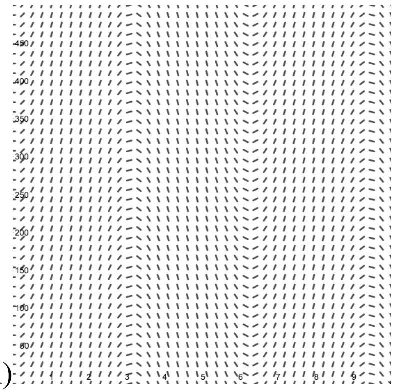
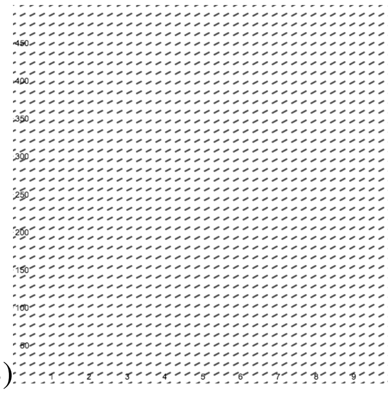
2.The slope field below models a differential equation. Sketch the solution curve through the point (0,1). Use this to estimate the particular solution when x = 2.

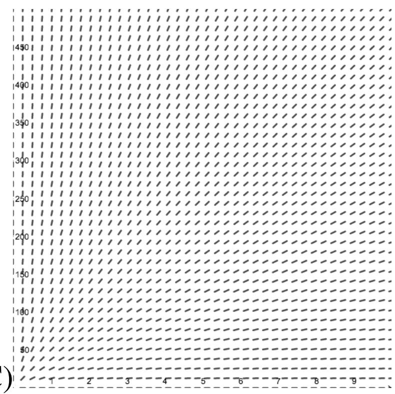
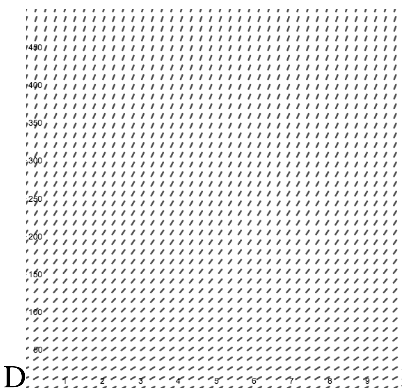



Use the following information regarding the population of sea lions to answer questions 3 and 4. Let $S(t)$ represent the number of sea lions in a population at time $t$ years. For $t \geq 0$, $S$ is increasing at a rate directly proportional to S+100.

3. If the initial population of sea lions is 200, set up an expression for the function $S(t)$.

4. Which of the following slope fields could represent the sea line population? Explain.

A)  B) 

C)  D) 

5. A chlorinated pool has water pumped in with a different concentration of chlorine. So as not to overflow the pool, water is also being drained. The concentration of chlorine in the pool changes at a rate modeled by the differential equation $\frac{dC}{dt} = \frac{C}{t+100}$ where $t$ is in minutes. Suppose the initial concentration of chlorine is 3 ppm (parts per million). Find the concentration of chlorine after an hour.

6. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-2}$. What is the domain of the slope field? Describe all points on the x-y plane for which the slopes are positive.

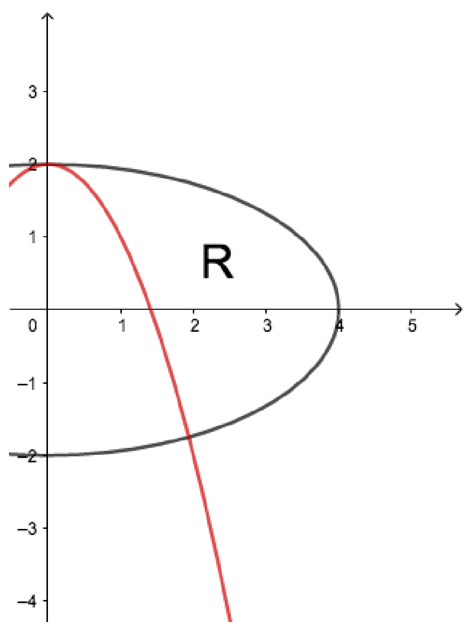
7. Suppose $y' = (x^2 + 3)e^y$. Find $y$ in terms of $x$.

8. Find $A$ and $B$ such that $y = x^2 + 3x$ is a solution to the differential equation $Ay'' - By' = 2x - 1$

Section 7

1. Find the volume of a solid if its base is bounded by the x axis, $x = 4$ and $y = x^2$ and the cross sections perpendicular to the x-axis are semi-circles.

Refer to the graph to answer questions 2 and 3. Let R be the region bounded by the function $f(x) = 2 - x^2$ and the ellipse $x^2 + 4y^2 = 16$ as shown below.




2. Write but do not solve, an expression for the area of R.

3. Find the volume of the solid generated by revolving R around the $y$-axis.

4. The amount of grain, in pounds, being milled into flour at a time $t$, in hours, at a certain factory is represented by the differentiable function $G(t)$. A sampling of the values of $G(t)$ at various times are given in the table below:

| t hours | 0 | 4 | 8 | 12 | 16 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| G(t) | 0 | 100 | 230 | 0 | 200 | 170 | 195 |

Set up an equation to find $\frac{1}{24}\int_0^{24} G(t)dt$ using a right Riemann sum and explain its meaning in the context.

Use the information below to answer questions 5 and 6.

At a certain bakery, cookies are being baked continuously throughout the workday, $0 \leq t \leq 8$, where $t$ represents the number of hours after 8 am. $C(t) = \frac{1}{20}e^{\frac{1}{2}x} + 3x^2 + 5$,represents the rate at which are produced. On a particular day, the rate that the cookies are sold is modeled by $S(t)$ given below.

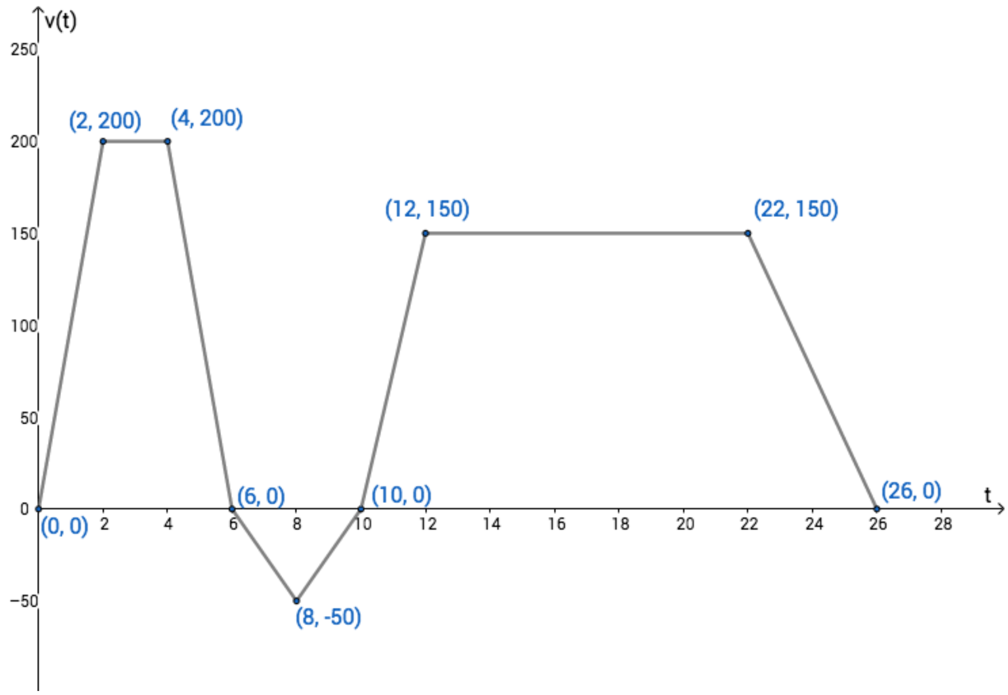
$$S(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 2 & \text{if } 1 \leq t < 3 \\ t+1 & \text{if } 3 \leq t \leq 8 \end{cases}$$

5. Are there any cookies left at the end of the day? If so, how many? Justify your answer.

6. Suppose that on another day the owners know the cookies are going to sell at a rate of $S(t) = 20t + 12$. While they will bake the cookies beginning at 8am, they will delay opening the doors to sell cookies until a later time so that they do not run out of cookies. If they want to open right at the beginning of an hour and sell as many cookies as possible without running out, what time should they open their bakery? Justify your answer.

Use the following information to answer questions 7 and 8.

Brian skateboards from his house along a straight road to get to the beach, which he reaches after 28 minutes. His velocity, measured in meters per minute, is modeled by the function $v(t)$ on the graph below.




7. How far did Brian travel? Is this result equal to the distance from his house to the beach, if not, how far is his house from the beach? Justify your answer.

8. What was Brian's average speed during the trip?

9. The following chart represents the velocity of particle (in meters per second) recorded at various times ($t$), where $t$ is in seconds.

| t | 0 | 1 | 3 | 4 | 6 | 8 |
|---|---|---|---|---|---|---|
| v(t) | 1 | 3 | 2 | 4 | 1 | 5 |

Find the average velocity and the average acceleration of the particle during the 8 second period. Use correct units. If/where necessary, use a midpoint sum.

# APPENDIX E: GREEN HIGH SCHOOL TESTS

GHS Section 1

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

NO CALCULATOR on this portion of the test

**To receive credit, show the mathematics behind your answer selection or make a statement justifying your answer.** For problems 1-4, us the following function.

$$f(x) = \begin{cases} 2 - x & x \leq 1 \\ \frac{x}{2} + 1 & x > 1 \end{cases}$$

1. What is the value of $f(1)$?

A) 0 B)1 C)$\frac{5}{2}$ D)$\frac{3}{2}$ E) Nonexistent

2. What is the value of $\lim_{x \to 1^+} f(x)$?

A) 0 B)1 C)$\frac{5}{2}$ D)$\frac{3}{2}$ E) Nonexistent

3. What is the value of $\lim_{x \to 1^-} f(x)$?

A) 0 B)1 C)$\frac{5}{2}$ D)$\frac{3}{2}$ E) Nonexistent

4. What is the value of $\lim_{x \to 1} f(x)$?

A) 0 B)1 C)$\frac{5}{2}$ D)$\frac{3}{2}$ E) Nonexistent

5. Which of the following functions is NOT continuous at x = 3?

A) $g(x) = \begin{cases} \frac{x^2+2x-15}{x-3} & x \neq 3 \\ 8 & x = 3 \end{cases}$

B) $h(x) = \begin{cases} \frac{x^2+2x-15}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$

C) $f(x) = \begin{cases} 5 - x & x \leq 1 \\ 2x - 4 & x > 1 \end{cases}$

D) $k(x) = \begin{cases} 4 - x & x \leq 3 \\ 2x - 5 & x > 3 \end{cases}$

E) All of the functions above are continuous at x = 3

6. $\lim_{x \to -\infty} \frac{x^2+x-7}{x^2-5x-3}$?

A) 0 B) $\frac{7}{3}$ C) 4 D) 1 E) Nonexistent

7. $\lim_{x \to \infty} \frac{-2x^2+7x-3}{x^2+2x-3}$?

A) -3 B)-2 C) 2 D) 3 E) Nonexistent

**Show your work.**

8. Use algebra to evaluate

A) $\lim_{x \to -3} \frac{x^2}{x+2}$

B) $\lim_{x \to 4} \frac{x^2-16}{x^2-6x+8}$

Read and follow all directions.

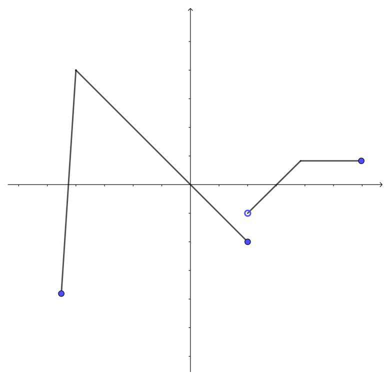
Use a pencil and show all your work (where necessary) for full credit.

You may use a calculator on this portion of the test

**Choose the appropriate answer. To receive credit, show the mathematics behind your answer**

**selection or make a statement justifying your answer.**

Let $g(x) = \frac{|4-x|}{4-x}$ and let $y = f(x)$ have the graph shown in the figure. Use these functions to answer problems 9-11.



9. $\lim\limits_{x \to 2^+} f(x) = ?$

A) 0 B)-1 C) -1.5 D) -2 E) Nonexistent

10. $\lim\limits_{x \to 2} f(x) = ?$

A) 0 B)-1 C) -1.5 D) -2 E) Nonexistent

11. $\lim\limits_{x \to 5} f(x) + g(x) = ?$

A) 0 B)-1 C) -1.5 D) -2 E) Nonexistent

12. $\lim\limits_{x \to 2^+} \frac{x}{x-2} = ?$

A) $-\infty$ B)$\infty$ C) 1 D) $-\frac{1}{2}$ E)-1

13.Which of the following is true about the graph of $f(x) = x^{\frac{2}{3}} at x = 0$?

A)$f(0)$ does not exist B) It has a discontinuity C) It has a cusp

D) $\lim\limits_{x \to 0} f(x)$ does not exist E) It has a horizontal tangent

**Answer each question. To receive credit, show the mathematics behind your answer selection**

**or make a statement justifying your answer.**

14. Is the following function continuous at x=4? Prove your answer by using the definition of continuity.

$$f(x) = \begin{cases} \sqrt{4-x} & x < 4 \\ (x-4)^2 & x \geq 4 \end{cases}$$

15. Find the following limit by completing the table: $\lim_{x\to 2} \frac{sin(x-2)}{x-2}$

| x | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| f(x) | | | | | | | |

16. Find each limit algebraically

A) $\lim_{x\to 0} \frac{sin(5x)}{3x}$

B)$\lim_{x\to 4} \frac{x^2+3x-28}{x-4}$

17.For the function $f(x) = \frac{9x^2-16}{x^2-25}$

A) Find $\lim_{x\to\infty} f(x)$

B) Find $\lim_{x\to-\infty} f(x)$

C) Find the vertical asymptotes of $f(x)$

18. Find $\lim_{x\to 0} \frac{\frac{1}{2}-\frac{1}{x+2}}{x}$

19. Explain the difference between a removable discontinuity and a non-removable discontinuity

20. Find $\lim_{x\to 0} \frac{sinxcosx}{x}$. Show all steps.

GHS Section 2

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

NO CALCULATOR on this portion of the test

**To receive credit, show the mathematics behind your answer selection or make a statement justifying your answer.**

1. Let $f(x) = x - \frac{1}{x}$. Find $f''(x)$.

A)$1 + \frac{1}{x^2}$ B)$1 - \frac{1}{x^2}$ C) $\frac{2}{x^3}$ D) $-\frac{2}{x^3}$ E)Nonexistent

2. Which of the following is $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{x+1}{x-1})$?

A)$\frac{2}{(x-1)^2}$ B) 0 C)$-\frac{x^2+1}{x^2}$ D)$2x - \frac{1}{x^2} - 1$ E)$-\frac{2}{(x-1)^2}$

3. Which of the following numbers is the slope of the line tangent to the curve $y = x^2 + 5x$ at x=1.5?

A) 24 B)$-\frac{5}{2}$ C) 11 D) 8 E)$-\frac{2}{5}$

4. Which of the following is an equation of the tangent to the graph for $f(x) = \frac{1}{x}$ at (1,1)?

A) $y = -2x$ B)$y = 2x$ C)$y = -2x + 4$ D)$y = -x + 2$ E)$y = x + 3$

5. Let $f(x) = 1 + 2x - 3x^2$. Which of the following is equal to $f'(x)$ when x=1?

A) -6 B) -4 C) 4 D) 6 E) Nonexistent

6. $\lim_{x \to -\infty} \frac{x^2+x-3}{7x^2-5x-4}$?

A) 0 B)$\frac{1}{7}$ C) $\frac{3}{4}$ D) 7 E) Nonexistent

7.Find the derivative of the function $y = \frac{4}{x^3}$

A) $-4x^2$ B) $-\frac{12}{x^2}$ C) $\frac{12}{x^2}$ D) $\frac{12}{x^4}$ E) $-\frac{12}{x^4}$

8.Find $\frac{\text{dy}}{\text{d}x}$ if $3xy = 4x + y^2$

A) $\frac{(4-3y}{2y-3x}$ B) $\frac{(3x-4}{2x}$ C) $\frac{3y-x}{2}$ D) $\frac{3y-4}{2y-3x}$ E) $-\frac{4+3y}{2y+3x}$

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

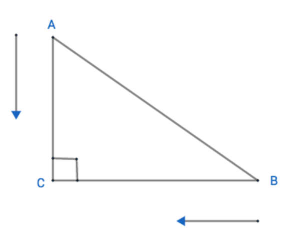
You may use a calculator on this portion of the test

**Choose the appropriate answer. To receive credit, show the mathematics behind your answer selection or make a statement justifying your answer.**

9.Find $\frac{\text{dy}}{\text{d}x}$ for $y = 4sin^2(3x)$.

A) $8sin(3x)$ B) $24sin(3x)$ C) $8sin(3x)cos(3x)$ D) $12sin(3x)cos(3x)$ E) $24sin(3x)cos(3x)$

10.In right triangle $\triangle ABC$, point A is moving along a leg of a right triangle toward point C at a rate of $\frac{1}{2}$ cm/sec and point B is moving toward point C at a rate of $\frac{1}{3}$ cm/sec along a line containing the other leg of a right triangle, as illustrated in the triangle shown below. What is the rate of change in the area of $\triangle ABC$, with respect to time, at the instant when AC=15 cm and BC= 20 cm? **(Formula: $A = \frac{1}{2}bh$; ***HINT: you have to use the product rule because of $bh$)**




A) $-0.0833cm^2/sec$ B) $-0.4167cm^2/sec$ C) $-0.833cm^2/sec$ D) $-7.5cm^2/sec$ E) $-15cm^2/sec$

11.Find the second derivative of $f(x)$ is $f(x) = (2x+3)^4$? A)$4(2x+3)^3$ B) $8(2x+3)^3$ C)$12(2x+3)^2$ D)$24(2x+3)^2$ E)$48(2x+3)^2$

**Show your work and use the correct calculus notation.**

12.Let $y = \frac{x^3-1}{2x}$

A) Find $\frac{dy}{dx}$

B) Find $\frac{d^2y}{dx^2}$

C)Find an equation of the line tangent to the curve $y$ at (2,0).

13.Use implicit differentiation to find $\frac{dy}{dx}$ for the following function: $x^3 + y^3 = 2xy$.

GHS Section 3

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

NO CALCULATOR on this portion of the test

**To receive credit, show the mathematics behind your answer selection or make a statement justifying your answer.**

1. What value of $c$ in the open interval (0,4) satisfies the Mean Value Theorem for $f(x) = \sqrt{3x+4}$?

A) 0   B)$\frac{3}{5}$   C)$\frac{5}{3}$   D) 2   E)3

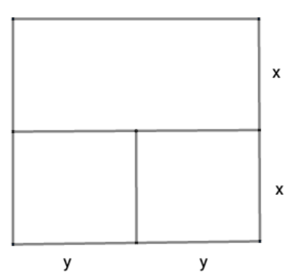
2. If $f'(x) = \frac{x^2(x+1)}{(x-1)^{\frac{1}{3}}}$, then on which interval(s) is the continuous function $f(x)$ increasing?

A) (-1,1)   B) $(-\infty,-1)\cup(1,\infty)$   C)$(-\infty,0)\cup(1,\infty)$   D) $(-\infty,-1)\cup(0,\infty)$   E)$(1,\infty)$

3. Evaluate $\lim\limits_{x\to\infty} \frac{\sqrt{x^2-14}}{3-2x}$?

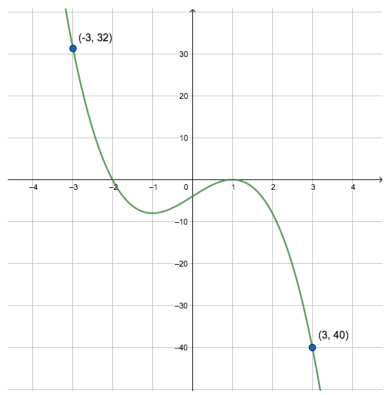
A) $-\infty$   B)$-\frac{1}{2}$   C)$\frac{1}{2}$   D) $\frac{\sqrt{14}}{3}$   E)$\infty$

4. A farmer has 100 yards of fencing to form two identical rectangular pens and a third pen that is twice as long as the other two pens, as shown in the diagram. All three pens have the same width $x$. Which value of $y$ produces the maximum total fences area? **Hint: Use** $5x + 6y = 100$ **and** $A = 2x(2y)$**.**



A) $\frac{25}{2}$ B) 10 C) $\frac{100}{11}$ D) $\frac{25}{3}$ E)None of these

5. The function $f(x)$ is defined as $f(x) = 02(x+2)(x-1)^2$ on the open interval (-3,3) as illustrated in the graph shown. **Justify each part of your answer.**




A) Determine the coordinates of the relative extrema of $f(x)$ on the open interval (-3,3).

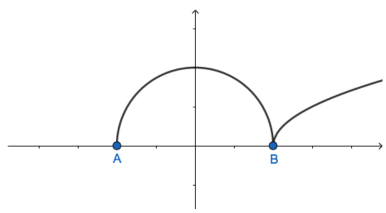
B)Find all values of x for which $f(x)$ is concave down. Explain your reasoning.

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

You may use a calculator on this portion of the test

**To receive credit, show the mathematics behind your answer selection or make a statement justifying your answer.**



6. The function F (shown above) satisfies the conclusion of Rolle's Theorem in the interval [a,b] because

I. F is continuous

II. F is differentiable on (a,b)

III. F(a) = F(b) = 0

A) I only   B) II only   C) I and III only

D) I, II, and III   E)F does not satisfy Rolle's Theorem

7. The value of $c$ guaranteed to exist by the Mean Value Theorem for $f(x) = x^2$ in the interval [0,3] is

A) 1   B) 2   C)$\frac{3}{2}$   D) $\frac{1}{2}$   E)None of these

8. Find the differential $dy$ of the given function: If $y = xsinx$, then $dy =$

A) $(sinx + cosx)dx$   B)$(sinx + xcosx)dx$   C)$(sinx - xcosx)dx$   D) $[x(sinx + cosx)]dx$

E) $[x(sinx - cosx)]dx$

9. Let $f$ be the function given by $f(x) = x^3 - 5x^2 + 3x$.

A) On what intervals is $f$ increasing? Justify your answer.

B) On what intervals is the graph of $f$ concave downward? Justify your answer.

10.If $f(x) = -\frac{8}{x^2}$, $f'(2) =$?

A) -2   B) 2   C) -5   D) 5   E) -1

11. Analyze and sketch a graph of the function: $y = x^5 - 5x$ x- intercepts:

y-intercepts:

Vertical Asymptote:

Horizontal/Slant Asymptote:

f'(x) =

Critical Numbers:

f"(x) =

Possible Points of Inflection:

Test Intervals - vertical asymptotes, critical numbers, inflection points

| Test Interval | f(x) | f'(x) | f"(x) | Characteristic of Graph |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |

12. Identify all relative extrema of $f(x) = 2x^3 + 3x^2 - 12x$

13.Use the following information to compare $\Delta y$ and $\Delta dy$.

$$\Delta y : f(x + \Delta x) - f(x)$$
$$dy : f'(x)dx$$

**Function:**$y = 6 - 2x^2$ **x-value:**$x = -2$ $\Delta x = dx = 0.1$

14. Give the value of x where the function has a local maximum. $f(x) = x^3 - 9x^2 + 24x + 4$

A) 4 B) -2 C) 2 D) -4 E) 3

15. Give a value of $c$ that satisfies the conclusion of the Mean Value Theorem for the function on the interval [1,3]. $f(x) = -2x^2 - x + 2$

A) $\frac{9}{4}$ B) $\frac{3}{2}$ C) 2 D) $\frac{5}{4}$ E) $\frac{1}{2}$

GHS Section 4

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

NO CALCULATOR on this portion of the test

**To receive credit, show the mathematics behind your answer selection or make a statement justifying your answer.**

1.Find $\int_1^2 \frac{1}{x^2} dx =$

A) $-\frac{1}{2}$ B) $\frac{7}{24}$ C) $\frac{1}{2}$ D) 1 E) 2

2. $\int_0^x sintdt =$ **Hint:** $\int sinxdx = -cosx + C$

A) sinx B) -cosx C) cosx D) cosx - 1 E) 1 - cosx

3.Determine the following indefinite integral: $\int (3x^5 - 5x^9)dx$

4. Evaluate the following indefinite integral. Show $u$ and $du$: $\int t^3 \sqrt{2t^4 + 3}dt$

5. Evaluate the integral using the fundamental theorem of calculus: $\int_0^2 (3x^2 + 2x)dx$

6. $\int (x^3 - 3x)dx =$

A) $3x^2 - 3 + C$ B) $4x^4 - 6x^2 + C$ C) $\frac{x^4}{3} - 3x^2 + C$ D) $\frac{x^4}{4} - 3x + C$ E) $\frac{x^4}{4} - \frac{3x^2}{2} + C$

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

You may use a calculator on this portion of the test

**To receive credit, show the mathematics behind your answer selection or make a statement**

**justifying your answer.**

| x | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| f(x) | .48 | 1.25 | 1.07 | .53 | .27 | 1.04 | 3.56 | 2.18 | 2 |

7. Selected values for the continuous function $f(x)$ are given in the table above. Using three left-hand rectangles of equal width, an approximation for $\int_{-3}^{3} f(x)dx$ is:

A) 9.90 B) 7.72 C) -5.64 D) 4.90 E) 2.82

8. A table of values for $g(t)$ is given.

| t | 0 | 40 | 70 | 90 | 100 |
|---|---|---|---|---|---|
| g(t) | 150 | 180 | 195 | 184 | 172 |

a Estimate $\int_{0}^{100} g(t)dt$ by using a left Reimann sum with four subintervals.

b Estimate $\int_{0}^{100} g(t)dt$ by using a right Reimann sum with four subintervals.

c Estimate $\int_{0}^{100} g(t)dt$ by using the trapezoidal rule with four subintervals.

9. $\int_{1}^{6} \sqrt{(x+3)}dx = ?$

A) $-\frac{5}{36}$ B) 1 C) $\frac{58}{5}$ D) $\frac{38}{3}$ E) 19

10.What is the average value of the function $3t^3 - t^2$ over the interval $-1 \leq t \leq 2$?

A) $\frac{11}{4}$ B) $\frac{7}{2}$ C) 8 D) $\frac{33}{4}$ E) 16

11.Find the value(s) of $c$ guarenteed by the Mean Value Theorem for integrals for the function over the given interval: $f(x) = \frac{9}{x^3}$, [1,3]

12. $\int_0^8 \frac{1}{\sqrt{1+x}} dx =$

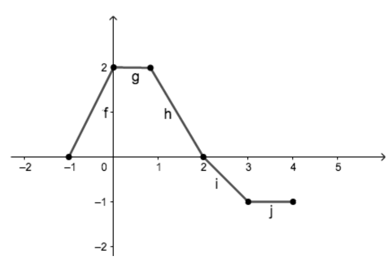
A) 1  B) $\frac{3}{2}$  C) 2  D) 4  E) 6

13. $\int sin(2x+3)dx =$  **Hint:** $\int sinx dx = -cosx + C$

A) $\frac{1}{2}cos(2x+3)+C$  B) $cos(2x+3)+C$  C) $-cos(2x+3)+C$  D) $-\frac{1}{2}cos(2x+3)+C$

E) $-\frac{1}{5}cos(2x+3)+C$

**Extra Credit: You must show all your work to receive credit.**




The graph of a piecewise-linear function $f$, for $-1 \leq x \leq 4$, is shown above. What is the value of $\int_{-1}^{4} f(x)dx$? A) 1  B) 2.5  C) 4  D) 5.5  E) 8

GHS Section 5

Use a pencil and show all your work to receive full credit.

You may use your calculator on this exam.

Integrals of Inverse Trigonometric Functions:

1. $\int \frac{du}{\sqrt{a^2-u^2}} = arcsin\frac{u}{a} + c$

2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}arctan\frac{u}{a} + c$

3. $\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a}arcsec\frac{|u|}{a} + c$

**Find each derivative.**

1.$y = 3ln(2x^3 + 4)$

2.$y = lncos4x$

3.$log_5(4x^3 - 3)^2$

4.$y = 6^{5x+3}$

5. $y = 2x^3lnx$

6.$y = e^{8x+1}$

**Find each integral.**

7. $\int 6e^{6x}dx$

8. $\int \frac{1+x}{\sqrt{1-x^2}}dx$

9. $\int \sqrt{x-1}dx$

10. $\int \frac{x^2}{x^3+1}dx$

11. $\int x^2 e^{x^3+1}dx$

**Find $\frac{dy}{dx}$.**

12. $4x^3 + lny^2 + 2y = 2x$

**Find an equation of the tangent line to the graph of the function at the given point.**

13. $3x^2 - lnx$, (1,3)

14. $y = e^{-2x}$, (0,1)

GHS Section 6

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

NO CALCULATOR on this portion of the test

**To receive credit, show the mathematics behind your answer selection or make a statement justifying your answer.**

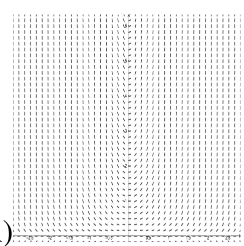
1. The general solution to the differential equation $\frac{dy}{dx} = y^2 sinx$ is

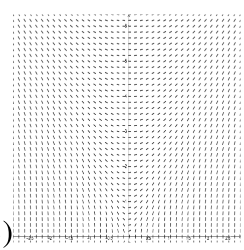
A) $y = \sqrt[3]{3cosx + C}$ B) $y = -cosx + C$ C) $y = \sqrt[3]{sinx + C}$ D) $y = \frac{1}{cosx+C}$

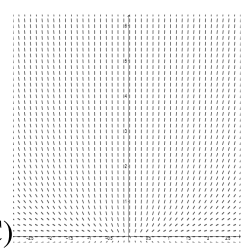
E) $y = \sqrt[3]{-2secx + C}$

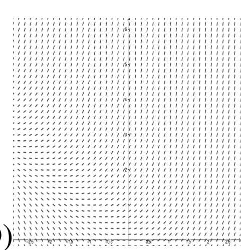
**Questions 2-3 refer to the following information: Consider the differential equation $\frac{dy}{dx} = \frac{4x}{y}$, for $y \geq 1$ only, with initial value y(0) = 1.**

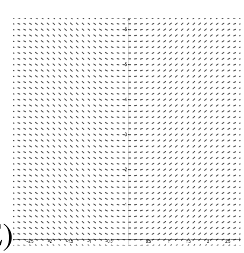
2. Which of the following is the slope field for the general solution to the given equation

A) 

B) 

C) 

D) 

E) 

3.The particular solution $y(x)$ is

A) $y = 2x$ B) $y = \sqrt{4x^2 - 4}$ C) $y = 2x^2 + 1$ D) $y = e^{2x^2}$ E) $y = \sqrt{4x^2 + 1}$

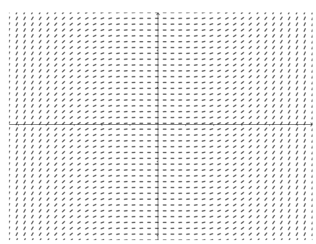
4.If $e^y \frac{dy}{dx} = 2x$ and y(1) = 2, then the particular solution $y(x)$ is

A) $y = ln(x^2) + 2$ B)$y = ln(x^2 + e^2 - 1)$ C) $y = 2e^{x^2-1}$ D) $y = x^2 + e^2 - 1$ E) $y = ln(x^2 - e - 4)$

5. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}ycosx$, for which the solution is $y = f(x)$. Let $f(0) = 2$. The particular solution is

A) $f(x) = x + 2$ B) $f(x) = 2e^{-\frac{1}{2}sin(x)}$ C)$f(x) = \sqrt{sin(x) + 4}$ D) $f(x) = e^{\frac{1}{2}sin(x)}$ E) $f(x) = 2e^{\frac{1}{2}sin(x)}$

6. The slope field for a certain differential equation is shown below. Which of the following could be the specific solution to the differential equation? Hint: The point (-2,2) is on the slope field.



A) $y = sinx$ B)$y = cosx$ C) $y = x^2$ D) $y = \frac{1}{6}x^3$ E) $y = lnx$

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

You may use a calculator on this portion of the test

**To receive credit, show the mathematics behind your answer selection or make a statement justifying your answer.**

**For questions 7-8 refer to the following information:**

Water flows continuously from a large tank at a rate proportional to the amount of water remaining in the tank; that is, $\frac{dy}{dx} = ky$. There was initially 10,000 cubic feet of water in the tank, and at a time $t = 4$ hours, 8000 cubic feet of water remained.

7. What is the value of k in the equation $\frac{dy}{dx} = ky$?

A) -0.050 B) -0.056 C) -0.169 D) -0.200 E) -0.223

8.To the nearest cubic foot, how much water remained in the tank at time $t = 8$ hours?

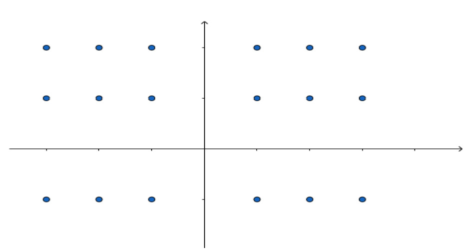
A) 5778 B) 6000 C) 6400 D) 6458 E) 6619

9. Water is being pumped continuously into a tank at a rate that is inversly proportional to the amount of water in the tank; that is, $\frac{dy}{dx} = \frac{k}{y}$, where $y$ is the number of gallons of water in the tank after $t$ minutes ($t \geq 0$). Initially there were 5 gallons of water in the tank, and after 3 minutes there were 7 gallons. How many gallons of water were in the tank at time $t$ =18 minutes. A) $\sqrt{61}$ B) $\sqrt{97}$ C) 13 D) $\sqrt{201}$ E) 17

**Free Response**

10. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$

A) On the axis provided, sketch a slope field for the given differential equation.



B) Let $f$ be the function that satisfies the given differential equation. Write an equation for the tangent line to the curve $y = f(x)$ through the point (1,1). then use your tangent line equation to estimate the value of $f(1.2)$

C) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 1$. Use your solution to find $f(1.2)$.

D) Compare your estimate of $f(1.2)$ found in part (b) to the actual value of $f(1.2)$ found in part c

GHS Section 7

Read and follow all directions.

Use a pencil and show all your work (where necessary) for full credit.

You may use a calculator on the entirety of the test.

**To receive credit, show the mathematics behind your answer selection or make a statement justifying your answer.**

1. Find the volume of the solid obtained by revolving about the x-axis the region bounded by the graphs $F(x) = x^3$ and $g(x) = x^2$?

2. The base of a solid is a region bounded by the circle $x^2 + y^2 = 4$. The cross sections of the solid perpendicular to the x-axis are equilateral triangles. Find the volume of the solid.

3. What is the area of the region(s) between the graphs of $y = x^2$ and $y = -x$ from x=0 to x=2?

A) $\frac{2}{3}$ B)$\frac{8}{3}$ C) 4 D) $\frac{14}{3}$ E) $\frac{16}{3}$

4. Identify the definite integral(s) that represents the area of the region(s) bounded by the graphs of $y = 4 - x^2$ and $y = x - 2$.?

A) $\int_{-3}^{2}(x^2 + x - 6)dx$ B)$\int_{-3}^{2}(-x^2 - x + 6)dx$ C) 4

D) $\int_{-3}^{2}(4 - x^2 - x - 2)dx$ E) $\int_{-3}^{0}(x - 2)dx + \int_{0}^{2}(4 - x^2)dx$

5. The base of a solid is the region in the first quadrant bounded by the line $x = -2y + 6$ and the coordinate axis. What is the volume of the solid if every cross section perpendicular to the y-axis is a square?

A) 15.75 B) 36 C) 18 D) 72 E) None of these

6. Consider $f(x) = -x^3 + 3x^2 + x$ and $g(x) = 4 - x^2$. Sketch a graph of the region(s) bounded by

the two curves. Then find the area of the region(s) bounded by the two curves. (Be sure to show the integrals(s) you use to find the area).

7.For the following question, region $R$ is bounded by $f(x) = x^2 - 3$ and $g(x) = 3x + 1$. Which of the following expressions gives the area of region $R$?

A) $\int_{-2}^{13}(3x+1)-(x^2-3)dx$  B) $\int_{-2}^{13}(x^2-3)-(3x+1)dx$  C) $\int_{-1}^{4}(3x+1)-(x^2-3)dx$

D) $\int_{-1}^{4}(x^2-3)-(3x+1)dx$  E) $\int_{-1}^{4}(3x+1)+(x^2-3)dx$

**Free Response**

8. Let $R$ be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$.

A) Find the are of $R$.

B) Find the volume of the solid generated when R is rotated about the vertical line x = -1.

C)The region $R$ is the base of a solid. For this solid, the cross sections perpendicular to the y-axis are squares. Find the volume of this solid.

# APPENDIX F: LITERATURE REVIEW CODING

Experience

| | Success | | | Attendance | | |
|---|---|---|---|---|---|---|
| References | AP Calculus | DE Calculus | Calculus | AP Calculus | DE Calculus | Calculus |
| Burton (1989) | | | | | | |
| Ferrini-Mundy, Gaudard (1992) | | | | | | |
| Wilhite, Windham, Munday(1998) | | | | | | |
| Fayowski,Hyndman, MacMillan (2009) | | | | | | |
| Laurent (2009) | | | | | | |
| Ubuz (2011) | | | | | | |
| Barnett,Sonnert,Sadler (2014) | | | | | | |
| Bressoud(2015) | | | | | | |

The study found this factor to be an indicator of success in college calculus.

The study found that this is not an indicator of success in college calculus.

The study did not test the factor.

Ability Factors

| References | GPA | Math GPA | HS Ranking | SAT/ACT Score | SAT/ACT Math Score |
|---|---|---|---|---|---|
| Scannicchio (1969) | | | | | |
| Edge,Friedberg (1984) | | | | | |
| Bridgeman,Wendler (1989) | | | | | |
| Stribling (1990) | | | | | |
| Wilhite,Windham,Munday (1998) | | | | | |
| Mwavita (2005) | | | | | |
| Messina (2008) | | | | | |
| Pyzdrowski,Sun,Curtis,Miller,Winn,Hensel (2013) | | | | | |
| Islam,Al-Ghassani (2015) | | | | | |

The study found this factor to be an indicator of success in college calculus.

The study found that this is not an indicator of success in college calculus.

The study found that this factor is an indicator of success only when combined with other factors.

The study did not test the factor.

Effort and Attitude

Effort

| References | Time Spent Studying | Time Spent Reading Math Textbook |
|---|---|---|
| Barnett,Sonnert,Sadler (2014) | | |

Attitudes

| References | Positive View | Perseverance | Confident in Math | View on Past Education | Motivation |
|---|---|---|---|---|---|
| Shaughnessy,Stockard,Moore(1994) | | | | | |
| House (2001) | | | | | |
| Pyzdrowski,Sun,Curtis,Miller,Winn,Hensel (2013) | | | | | |
| Worthley,Gloeckner,Kennedy(2016) | | | | | |

The study found this factor to be an indicator of success in college calculus.

The study found that this is not an indicator of success in college calculus.

The study found that this factor is an indicator of success in college calculus only when combined with other factors.

The study did not test the factor.

# APPENDIX G: CONTENT MATRIX FOR CUMULATIVE ASSESSMENTS

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 1.1A Definition/Existence of limits | | | | |
| • 1.1A1 Notation | 1.2, 1.3 | 1.1 | 1.9 | |
| • 1.1A2 Concept of limits | | 1.1, 1.3 | 1.11 | |
| 1.1C Algebraic Rules for Limits | | | | |
| • 1.1C1 Sums and Products | 1.6 | 1.6 | | |
| • 1.1C2 Algebraic manipulation | 1.8 | | | 1.2 |
| • 1.1C3 L'Hopitals rule | 2.2 | | 2.13 | |
| 1.1D Limits Relate to Functions | | | | |
| • 1.1D1 Asymptotes | | 1.5 | 1.5 | 1.12 |
| • 1.1D2 Magnitude, End Behavior | | 1.9 | 1.9, 1.11 | 1.9 |
| 1.2A Limits and continuity | | | | |
| • 1.2A1 Continuity at a point | 1.2, 1.8 | | 1.8 | 1.2, 1.10 |
| • 1.2A3 Types of Discontinuities | | 1.4, 1.7, 1.8 | 1.4, 1.8 | 1.7 |
| 1.2B Continuity condition for theorem | 3.4 | 3.2, 3.4 | 3.2 | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 2.1A Definition of derivative | | | | |
| • 2.1A2 Limit definition at a point | 2.1, 2.12 | | 2.1 | |
| • 2.1A3 Limit definition of function | 2.12 | | 2.12 | |
| • 2.1A4 Notational variety | 2.1, 2.5 | | | |
| 2.1B Estimation of derivatives | 2.8 | 2.6, 2.8, 3.2, 3.4, 7.9 | 2.8 | 2.6 |
| 2.1C Rules for calculating derivatives | | | | |
| • 2.1C2 Solving directly | 2.3, 5.1, 5.2, 5.6, 5.8, 5.9 | 3.8 | 5.1, 5.2 | 5.6 |
| • 2.1C3 Product/Quotient Rule | 2.3 | | 2.5 | |
| • 2.1C4 Chain Rule | 2.3, 2.10 | 2.4, 2.5 | | 2.4, 2.5 |
| • 2.1C5 Implicit Differentiation | 2.10 | | | |
| • 2.1C6 Inverse Function | | | | 2.7 |
| 2.1D Higher order derivatives | | | | |
| • 2.1D1 Definition | 2.3 | 2.6 | | 2.6 |
| • 2.1D2 Notational variety | 2.3, 2.6 | | | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 2.2A Derivatives relationship to a graph | 4.7 | 3.5, 3.6, 3.9, 5.8 | 3.5, 3.6, 3.9, 4.2 | 3.7 |
| 2.2B Continuity and derivative | | | | |
| • 2.2B1 Points not differentiable | 5.6 | 2.11, 6.6 | 2.11 | 2.10 |
| • 2.2B2 Differentiability implies continuity | 2.11, 4.7 | | | |
| 2.3B Derivative as tangent line | | | | |
| • 2.3B1 derivative as slope | 4.7 | 2.4, 2.5 | 2.1, 2.9 | 2.10, 5.6 |
| • 2.3B2 approximation | | | 2.9 | |
| 2.3C Derivatives as Related Rates | | | | |
| • 2.3C1 Motion | | 3.5, 3.6, 5.1 | 3.5, 3.65.1, 5.3 | |
| • 2.3C2 Related Rates | 3.3, 3.8 | 3.8 | 3.3, 3.8, 5.2 | |
| • 2.3C3 Optimization | | 3.1, 3.5 | 3.1 | |
| 2.3E solution of differential equations | 6.1 | | 6.1 | 6.8 |
| 2.3F Slope Field | 6.6 | 6.2, 6.4, 6.6 | 6.6 | |
| 2.4A Mean Value Theorem | 3.4 | 3.2, 3.4 | 3.2, 3.4 | |
| 3.2A Integral definition | | | | |
| • 3.2A1 Reimann Sum | 4.1, 4.6 | 4.1 | | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| • 3.2A2 Limit of Reimann Sum | 4.6 | | | 4.1 |
| 3.2B Approximations | 4.6 | 4.12 | 4.12 | 4.1 |
| 3.2C Properties from graph | | | | |
| • 3.2C1 Evaluate using Area | | 4.2, 4.3, 4.10, 7.7 | | 4.3 |
| • 3.2C2 Add/subtract Integrals | 4.5 | 4.5, 4.10, 7.7 | | |
| • 3.2C3 Removable/Jump Discontinuity | 4.5, 4.8, 5.1, 5.9 | 4.10, 5.1 | | |
| 3.3A Integral and Functions | | | | |
| • 3.3A1 Integral as functions | | 4.10 | | 4.7 |
| • 3.3A2 Fundamental theorem | 4.11 | 4.2 | 4.9 | 4.7 |
| 3.3B Computing integrals | | | | |
| • 3.3B1 Vocabulary | 4.2, 4.9 | | | |
| • 3.3B2 Definite | 4.8, 5.5, 5.9 | | | 4.4, 5.4 |
| • 3.3B3 Indefinite | 4.9, 5.3, 5.7 | | 4.9, 5.3 | |
| • 3.3B5 Algebraic manipulations | 4.8 | | | 4.4 |
| 3.4A Integral as the net change | | | 7.5, 7.6 | 7.6 |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 3.4B Integrals as average value | 7.8 | 7.4, 7.12 | 7.4 | |
| 3.4C Integral and motion | | 7.7, 7.8 | 7.7, 7.8 | |
| 3.4D Area/Volume of regions | | | | |
| • 3.4D1 Area of region | | 7.2 | | |
| • 3.4D2 Volume | 7.1, 7.3 | 7.1, 7.3 | | |
| 3.5A Separable DEs | | | | |
| • 3.5A1 Initial values | 6.3 | | 6.3, 6.5 | |
| • 3.5A2 Separation of variables | 6.3, 6.7 | | 6.5 | |
| • 3.5A3 Domain restrictions | 6.6 | 6.6 | | |
| • 3.5A4 General vs particular soln | 6.3, 6.7 | 6.2, 6.4 | 6.3, 6.5 | |
| 3.5B Exponential Growth and Decay | | 6.4 | 6.3 | |

# APPENDIX H: CONTENT MATRIX FOR GHS CUMULATIVE ASSESSMENTS

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 1.1A Definition/Existence of limits | | | | |
| • 1.1A1 Notation | 1.2, 1.4 | | | |
| • 1.1A2 Concept of limits | 1.2, 1.3, 1.12, 1.15 | 1.9 | | |
| 1.1C Algebraic Rules for Limits | | | | |
| • 1.1C1 Sums and Products | | 1.11 | | |
| • 1.1C2 Algebraic manipulation | 1.8, 1.16, 1.18, 1.20 | | | |
| • 1.1C3 L'Hopitals rule | | | | |
| 1.1D Limits Relate to Functions | | | | |
| • 1.1D1 Asymptotes | 1.17, 2.6, 3.3 | | | |
| • 1.1D2 Magnitude, End Behavior | 1.6, 1.7 | | | |
| 1.2A Limits and continuity | | | | |
| • 1.2A1 Continuity at a point | 1.4, 1.5, 1.14 | 1.10, 1.13 | | |
| • 1.2A3 Types of Discontinuities | | | 19 | |
| 1.2B Continuity condition for theorem | 3.7 | | | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 2.1A Definition of derivative | | | | |
| • 2.1A2 Limit definition at a point | | | | |
| • 2.1A3 Limit definition of function | | | | |
| • 2.1A4 Notational variety | 2.2, 2.5, 2.7 | | | |
| 2.1B Estimation of derivatives | | | | |
| 2.1C Rules for calculating derivatives | | | | |
| • 2.1C2 Solving directly | 2.5, 2.7, 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.12, 5.13, 5.14 | | | |
| • 2.1C3 Product/Quotient Rule | 2.2, 2.10, 2.12 | | | |
| • 2.1C4 Chain Rule | 2.9, 2.11 | | | |
| • 2.1C5 Implicit Differentiation | 2.8, 2.13 | | | |
| • 2.1C6 Inverse Function | | | | |
| 2.1D Higher order derivatives | | | | |
| • 2.1D1 Definition | 2.1, 2.11, 2.12 | | | |
| • 2.1D2 Notational variety | 2.1, 2.11 | | | |

| Topic | Routine | External | Application | Nonroutine |
| --- | --- | --- | --- | --- |
| 2.2A Derivatives relationship to a graph | 3.2, 3.5, 3.9, 3.12, 3.14 | 3.11 | | |
| 2.2B Continuity and derivative | | | | |
| • 2.2B1 Points that fail differentiabilty | | | | |
| • 2.2B2 Differentiabilty implies continuity | | | | |
| 2.3B Derivative as tangent line | | | | |
| • 2.3B1 derivative as slope | 2.3, 2.4, 2.12 | | | |
| • 2.3B2 tangent line as approximation | | | | |
| 2.3C Derivatives as Related Rates | | | | |
| • 2.3C1 Motion | | | | |
| • 2.3C2 Related Rates | | | | |
| • 2.3C3 Optimization | 3.4 | | | |
| 2.3E solution of differential equations | | | | |
| 2.3F Slope Field | | 6.2, 6.6, 6.10 | | |
| 2.4A Mean Value Theorem | 3.1, 3.15 | 3.6 | | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 3.2A Integral definition | | | | |
| • 3.2A1 Reimann Sum | | | | |
| • 3.2A2 Limit of Reimann Sum | | | | |
| 3.2B Approximations | | 4.7, 4.8 | | |
| 3.2C Properties from graph | | | | |
| • 3.2C1 Evaluate using Area | | 4.B | | |
| • 3.2C2 Add/subtract Integrals | | | | |
| • 3.2C3 Removable/Jump Discontinuity | | | | |
| 3.3A Integral and Functions | | | | |
| • 3.3A1 Integral as functions | | | | |
| • 3.3A2 Fundamental theorem | | | | |
| 3.3B Computing integrals | | | | |
| • 3.3B1 Vocabulary | | | | |
| • 3.3B2 Definite | 4.1, 4.2, 4.5, 4.9, 4.12 | | | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| • 3.3B2 Definite | 4.1, 4.2, 4.5, 4.9, 4.12 | | | |
| • 3.3B3 Indefinite | 4.3, 4.4, 4.6, 4.13, 5.7, 5.8, 5.9, 5.10, 5.11 | | | |
| • 3.3B5 Algebraic manipulations | 4.3, 4.4, 4.5, 4.6, 4.13 | | | |
| 3.4A Integral as the net change | | | | |
| 3.4B Integrals as average value | 4.10 | | | |
| 3.4C Integral and motion | | | | |
| 3.4D Area/Volume of regions | | | | |
| • 3.4D1 Area of region | | 7.3, 7.4, 7.6, 7.7, 7.8 | | |
| • 3.4D2 Volume | | 7.1, 7.2, 7.5, 7.8 | | |
| 3.5A Separable DEs | | | | |
| • 3.5A1 Initial values | 6.10 | | 6.7, 6.9 | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| • 3.5A2 Separation of variables | 6.3, 6.4, 6.5, 6.9, 6.10 | | | |
| • 3.5A3 Domain restrictions | | | | |
| • 3.5A4 General vs particular soln | 6.1, 6.3, 6.4, 6.5, 6.9 | | | |
| 3.5B Exponential Growth and Decay | 6.7, 6.8 | | | |

# APPENDIX I: CONTENT MATRIX FOR AP FRQ'S 2007-2016

| Topic | Routine | Graphical | Application | Nonroutine |
|---|---|---|---|---|
| 1.1A Definition/Existence of limits | | | | |
| • 1.1A2 Concept of limits | | | | |
| • 1.1A3 Limits that do not exist | | | | |
| 1.1C Algebraic Rules for Limits | | | | |
| • 1.1C1 Sums and Products | | | | |
| • 1.1C2 Algebraic manipulation | | | | |
| • 1.1C3 L'Hopitals rule | | | | |
| 1.1D Limits Relate to Functions | | | | |
| • 1.1D1 Asymptotes | 8.6 | | | |
| • 1.1D2 Magnitude, End Behavior | 8.5 | | | |
| 1.2A Limits and continuity | | | | |
| • 1.2A1 Continuity at a point | 11.6, 12.4 | | | |
| • 1.2A2 Continuous functions | | | | |
| • 1.2A3 Types of Discontinuities | | | | |
| 1.2B Continuity and theorems | | | | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| 2.1A Definition of derivative | | | | |
| • 2.1A2 Limit definition at a point | | | | |
| • 2.1A3 Limit definition of function | | | | |
| • 2.1A4 Notational variety | | | | |
| 2.1B Estimation of derivatives | 8.2, 9.4, 10.2, 11.2, 12.1, 13.3, 15.3, 16.1 | 11.4 | 8.2, 12.1, 14.1, 14.4, 16.1 | |
| 2.1C Rules for calculating derivatives | | | | |
| • 2.1C2 Solving directly | 11.6 | | | |
| • 2.1C3 Product/Quotient Rule | | | | 14.3, 16.6 |
| • 2.1C4 Chain Rule | 12.4 | | | |
| • 2.1C5 Implicit Differentiation | | | | |
| • 2.1C6 Inverse Function | | | | 7.3 |
| 2.1D Higher order derivatives | 11.5, 15.6 | | | |
| • 2.1D1 Definition | | | | |

| Topic | Routine | External | Application | Nonroutine |
| --- | --- | --- | --- | --- |
| 2.2A Derivatives relationship to graphs | 8.6, 15.4 | 9.6, 10.3, 10.5, 11.4, 12.3, 12.5, 13.4, 14.3, 15.5, 16.3 | 7.5, 9.1, 9.6, 10.3,15.1 | 7.6, 10.5, 12.5, 13.4, 14.5 |
| 2.2B Continuity and derivative | | | | |
| • 2.2B1 Points not differentiable | | | | |
| • 2.2B2 Differentiable is continuous | | | | |
| 2.3B Derivative as tangent line | | | | |
| • 2.3B1 derivative as slope | 7.3, 10.6, 11.3, 12.4, 13.4, 15.6 | | | 14.3, 15.6, 16.6 |
| • 2.3B2 approximations | 7.5, 8.6, 10.6, 13.6, 14.6, 16.4 | | 7.5, 9.5, 10.6, 11.5 | |
| 2.3C Derivatives as Related Rates | | | | |
| • 2.3C1 Motion | | 8.4, 9.1 | 7.4, 11.1, 12.6, 13.2, 14.4, 15.3, 16.2 | |

| Topic | Routine | External | Application | Nonroutine |
| --- | --- | --- | --- | --- |
| • 2.3C2 Related Rates | 13.1 | 15.2 | 7.2, 7.5, 8.3, 10.2, 12.5, 13.1, 13.3, 14.1, 16.5 | |
| • 2.3C3 Optimization | | 7.2, 10.3 | 7.3, 7.4, 8.3, 9.2, 9.3 | |
| 2.3E solution of DEs | 7.4 | | | 15.4 |
| 2.3F Slope Field | | 8.5, 14.6, 15.4, 16.4 | | |
| 2.4A Mean Value Theorem | 7.3, 8.2 | | 13.3, 14.5, 16.1 | |
| 3.2A Integral definition<br>• 3.2A1 Understand Reimann Sum<br>• 3.2A2 Integral as Reimann Sum | | | | |
| 3.2B Approximations | 7.5, 8.2, 9.5, 10.2, 13.3 | 15.3, 16.1 | 12.1, 16.1 | |
| 3.2C Properties from graph | | | | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| • 3.2C1 Evaluate using Area | | 8.4, 9.6, 10.5, 11.4 | | |
| • 3.2C2 Add/subtract Integrals | | 12.3, 14.3 | | |
| • 3.2C3 Removable/Jump Discontinuity | 10.1 | | | |
| 3.3A Integral and Functions | | | | |
| • 3.3A1 Integral can be a function | 7.5 | 16.3 | 7.5 | |
| • 3.3A2 Fundamental theorem of Calc | 12.1, 15.5 | | 9.2, 11.2 | 7.3, 9.5, 14.5, 16.6 |
| 3.3B Computing of integrals | | | | |
| • 3.3B2 Definite | | | | |
| • 3.3B3 Indefinite | | | | |
| • 3.3B5 Algebraic manipulations | 12.4 | | | |
| 3.4A Integral as the net change | | 10.3 | 7.2, 8.2, 8.3, 9.2, 9.3, 10.1, 10.2, 12.1, 13.1, 15.1, 16.1 | |

| Topic | Routine | External | Application | Nonroutine |
| --- | --- | --- | --- | --- |
| 3.4B Integrals as average value | 11.6 | | 8.2, 9.2, 10.2, 11.2, 13.3, 14.1, 15.3, 16.5 | |
| 3.4C Integral and to motion | | | 9.1, 11.1, 12.6, 13.2, 14.4, 15.3, 16.2 | |
| 3.4D Area/Volume of regions | | | | |
| • 3.4D1 Area of region | | 7.1, 8.1, 9.4, 10.4, 11.3, 12.2, 13.5, 14.2, 15.2 | | |
| • 3.4D2 Volume | | 7.1, 8.1, 9.4, 10.4, 11.3, 12.2, 13.5, 14.2, 15.2, 16.5 | | |
| 3.5A Separable DEs | | | | |
| • 3.5A1 Initial values | 8.5, 14.6, 16.4 | | | |

| Topic | Routine | External | Application | Nonroutine |
|---|---|---|---|---|
| • 3.5A2 Separation of variables | 8.5, 10.6, 11.5, 12.5, 13.6, 14.6, 16.4 | | | |
| • 3.5A3 Domain restrictions | | | | |
| • 3.5A4 General vs particular soln | 10.6, 11.5, 12.5, 13.6, 14.6, 16.4 | | | |
| 3.5B Exponential Growth and Decay | | | | |

# APPENDIX J: ANSWER KEY FOR ASSESSMENTS

Unit 1

*Section 1*

1. Write an equation that summarizes the following information.

Solution: $\lim_{x\to -4} f(x) = 3$

2. Find $\lim_{x\to 5} \frac{6x-30}{x^2-25}$. Is the function continuous at $x = 5$? Explain.

Solution: $\lim_{x\to 5} \frac{6x-30}{x^2-25} = \lim_{x\to 5} \frac{6(x-5)}{(x-5)(x+5)} = \lim_{x\to 5} \frac{6}{x+5} = \frac{6}{10} = \frac{3}{5}$.
The function $\frac{6x-30}{x^2-25}$ is not continuous at $x = 5$ since $f(5) = \frac{0}{0}$=DNE and therefore $\lim_{x\to 5} f(x) \neq f(5)$. However, the graph of $\frac{6x-30}{x^2-25}$ has a removable discontinuity, meaning it is continuous everywhere around $(5, \frac{3}{5})$ but not at the point itself. The function approaches the point $(5, \frac{3}{5})$ from both sides, but there is a hole in the graph of the function $\frac{6x-30}{x^2-25}$ at this point.

3. $\lim_{x\to 1^+} h(x) =$

Solution: $\lim_{x\to 1^+} h(x) = -3$.

4. $\lim_{x\to 1} h(x) =$

Solution: $\lim_{x\to 1} h(x) = DNE$. $\lim_{x\to 1^+} h(x) = -3$ and $\lim_{x\to 1^-} h(x) = -1$, since h(x) approaches -1 from the left and -3 from the right, there is no value being squeezed in on. In other words, there is a gap or jump between the values of h(x) as x moves toward 1 from either side. This is classified as a jump discontinuity.

5. $\lim_{x\to -1^+} h(x) =$

Solution: $\lim_{x\to -1^+} h(x) = \infty$ or $DNE$. The limit does not exist because as $x$ gets closer to -1 from the right, the function h(x) continually get larger at an increasing rate. This is classified as an asymptote.

6. $\lim_{x\to 3}(h(x) + g(x)) =$

Solution: $\lim_{x\to 3}(h(x) + g(x)) = \lim_{x\to 3} h(x) + \lim_{x\to 3} g(x) = -3 + \frac{cos(-3\pi)}{-3} = -3 + \frac{1}{3} = -\frac{8}{3}$

7. Find all values of c, where both $h(c)$ and $\lim_{x\to c} h(x)$ exist but $\lim_{x\to c} h(x) \neq h(c)$.

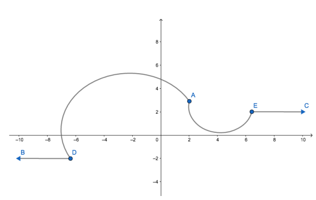
Solution: $c = -5$ $h(-5) = 3$ but $\lim_{x\to -5} h(x) = 2$. Therefore, while $h(-5)$ and $\lim_{x\to -5} h(x)$ exist, $h(-5) \neq \lim_{x\to -5} h(x)$.

8. Find the polynomial $2x^2 + bx + c$, such that $\lim_{x\to -2} \frac{2x^2+bx+c}{x^2+x-2} = \frac{7}{3}$.

Solution: $\lim_{x\to -2} \frac{2x^2+bx+c}{x^2+x-2} = \frac{7}{3}$ factoring the denominator, $\lim_{x\to -2} \frac{2x^2+bx+c}{(x+2)(x-1)} = \frac{7}{3}$ the numerator can likewise be factored by $(x-2)$, $\lim_{x\to -2} \frac{(x+2)(2x+g)}{(x+2)(x-1)}$. Now, we can cancel $x + 2$ and thus remove the discontinuity, leaving $\lim_{x\to -2} \frac{(2x+g)}{(x-1)}$ This can be evaluate at the point $x = -2$, $\frac{-4+g}{-3} = \frac{7}{3}$ by cross multiplication, $-4 + g = -7 \to g = -3$. This can be substituted into the factored version of the equation, $(2x - 3)(x + 2)$ and through distribution, we get $2x^2 - x - 6$. This means that $b = -1$ and $c = -6$.

9. A certain function on the coordinate axis has a limit of one as $x$ approaches infinity and a limit of negative two as $x$ approaches negative infinity. Additionally, the limit as $x$ approaches two is three. Sketch a possible drawing of the function on the coordinate axis below.

Solution: There are a multitude of graphs that fit the description above. Below is one example.



10. For what values of a and b is the function $f(x)$ continuous, where a and b are integers?

$$f(x)=\begin{cases} \text{a - x} & x \leq 1 \\ \frac{x}{b}+3 & x > 1 \end{cases}$$

Solution: The function is continuous which means that, $f(1) = f(1^+)$. $f(1) = a - 1$ and $f(1^+) = \frac{1}{b} + 3$, so $a - 1 = \frac{1}{b} + 3$. Simplifying, $a - 4 = \frac{1}{b} \rightarrow b(a - 4) = 1$ since a and b are integers $b = 1, -1$. Substituting the solutions for b into the equation we can solve for a. $1(a - 4) = 1, -1(a - 4) = 1 \rightarrow (a, b) = (5, 1), (3, -1)$.

11. Suppose that the revenue (in millions of dollars) that a certain band makes while touring can be modeled by the function $r(t) = \frac{82t^2-3t+7}{2t^2+6t}$ where $t$ represents the number of months the band is on tour. If the band continues to tour, what will they make ultimately?

Solution: $\lim_{t\to\infty} \frac{82t^2-3t+7}{2t^2+6t} = \frac{82}{2} = 41$ million dollars.

12. If the graph of $\frac{ax^2+b}{x^2+c}$ has a horizontal asymptote at $y = 3$ and a vertical asymptote at $x = 4$, find $a - c$.

Solution: vertical asymptote:$x^2 + c = 0 \rightarrow c = -16$

horizontal asymptote:$\lim_{x\to\infty} \frac{ax^2+b}{x^2-16} = 3 \rightarrow a = 3$

$a - c = 3 - (-16) = 19$

Unit 2

*Section 2*

1. Let $f(x) = x^2 - 2x + 3$. Use the definition of derivative to find $f'(-3)$. Explain what your solution means in terms of the graph of $f(x)$.

Solution: $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) + 3 - (x^2 - 2x + 3)}{h}$ distributing, we get $\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x + 2h + 3 - -x^2 + 2x - 3}{h}$. By combining like terms, $\lim_{h \to 0} \frac{h^2 + 2xh + 2h}{h}$. The numerator can be factored to $\lim_{h \to 0} \frac{h(h+2x+2)}{h}$ and then $h$ can be canceled. $\lim_{h \to 0}(h + 2x + 2) = 2x + 2$

$f'(-3) = 2(-3) + 2 = -4$ The slope of the line tangent to the function f(x) at the point (-3, 18) is -4.

2. Find $\lim_{x \to 0} \frac{3sin(3x)}{2x}$

Solution. $\lim_{x \to 0} \frac{3sin(3x)}{2x} = \frac{0}{0}$ This is indeterminate, meaning that the numerator and denominator are changing but we can not determine the proportionality with which they change in this current form. We will use L'Hopital's rule to better examine the way the rates change comparatively. $\lim_{x \to 0} \frac{3sin(3x)}{2x} = \lim_{x \to 0} \frac{9cos(3x)}{2} = \frac{9 \cdot 1}{2} = \frac{9}{2}$.

3. Find $\frac{d^2}{dx}(3sin^2(2x)$.

Solution:$\frac{dy}{dx} = (2 \cdot 3sin(2x) \cdot (cos(2x)) \cdot 2) = 12sin(2x)cos(2x)$

$\frac{dy^2}{d^2x} = 12sin(2x) \cdot (-sin(2x)) \cdot 2 + 12cos(2x) \cdot cos(2x) \cdot 2 = -24sin^2(2x) + 24cos^2(2x) = 24(cos^2(2x) - sin^2(2x))$

4. Let $g(x) = h(f(x))$. Find $g'(-3)$.

Solution: $g'(x) = h'(f(x)) \cdot f'(x); g'(-3) = h'(f(-3)) \cdot (f'(-3)) = h'(0) \cdot (-2) = -\frac{1}{2} \cdot (-2) = 1$

5. Let $k(x) = \frac{h(3x)}{f(x)}$. Find $k(-1)$.

Solution:$k'(x) = \frac{f(x)\cdot h'(3x)\cdot 3 - f'(x)\cdot h(x)}{(f(x))^2} = \frac{3f(x)h'(3x) - f'(x)h(3x)}{(f(x))^2}$ $k'(-1) = \frac{3f(-1)h'(-3) - f'(-1)h(-3)}{(f(-1))^2} = \frac{(3\cdot 8\cdot 1)-(4\cdot 2)}{(4)^2} = \frac{16}{16} = 1$

6. Estimate $f''(-\frac{1}{2})$

Solution: $f''(-\frac{1}{2}) \approx \frac{f'(0)-f'(-1)}{0-(-1)} = \frac{-1-4}{1} = -5$

7. Let $p(x) = f^{-1}(x)$. Find $p'(0)$.

Solution: $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(-3)} = -\frac{1}{2}$

8. As part of a cyber Monday special, an electronics store started selling smartphones online at midnight and sold out 8 hours later. The function S(t) represents the number of smartphones the store has in stock at time $t$, where t represents the hours after the sale began. Values of S(t) at various times $t$ are shown in the table above. Use the data in the table above to estimate $S'(3)$. Explain what the result means in context.

Solution: $S'(3) \approx \frac{f(4)-f(2)}{4-2} = \frac{610-724}{2} = \frac{-114}{2} = -57$
This means that the average rate that smartphones are sold at 3am was 57 smartphones per hour.

9. At Willy Wonka's chocolate factory a certain vat contains 150 liters of corn syrup. More corn syrup is pumped into the vat. The function $C$ models the amount of corn syrup in the vat after $t$ minutes. The umpa lumpas believe that $C$ satisfies the differential equation $\frac{dC}{dt} = \frac{1}{10}(C - 50)$ for as long as the corn syrup is being pumped. Use the line tangent to the graph of $C$ at t = 0 to approximate the amount of corn syrup in the vat after the pump has run for 2 minutes and 30 seconds (make sure to include appropriate the units).

Solution: $C(0) = 150$; $\frac{dC(0)}{dt} = \frac{1}{10}(150 - 50) = 10$; $m = 10$

Tangent line: $y = 10x + 150$;   $y = 10(2.5) + 150 = 175$ liters

10. Let $(x + y)^2 + y^2 = 8$. Find all points on the curve at which the tangent line to the curve at that point is vertical.

Solution:The derivative of the curve: $2 \cdot (x + y) \cdot (1 + \frac{dy}{dx}) + 2y\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{x+y}{x+2y}$

We are looking for where the tangent line is vertical, that is, where the slope is undefined, thus the denominator equals zero, $x + 2y = 0 \rightarrow x = -2y$

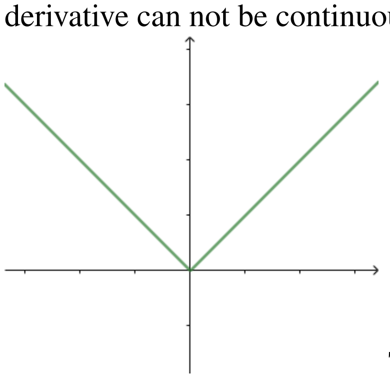
Substituting this into the curve, $(-2y + y)^2 + y^2 = 8 \rightarrow 2y^2 = 8 \rightarrow y = 2, -2$

$x = -2(2)$ and $x = -2(-2)$ generating the points (4,-2),(-4,2).

The lines tangent to the curve at (4,-2) and (-4,2) is vertical.

11. Sketch the graph of a function f(x) that is continuous but not differentiable. Write a sentence(s) explaining why the function is continuous but not differentiable. Is it possible for the function to be differentiable but not continuous at a point?

Solution: Students should have a continuous function (no jumps, etc.) that has a cusp for which a derivative can not be continuous at a point. Sample solution:



This function above, f(x) is continuous for all points, however, the function is not differentiable at x = 0, since the $\lim_{x\to0^-} f'(x) = -1$ but $\lim_{x\to0^+} f'(x) = 1$, so $\lim_{x\to0^-} f'(x) \neq \lim_{x\to0^+} f'(x)$. Since the function is not differentiable at x=0, the derivative function f'(x) is not continuous at x=0. If a function were differentiable then that implies it is also continuous so it is not possible for the function to be differentiable but not continuous at a point.

12. Suppose $f(x)$ is differentiable at the point $x = 2$. Which of the following must be true? Justify.

A)$\lim\limits_{h\to 0} \frac{f(2+h)-f(2)}{h}$

B)$\lim\limits_{h\to 0} \frac{f(2-h)-f(2)}{h}$

C)$\lim\limits_{a\to 2} \frac{f(a+h)-f(a-h)}{h}$

D) $\lim\limits_{x\to 2} \frac{f(a+h)-f(a)}{h}$

E)$\lim\limits_{x\to 2} \frac{f(x)-f(2)}{x-2}$

Solution.

A) This must be true as it measures the instantaneous rate of change at x=2. In this equation h is the distance a point is from 2, as this goes to zero, the instantaneous change generated.

B) This must be true as it measures the instantaneous rate of change at x=2. In this equation h is the distance a point is from 2, as this goes to zero, the instantaneous change generated.

C) This need not be true as h, the distance a point is from $a$ as $a$ goes to 2, need not approach zero, thus an average of some sort is calculated.

D) This need not be true as h, the distance a point is from $a$ as $x$ goes to 2, need not approach zero, thus an average of some sort is calculated.

E) This must be true as it measures the instantaneous rate of change at x=2. This is the average rate of change as x get infinitely close to 2.

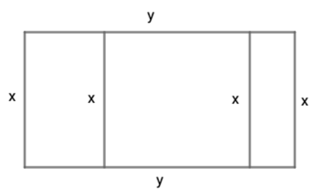
13. Can you use L'hopital's rule to compute $\lim\limits_{x\to 3} \frac{cos(\pi x)}{x-3}$? If so, evaluate the limit, if not, explain.

Solution: $\lim\limits_{x\to 3} \frac{cos(\pi x)}{x-3} = \frac{-1}{0}$. L'Hopital's rule can only be employed when the limit evaluates to a fraction in indeterminate form, e.g., $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $\frac{-\infty}{-\infty}$. Therefor the conditions are not met and L'Hopital's rule can not be used.

*Section 3*

1. A farmer wants to construct a large rectangular fence. He wants to subdivide the enclosed area into 3 smaller, parallel rectangles using the same fencing. His goal is to enclose as much area as possible. If the farmer has 1200 feet of fencing available, what are the optimal dimensions of the larger rectangular fence?

Solution:




Total fencing needed is represented by 4x + 2y = 1200. We want to optimize the Area = xy. To do this, we need to get the equation in terms of one variable $4x + 2y = 1200 \rightarrow y = 600 - 2x$. Substituting into the area equation $A = x(600-2x) = -2x^2+600x$ Taking the derivative and setting this equal to zero, we get $A' = -4x + 600 = 0 \rightarrow x = 150$, substituting $y = 600 - 2(150) = 300$. The dimensions of the optimal rectangle are 300ft by 150ft.

2. Find the average rate of change of $f$ from $-2 \leq x \leq 4$ using the graph above. Mark believes that there does not exist a value $c$ such that $f'(c)$ is equal to the average rate of change. Is Mark correct? If so, does this contradict the Mean Value Theorem? Explain.

Solution: $\frac{f(4)-f(-2)}{4-2} = \frac{1}{6}$. Note from the graph that the slope (rate of change) is $\frac{3}{4}$ from 0 <x <2. Then from 2<x <4, the rate of change is always negative. So Mark is correct that there is no point, c, for which $f'(c) = 16$. While this does contradict the conclusion of the MVT, the conditions to apply the MVT are not met so the theorem does not apply. Namely, for the MVT to apply to a situation the function must be differentiable everywhere on the interval. However, $f$ is not differentiable at 2.

3. An hourglass consists of two right cones. Sand falls through the top cone into the bottom cone, filling the cone at a rate of 1.2 $cm^3/min$. The hourglass is constructed such that height of the sand is always equal to one-fourth of the diameter of the base in the bottom portion of the hourglass. Find how fast the height is changing when the height is 3 cm. (Volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

Solution: $\frac{dV}{dt} = 1.2 \qquad h = \frac{1}{4}2r = \frac{1}{2}r \rightarrow r = 2h$
$V = \frac{1}{3}\pi(2h)^2h = \frac{4\pi}{3}h^3$. Taking the derivative, we get $\frac{dV}{dt} = 4\pi h^2\frac{dh}{dt} = 1.2 \rightarrow \frac{dh}{dt} = \frac{1.2}{4\pi h^2}$ when h = 3, $\frac{dh}{dt} = \frac{1.2}{36\pi} = \frac{1}{30\pi}$ cm/min

4. Is there a time between 30 and 40 minutes where the rate at which the water is pumped out of the cistern is 4.5 liters/minute? Justify your answer.

Solution: $\frac{W(40)-W(30}{40-30} = \frac{45}{10} = 4.5$ Since the functions is twice differentiable, it is continuous everywhere on the derivative. So by the Mean Value Theorem, we know that there exists a time $t$ where 30 <t <40 such that $R(t) = 4.5$ liters/min.

5. At what time on the interval $0 \leq t < 3$ is the taxi furthest from the airport? Explain your answer.

Solution: $v(t) = 0$ at t = 2. Since velocity is the derivative of position, when the velocity equals zero we have a potential maximum. This occurs after 2 minutes. The other potential maximum values occur at the endpoints of closed intervals but at time $t = 0$, the cab is at the ariport and thus the distance away form the airport is zero, which can not be the furthest. This means that after 2 minutes the car is further from the airport than at any other time between the start until 3 minutes (non-inclusive).

6. On what intervals is the car slowing down? Explain.

Velocity and acceleration both have a magnitude and direction, if the velocity and acceleration are moving in the same direction, the car is speeding up. This makes sense because if the velocity is positive and then the acceleration is also positive, the acceleration would contribute to the positivity (working with the velocity) and thus increase the speed. If acceleration is working against velocity, i.e. going the opposite way, the car is slowing down. Likewise if the velocity is negative but the acceleration is positive, then the negative velocity is being acted on in the opposite manner, which makes it less negative, so the car is moving in the negative direction slower. Thus the car is slowing down on the intervals (1,2)U(3,4).

7. The function $f(x) = x^3 + bx^2 + cx$ has a local maximum of 9 at $x = -3$. Find the local minimum.

Solution: The function has a maximum at x =-3, so $f'(-3) = 0 \quad \rightarrow f'(-3) = 3(-3)^2 + 2b(-3) + c = 0 \quad \rightarrow -6b + c = -27 \quad \rightarrow c = -27 + 6b$

The maximum is 9, so $f(-3) = 9 \quad \rightarrow (-3)^3 + b(-3)^2 + c(-3) = -27 + 9b - 3c = 9 \quad \rightarrow 9b - 3c = 36$

Using substitution $9b - 3(-27 + 6b) = 36 \quad \rightarrow 9b + 81 - 18b = 36 \quad \rightarrow -9b = -45 \quad \rightarrow b = 5$ and by substitution $c = 3$. So the original equation is $f(x) = x^3 + 5x^2 + 3x$.

Now we will find the minimum by taking the derivative and setting it equal to zero, $f'(x) = 3x^2 + 10x + 3 = 0 \quad \rightarrow (x + 3)(3x + 1) = 0 \quad \rightarrow x = -3, -\frac{1}{3}$.

Note that a possible minimum is at $-\frac{1}{3}$. The second derivative is $f''(x) = 6x + 10$, $f''(-\frac{1}{3}) = -2 + 10 = 8 > 0$, thus there is a minimum at $x = -\frac{1}{3}$. To find the value of the minimum, substitute $x = -\frac{1}{3}$ into f(x), $f(\frac{1}{3}) = (-\frac{1}{3})^3 + 5(-\frac{1}{3})^2 + 3(-\frac{1}{3}) = -\frac{1}{27} + \frac{5}{9} - 1 = -\frac{13}{27}$

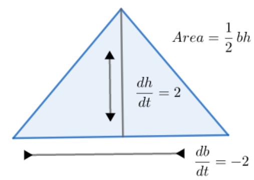
8. The base of a triangle is decreasing at a rate of 2 centimeters per minute while its height is increasing at a rate of 2 centimeters per minute. Which of the following is true about the area of the triangle? Show the work that leads to your answer.

A) The area is always increasing B) The area is always decreasing

C) The area is increasing only when b<h D) The area is increasing only when h>b

E) The area remains constant

Solution:




$\frac{dA}{dt} = \frac{1}{2}(b \cdot \frac{dh}{dt} + h\frac{db}{dt} = \frac{1}{2}(2h - 2b) \rightarrow \frac{1}{2}(2h - 2b)$. This means that the rate of change of the area depends on both $h$, the height, and $b$, the base. This eliminates A, B and E, as these answers express an understanding of area that does not depend on $h$ or $b$. To see when the area is increasing, $\frac{1}{2}(2h - 2b) > 0 \rightarrow h > b$. This corresponds to answer choice D.

9. The six graphs below represent three functions and their corresponding derivatives. Match the function to its derivative and briefly justify your choice. Make sure to specify which graph is the original function and which is the derivative.

Solution: Graph F shows the original function and B shows its derivative. Graph F has a function with horizontal tangent lines around $x \approx -1.5, .8$. Graph B has zeros at $x \approx -1.5, .8$.

Graph E shows the original function and D shows its derivative. Graph E has a function with horizontal tangent lines around $x = 0$. Graph D has zeros at $x = 0$. This is not enough since both function are actually 0 at 0. Graph E has a negative tangent lines from -2 to 0. Graph D is negative on the interval (-2,0).

Graph A shows the original function and C shows its derivative. Graph A has a function with horizontal tangent lines around $x \approx -2.2, 2.2$. Graph C has zeros at $x \approx -2.2, 2.2$.

Unit 3

*Section 4*

1. Let $f(x)$ be a continuous, positive, concave up, increasing function for all $x \geq 0$. Rank the following in increasing order and explain your reasoning:

A) $\int_0^5 f(x)dx$ B) $\sum_{i=1}^{5} f(x_i)$ C) $\sum_{i=0}^{4} f(x_i)$ D) $\sum_{i=1}^{5} \frac{f(x_i)+f(x_{i-1})}{2}$

Solution:

A) gives the exact solution for the area under the curve

B) gives a right hand approximation which overestimates the area

C) gives a left hand approximation which underestimates the area

D) gives a midpoint approximation that underestimates the area but by less than the left hand point does

Therefore, C <D <A <B

2. Let $f(x) = \int_0^x g(x)dx$ and let $g(x)$ be the function shown in the graph above. Find $f(6)$, $f'(4)$ and $f''(3)$.

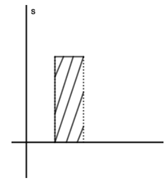
Solution: $f(6) = \int_0^6 g(x)dx = \int_0^4 g(x)dx + \int_4^6 g(x)dx = 4 + -3 = 1$

$f'(4) = g(4) = -2$

$f''(3) = g'(3) =$ slope of tangent line at $g(3) = -2$

3. Suppose that $h$ is a continuous function on the interval [1,3] and that $2 \leq h(x) \leq 6$. What is the greatest possible value of $\int_1^3 h(x)dx$?

Solution: The area under the curve is largest when functions is greatest i.e. $h(x) = 6$. So the largest area would be the rectangle created by a base of (3-1) and height of 6 = 12.

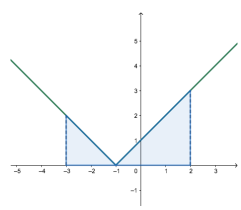


4. Find $\int_{-4}^{2}(f(x+1)-g(x))dx =$

Solution: $\int_{-4}^{2}(f(x+1)-g(x))dx = \int_{-4}^{2} f(x+1)dx - \int_{-4}^{2} g(x))dx = \int_{-3}^{3} f(u)du - \int_{-4}^{2} g(x))dx = \int_{-3}^{0} f(u)du + \int_{0}^{3} f(u)du - \int_{-4}^{2} g(x))dx = -\int_{0}^{-3} f(u)du + \int_{0}^{3} f(u)du - \int_{-4}^{2} g(x))dx = -(-7)+2-3 = 6$

5. $\int_{-3}^{2} |x+1|dx$

Solution: The graph of the function reveals that the area is apportioned in two triangles.



6. Express $\lim\limits_{max\Delta x_i \to 0} 9\sum\limits_{i=1}^{\infty} x_i^2 \Delta x_i$ where $\Delta x_i = \frac{2-0}{n}$ as an integral expression. Then, use a right Riemann sum with three sub-intervals to estimate it.

Solution: $\lim\limits_{max\Delta x_i \to 0} 9\sum\limits_{i=1}^{\infty} x_i^2 \Delta x_i = \int_{0}^{2} 9x^2 dx$
Partitioning the base of 2 into 3 subintervals means each with have a width of $\frac{2}{3}$. The right endpoints are $\frac{2}{3}$, $\frac{4}{3}$ and 2, with the respective heights of 4, 16, and 36. Thus the estimation is $\frac{2}{3}(4+16+36) = \frac{122}{3}$

7. Suppose that $f$ has a positive derivative for all $x$ and that $f(1) = 0$. Which of the following statements must be true of the function $g(x) = \int_{0}^{x} f(t)dt$. Justify your answers.

A) $g$ is a differentiable function of $x$

B) $g$ is a continuous function of $x$

C) The graph of $g$ has a horizontal tangent at $x = 1$

D) $g$ has a local maximum at $x = 1$

E) The graph of $g$ has an inflection point at $x = 1$

F) The graph of $\frac{dg}{dx}$ crosses the $x$-axis at $x = 1$

Solution: A) True. $g'(x) = f(x)$ and since f is differentiable and therefore continuous, g is differentiable.

B) True. $g$ is differentiable from (A) which means is it continuous.

C) True. $g'(1) = f(1) = 0$ the slope of the tangent line at $x = 1$ is zero so $g$ has a horizontal tangent at $x = 1$.

D) False. $g'(1) = f(1) = 0$, so there is an extrema at $x = 1$. The graph of g is also concave up (since $g''(1) = f'(1) > 0$) so it is a minimum not a maximum.

E) False. $g''(1) = f'(1) > 0$ so $g''(0) \neq 0$ so there is not an inflection point at $x = 1$

F)True. $\frac{dg}{dx} = f(x)$ and since $g(x)$ has a minimum at 1, the $\frac{dg}{dx} < 0$ just before $x = 1$ and $\frac{dg}{dx} > 0$ just after $x = 1$, so it crosses the x-axis at x=1.

8. $\int_0^5 \frac{4x^2-1}{2x-1}dx$

Solution: $\frac{4x^2-1}{2x-1}$ is discontinuous at $x = \frac{1}{2}$ but $\frac{4x^2-1}{2x-1} = \frac{(2x+1)(2x-1)}{2x-1} = (2x-1)$ where $x \neq \frac{1}{2} \rightarrow \int_0^5 \frac{4x^2-1}{2x-1}dx = \int_0^5 (2x+1)dx$ where $x \neq \frac{1}{2}$. But since $\int_{\frac{1}{2}}^{\frac{1}{2}} g(x)dx = 0$, we can eliminate the point $x = \frac{1}{2}$ without changing the area. So $\int_0^5 (2x+1)dx = x^2 + x\Big|_0^5 = 30$

9. Find the anti-derivative of $c(t)$. Where $c(t)$ models the number of cars per hour that were washed at a certain car wash and $c(t) = 4t^3\sqrt{2t^4+7}dt$. Explain the result in terms of the context.

Solution: $\int 4t^3\sqrt{2t^4+7}dt$ making a u substitution $\rightarrow u = 2t^4+7, du = 8t^3dt \quad \rightarrow \int 4t^3\sqrt{2t^4+7}dt = \frac{1}{4}\int u^{\frac{1}{2}}du = \frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}} + C =$ resubstituting for u, $\frac{1}{6}(2t^4+7)^{\frac{3}{2}} + C$. The rate of change, given in cars per hours, is the derivative of the number of cars with respect to time. We are looking at cars getting washed and the rate tells us how many per hour. The antiderivative reverses what the derivative does, so instead of looking at the rate of cars per hour we found a function for cars that were washed at a time *t*.

10. Set up an integral expression that represents the area under the curve in the following graph. then evaluate the expression.

Solution: $\int_a^b r dx + \int_b^c s dx$

$rx\Big|_a^b + sx\Big|_b^c = rb - ra + sc - sb$

11. Evaluate $\frac{d}{dx}\int_\pi^{x^2} sin(t^2)dt$

Solution: $2xsin(x^4)$

12. Estimate the amount of rain that fell from 8:00 am to 2:00 pm using a left Riemann sum.

Solution: $(r(t))(\Delta t) = (1.2)(1.5) + (1)(2) + (3)(1.5) + (1.8)(1) = 3.75 + 2 + 4.5 + 1.8 = 12.05$. Approximately 12.05 inches of rain fell from 8 am to 2 pm.

*Section 5*

1. A rocket is launched straight up into the air. Its velocity was recorded to be $v(t) = ln(lnt)$ on the interval $1 < t < 4$. Find a function for the acceleration, is this function continuous? Explain.

Solution: $a(t) = v'(t) = \frac{1}{lnt} \cdot \frac{1}{t} = \frac{1}{tlnt}$. The function $v(t)$ only holds for $1 < t < 4$ so $a(t) = \frac{1}{tlnt}$ when $1 < t < 4$. $a(t)$ has a discontinuity when $t = 0$, but that is not in the interval that it is defined for. In the interval where $a(t)$ exists, it is continuous.

2. Suppose that a car drives on a straight road in one direction such that the position of the car, at a given time, $t$, is given by the function $x(t) = 5^{t^2-1}$, where t is in hours. The number of gallons of gas used by the car to travel $x$ miles is given by the function $\frac{dg}{dx} = \frac{2}{3xln5}$ gallons per mile. How many gallons per hour is the car getting when t = 2, indicate the units.

Solution: $\frac{dx}{dt} = 5^{t^2-1} \cdot ln5 \cdot 2t$ $\quad \frac{dg}{dt} = \frac{dg}{dx} \cdot \frac{dx}{dt}$ $\quad x(2) = 5^{2^2-1} = 125$

$\frac{dg}{dt}\Big|_{t=2} = \frac{dg}{dx}\Big|_{x=125} \cdot \frac{dx}{dt}\Big|_{t=2} \rightarrow \frac{2}{375ln5} \cdot 500ln5 = \frac{1000ln5}{375ln5} = \frac{8}{3}$ gallons per hour.

3. The acceleration of a particle along the x-axis is given by $a(t) = \frac{4}{\sqrt{49-16t^2}}$. Find the velocity.

Solution: $v(t) = \int a(t)dt = \int \frac{4}{\sqrt{49-16t^2}} = 4arcsin\frac{4t}{7} + C$

4. Find the value of $a$, $a$>1, for which $\int_1^a \frac{1}{1+x^2}dx = \frac{\pi}{12}$.

Solution:$\int_1^a \frac{1}{1+x^2}dx = tan^{-1}x\Big|_1^a = \frac{\pi}{12} \quad \rightarrow tan^{-1}a - \frac{\pi}{4} = \frac{\pi}{12} \quad \rightarrow tan^{-1}a = \frac{\pi}{3} \quad \rightarrow a = tan(\frac{\pi}{3})$

$\rightarrow a = \sqrt{3}$

5. $\int_0^1 \frac{x}{x^2+1}dx$

Solution: $\int_0^1 \frac{x}{x^2+1}dx \rightarrow$ let $u = x^2 + 1, du = 2xdx$. The bounds of integration become $0(x) = 1(u), 1(x) = 2(u)$ Substituting, we see $\rightarrow \frac{1}{2}\int_1^2 \frac{1}{u}du = \frac{1}{2}ln|u|\Big|_1^2 = \frac{1}{2}ln2 = ln(\sqrt{2})$

6. Is there a common value of $x$ for which the tangent lines of $g(x) = ln(\frac{3}{x^2})$ and $h(x) = ln(-\frac{x+1}{x})$ are parallel to one another?

Solution: The slope of the tangent line to g(x) at a point x = $m_g = g'(x) = \frac{x^2}{3} \cdot \frac{6}{x^3} = -\frac{2}{x}$
The slope of the tangent line to h(x) at a point x = $m_h = h'(x) = \frac{x}{x+1} \cdot -\frac{x-(x+1)}{x^2} = -\frac{1}{x^2+x}$
The slopes are parrallel when $m_g = m_h$, that is, $g'(x) = h'(x) \rightarrow -\frac{2}{x} = -\frac{1}{x^2+x} \quad \rightarrow -2x^2 - 2x = x$ by cross multiplication$\rightarrow$ $2x^2 + x = 0$, solving for x $\rightarrow x = 0, -\frac{1}{2}$. The value $x$=0 is not in the domain of either function. However, $x = -\frac{1}{2}$ is within the domain of both function. Moreover, the slope of the tangent line of $g(x)$ and $h(x)$ at $x = -\frac{1}{2}$ is -4, so the tangent lines are parallel to one another at $x = -\frac{1}{2}$.

7. Find the function $f$ that goes through the point (0,0) and has a derivative $f'(x) = sin(\frac{\pi}{3}x) - cos(\frac{\pi}{3}x)$.

Solution: $f(x) = \int(sin(\frac{\pi}{3}x) - cos(\frac{\pi}{3}x))dx = -\frac{\pi}{3}cos(\frac{\pi}{3}x) - \frac{\pi}{3}sin(\frac{\pi}{3}x) + C$
$f(0) = -\frac{\pi}{3} + C = 0$ solving for C, $\rightarrow C = \frac{\pi}{3}$ now we can substitute $C$ into our solution equation, $f(x) = -cos(\frac{\pi}{3}x) - sin(\frac{\pi}{3}x) + \frac{\pi}{3}$

8. A function $f$ has a derivative $f'(x) = x6^x$. Find the second derivative and use it to explain why the graph of below can not model the function $f$.

Solution: $f''(x) = x6^x \cdot ln6 + 6^x = 6^x(xlnx + 1)$
$f''(x) = 6^x(xlnx + 1)$. Note that $6^x > 0$ for all $x$. The second portion, $(xlnx + 1)$ is only defined for $x > 0$ and on that domain $(xlnx + 1) > 0$.
Thus $f''(x)$ is always positive and so the graph of $f(x)$ will be concave up on the entire domain. In other words, since $f''(x)$ never has a sign change, $f(x)$ has no points of inflection. The graph below has a change in concavity and a corresponding point of inflection so it can not be the graph of $f(x)$. (Students might also set $f''(x) = 0$, see that there are no points where this is true and extrapolate

that there is no sign change and therefore no inflection points.)

9. Suppose the piece-wise differentiable function $f(x)$ has a derivative as represented below and $f(1) = 5$.

$$f'(x) = \begin{cases} \frac{1}{2}e^{x^2} & -4 \leq x \leq 1 \\ x^3 cos(x) + 3 & 1 < x \leq 7 \end{cases}$$

A) Find $f''(5)$. Suppose that a piece-wise twice-differentiable function $g(x)$ has the same derivative as $f(x)$ but has the condition that $g(1) = -100$, compare $f''(5)$ to $g''(5)$.

B) Find $f(5)$. Suppose that a piece-wise twice-differentiable function $g(x)$ has the same derivative as $f(x)$ but has the condition that $g(1) = -100$, compare $f(5)$ to $g(5)$.

Solution:

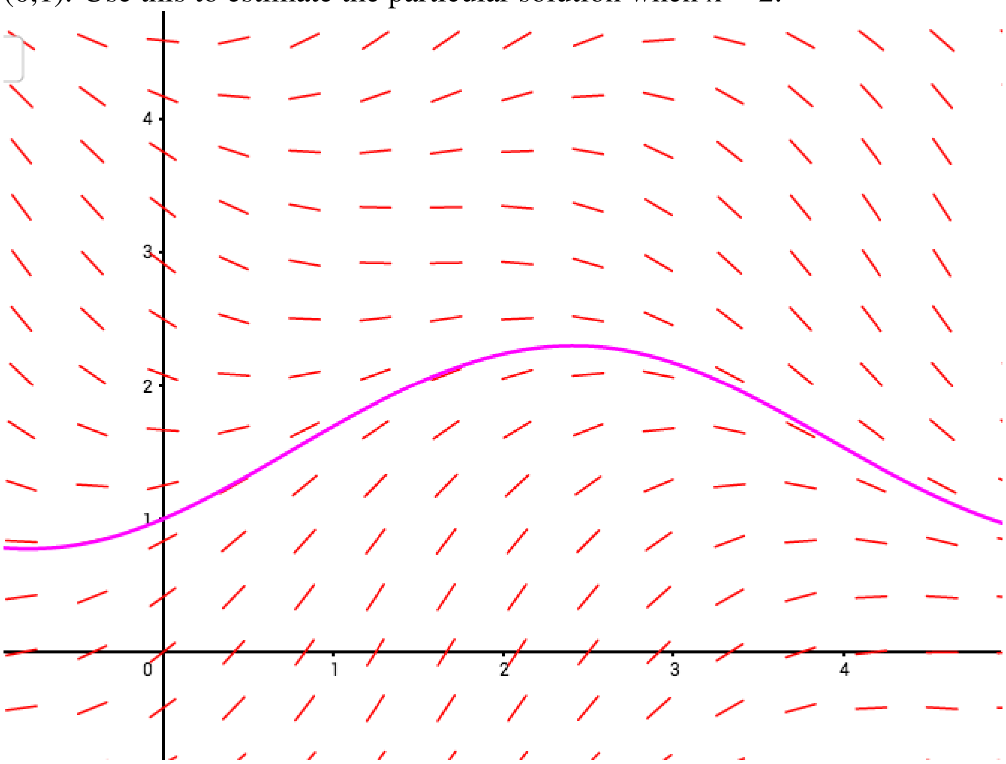
A)$f'(x) = x^3 cos(x) + 3$ for $x = 5$. $f''(x) = -x^3 sin(x) + 3x^2 cos(x) = 63.820$ The derivative, and second derivative, measure rates of change and therefore are unaffected by the initial conditions. So $f''(5) = g''(5)$.

B) $5 + \int_0^1 \frac{1}{2}e^{x^2} dx + \int 1^5 (x^3 cos(x) + 3) dx = 5 + .731326 + -50.525 = -44.794 or -44.793$. The function is measured by the area between the curve of the derivative function and the x-axis. However, these functions of rate of change can be moved up and down on the graphs without affecting the slope, thus when integrating the initial conditions must be taken into account. The initial conditions of $f(x)$ is greater than the initial condition of $g(x)$ by 105 units. This means that $f(5)$ is 105 units larger than $g(5)$.

1.Is $y = sin(2x)$ a solution of the differential equation $y'' + 4y = 0$? Justify.

Solution: $y = sin(2x)$, $y' = 2cos(2x)$, $y'' = -4sin(2x)$. Substituting into the differential equation, $-4sin(2x) + 4 \cdot sin(2x) = 0$. $y = sin(2x)$ is a solution for the differential equation $y'' + 4y = 0$, for all values of x.

2.The slope field below models a differential equation. Sketch the solution curve through the point (0,1). Use this to estimate the particular solution when x = 2.




The particular solution $\approx (2, 2)$

3. Let $B(t)$ represent the number of sea lions in a population at time $t$ years. For $t \geq 0$, $B$ is increasing at a rate directly proportional to B+100. If the initial population of sea lions is 200, set up an expression for the function $B(t)$.

Solution: $\frac{dB}{dt} = k(B + 100) \rightarrow \frac{1}{B+100}dB = kdt \rightarrow \int \frac{1}{B+100}dB = \int kdt \rightarrow ln|B + 100| = kt + C_0 \rightarrow$
$B + 100 = C_1 e^{kt} \rightarrow B(t) = C_1 e^{kt} - 100$
$B(0) = C_1 e^0 - 100 = C_1 - 100 = 200 \rightarrow C_1 = 300 \rightarrow B(t) = 300e^{kt} - 100$

4. Which of the following slope fields could represent the sea line population? Explain.

Solution:

5. A chlorinated pool has water pumped in with a different concentration of chlorine. So as not to overflow the pool, water is also being drained. The concentration of chlorine in the pool changes at a rate modeled by the differential equation $\frac{dC}{dt} = \frac{C}{t+100}$ where $t$ is in minutes. Suppose the initial concentration of chlorine is 3 ppm (parts per million). Find the concentration of chlorine after an hour.

Solution: $\frac{dC}{dt} = \frac{C}{t+100} \quad \rightarrow \int \frac{1}{C}dC = \int \frac{1}{t+100}dt \quad \rightarrow ln|C| = ln|t + 100| + c_0 \rightarrow e^{ln|C|} = e^{ln|t+100|)+c_0}$
$\rightarrow e^{ln|C|} = c_1 e^{ln|t+100|} \quad \rightarrow C = c_1(t + 100)$ plugging in the initial condition, $3 = c_1(0 + 100)$
$\rightarrow c_1 = \frac{3}{100}$ substituting this into the equation$\rightarrow C = \frac{3}{100}(t + 100)$ evaluating at $t = 60$
$C = \frac{3}{100}(60 + 100) = \frac{480}{100} = 4.8$ppm.

6. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-2}$. What is the domain of the slope field? Describe all points on the x-y plane for which the slopes are positive.

Solution: Domain: $(-\infty, 2) \bigcup (2, \infty)$.
$\frac{dy}{dx} > 0 \rightarrow y^2 > 0$ when $y \neq 0$, $x - 2 > 0$ when $x > 2$. Slopes are positive for all $x > 2$ and $y \neq 0$.

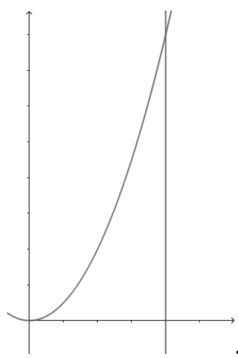
7. Suppose $y' = (x^2 + 3)e^y$. Find $y$ in terms of $x$.

Solution: $y' = (x^2 + 3)e^y$ can be written $\frac{dy}{dx} = (x^2 + 3)e^y$, we can separate the variables and integrate
$\frac{1}{e^y}dy = (x^2 + 3)dx \quad \rightarrow \int \frac{1}{e^y}dy = \int (x^2 + 3)dx \quad \rightarrow -e^{-y} = \frac{1}{3}x^3 + 3x + C$. Now we can solve for y,
$e^{-y} = -\frac{1}{3}x^3 - 3x - C \quad \rightarrow -y = ln| - \frac{1}{3}x^3 - 3x - C| \quad \rightarrow y = -ln| - \frac{1}{3}x^3 - 3x - C|$

8. Find $A$ and $B$ such that $y = x^2 + 3x$ is a solution to the differential equation $Ay'' - By' = 2x - 1$

Solution: $y = x^2 + 3x$ $y' = 2x + 3$ $y'' = 2$. Substituting into the differential equation $A(2) - B(2x + 3) = 2A - 2Bx - 3B = 2x - 1$ Solving for B,$-2Bx = 2x; B = -1$ Substitute B in to solve for A, $2A - 3B = -1$ $\rightarrow 2A + 3 = -1; A = -2$. The differential equation $-2y'' + y' = 2x - 1$ has the solution $y = x^2 + 3x$

1. Find the volume of a solid if its base is bounded by the x axis, $x = 4$ and $y = x^2$ and the cross sections perpendicular to the x-axis are semi-circles.

Solution: . The semicircles perpendicular to the x-axis have area of $= \frac{1}{4}\pi(x^2)^2$.
The area of the region $= \frac{1}{4}\pi \int_0^4 x^4 dx = \frac{1}{4}\pi\ \frac{1}{5}x^5\Big|_0^4 = \frac{256\pi}{5}$

2. Let R be the region bounded by the function $f(x) = 2 - x^2$ and the ellipse $x^2 + 4y^2 = 16$ as shown below.Write but do not solve, an expression for the area of R.

Solution: Area oriented horizontally:

Bounds of integration: We will find where they intersect through substitution, solving $y = 2 - x^2$ for $x^2$, $x^2 = 2 - y$. We will subsitute this into the second equation $x^2 + 4y^2 = 16$, and solve for y,
$2 - y + 4y^2 = 16 \quad \rightarrow 4y^2 - y - 14 = 0 \quad \rightarrow 4y^2 - 8y + 7y - 14 = 0 \quad \rightarrow (4y + 7)(y - 2) = 0$
$y = 2, -\frac{7}{4}$.

Transformation of Equation:Since the bounds are in terms of y, we will integrate with respect to y and thus the equations must be written in terms of y (i.e. solved for x).

The first equation: $y = 2 - x^2 \quad \rightarrow x^2 = 2 - y \quad \rightarrow x = (2 - y)^{\frac{1}{2}}$

The second equation: $x^2 + 4y^2 = 16 \quad \rightarrow x^2 = 16 - 4y^2 \quad \rightarrow x = (16 - 4y^2)^{\frac{1}{2}}$

Area $= \int_{-\frac{7}{4}}^{2}((16 - 4y^2)^{\frac{1}{2}} - (2 - y)^{\frac{1}{2}})dy$

Area oriented vertically:

Bounds of integration: We will find where they intersect through substitution, this time substituting in for y instead of x, so we will solve the first equation for $y^2$. $y = 2 - x^2$ $y^2 = 4 - 4x^2 + x^4$. Substituting this into the second equation, $x^2 + 4y^2 = 16 \rightarrow x^2 + 4(4 - 4x^2 + x^4) = 16 \rightarrow 4x^4 - 15x^2 = 0 \rightarrow x^2(x^2 - 15) = 0 \rightarrow x^2 = 0, 15 \rightarrow x = 0, -\sqrt{15}, \sqrt{15}$ but $x>0$, so $x = 0, \sqrt{15}$.

Transformation of Equation:Since the bounds are in terms of x, we will integrate with respect to x and thus the equations must be written in terms of x (i.e. solved for y).

The first equation is already solved for y:$y = 2 - x^2$

The second equation: $x^2 + 4y^2 = 16 \rightarrow 4y^2 = 16 - x^2 \rightarrow y = (4 - \frac{x^2}{4})^{\frac{1}{2}}$

Area = $\int_0^{\sqrt{15}}((4 - \frac{x^2}{4})^{\frac{1}{2}} - (2 - x^2))dx + 2\int_{\sqrt{15}}^{4}(4 - \frac{x^2}{4})^{\frac{1}{2}} - 0)dx$

3. Find the volume of the solid generated by revolving R around the $y$-axis.

Solution: Volume = $\pi\int_{-\frac{7}{4}}^{2}((16 - 4y^2)^{\frac{1}{2}})^2 - ((2 - y)^{\frac{1}{2}})^2 dy = \pi\int_{-\frac{7}{4}}^{2}(16 - 4y^2) - (2 - y)dy = \pi\int_{-\frac{7}{4}}^{2}(14 + y - 4y^2)dy = \pi\ (14y + \frac{1}{2}y^2 - \frac{4}{3}y^3)\Big|_{-\frac{7}{4}}^{2} = \pi[(28 + 2 - \frac{32}{3}) - (-\frac{49}{2} + \frac{49}{32} + \frac{1372}{192})] = \frac{1125}{32}$

4. Set up an equation to find $\frac{1}{24}\int_0^{24} G(t)dt$ using a right Riemann sum and explain its meaning in the context.

Solution: $\frac{1}{24}\int_0^{24} G(t)dt = \frac{1}{24} \cdot 4(100 + 230 + 0 + 200 + 170 + 195) = \frac{895}{6}$ pounds. This gives the average amount of grain processed into flour each hour in the factory.

5. Assume the bakery only sells cookies made that day, how many cookies are unsold at the end of the day? Justify your answer.

Solution: Amount of cookies baked: $\int_0^8(\frac{1}{20}e^{\frac{1}{2}x} + 3x^2 + 5)dt = (\frac{1}{10}e^{\frac{1}{2}x} + x^3 + 5x)\Big|_0^8 = 552$ cookies.
Amount of cookies sold: $\int_0^1 0dt + \int_1^3 20dt + \int_3^8(3t^2 + 1)dt = 0\Big|_0^1 + 20t\Big|_1^3 + (t^3 + t)\Big|_3^8 = 0 + (60 - 20) + (520 - 30) = 530$ cookies The amount of cookies left at the end of the day is equal to the amount of cookies bakes minus the amount of cookies sold, which is 552-530 = 22 cookies.

6. Suppose that on another day the owners know the cookies are going to sell at a rate of $S(t) = 20t + 12$. While they will bake the cookies beginning at 8am, they will delay opening the doors to sell cookies until a later time so that they do not run out of cookies. They will close the doors at the normal hour. If they want to open right at the beginning of the hour and sell as many cookies as possible without running out, what time should they open their bakery? Justify your answer.

Solution: $\int_t^8 (20t + 12)dt = 10t^2 + 12t\Big|_t^8 = (640 + 96) - (10t^2 + 12t) = 552 \quad \to 10t^2 + 12t - 184 = 0 \to t = 3.73 \approx$ 3 hours 45 minutes. In order to ensure they do not run out of cookies, they can open no earlier than 11:45, but since they want to open on the hour they should open at 12.

7. How far did Brian travel? Is this result the distance from his house to the beach, if not, how far is his house from the beach? Justify your answer.

Solution: distance $= \int_0^{26} |v(t)|dt = \frac{1}{2}2(200) + 2(200) + \frac{1}{2}2(200) + \frac{1}{2}4(50) + \frac{1}{2}2(150) + 10(150) + \frac{1}{2}4(150) = 200 + 400 + 200 + 100 + 150 + 1500 + 300 = 2850$meters. This is not the distance from his house to the beach, since he turned around after six minutes. Distance from the beach to his house: $\int_0^{26} v(t)dt = \frac{1}{2}2(200) + 2(200) + \frac{1}{2}2(200) - \frac{1}{2}4(50) + \frac{1}{2}2(150) + 10(150) + \frac{1}{2}4(150) = 200 + 400 + 200 - 100 + 150 + 1500 + 300 = 2650$meters.

8. What was Brian's average speed during the trip?

Solution: Average speed $= \frac{1}{26}\int_0^{26} |v(t)|dt = \frac{1}{26}(2850) = \frac{2850}{26}$ m/min.

9. Find the average velocity and the average acceleration of the particle during the 8 second period.

Solution: Average velocity $= \frac{1}{8}\int_0^8 v(t)dt \approx \frac{1}{8}[(1)(2) + (2)(\frac{3}{2}) + (1)(3) + (2)(\frac{5}{2}) + 2(3)] = \frac{1}{8}[2 + 3 + 3 + 5 + 6] = \frac{19}{8}$. The average velocity is approximately $\frac{19}{8}$ meters per second

Average acceleration $= \frac{v(8)-v(0)}{8-0} = \frac{4}{8}$. The average acceleration is $\frac{1}{2}$ meters per second squared

# APPENDIX K: IRB EXEMPTION

University of Central Florida Institutional Review Board
Office of Research & Commercialization
12201 Research Parkway, Suite 501
Orlando, Florida 32826-3246
Telephone: 407-823-2901 or 407-882-2276
www.research.ucf.edu/compliance/irb.html

# Approval of Exempt Human Research

From: **UCF Institutional Review Board #1**
**FWA00000351, IRB00001138**

To: **Lorna Wenzel Harris**

Date: **August 08, 2016**

Dear Researcher:

On 08/08/2016, the IRB approved the following activity as human participant research that is exempt from regulation:

| | |
|---|---|
| Type of Review: | Exempt Determination |
| Project Title: | Formal Assessment Predictors of AP Calculus Success |
| Investigator: | Lorna Wenzel Harris |
| IRB Number: | SBE-16-12424 |
| Funding Agency: | |
| Grant Title: | |
| Research ID: | n/a |

This determination applies only to the activities described in the IRB submission and does not apply should any changes be made. If changes are made and there are questions about whether these changes affect the exempt status of the human research, please contact the IRB. When you have completed your research, please submit a Study Closure request in iRIS so that IRB records will be accurate.

In the conduct of this research, you are responsible to follow the requirements of the Investigator Manual.

On behalf of Sophia Dziegielewski, Ph.D., L.C.S.W., UCF IRB Chair, this letter is signed by:

Signature applied by Patria Davis on 08/08/2016 03:59:48 PM EDT

IRB Coordinator

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