SPECTRAL PROPERTIES OF THE FINITE HILBERT TRANSFORM ON TWO ADJACENT INTERVALS VIA THE METHOD OF RIEMANN-HILBERT PROBLEM

by

ELLIOT BLACKSTONE
M.S. University of Central Florida, 2014
B.S. Penn State Erie, 2011

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Major Professors: Alexander Katsevich and Alexander Tovbis

ABSTRACT

In this dissertation, we study a self-adjoint integral operator $\hat{K}$ which is defined in terms of finite Hilbert transforms on two adjacent intervals. These types of transforms arise when one studies the interior problem of tomography. The operator $\hat{K}$ possesses a so-called "integrable kernel" and it is known that the spectral properties of $\hat{K}$ are intimately related to a 2×2 matrix function $\Gamma(z;\lambda)$ which is the solution to a particular Riemann-Hilbert problem (in the z plane). We express $\Gamma(z;\lambda)$ explicitly in terms of hypergeometric functions and find the small λ asymptotics of $\Gamma(z;\lambda)$. This asymptotic analysis is necessary for the spectral analysis of the finite Hilbert transform on multiple adjacent intervals. We show that $\Gamma(z;\lambda)$ also has a jump in the λ plane which allows us to compute the jump of the resolvent of $\hat{K}$. This jump is an important step in showing that the finite Hilbert transforms has simple and purely absolutely continuous spectrum. The well known spectral theory now allows us to construct unitary operators which diagonalize the finite Hilbert transforms. Lastly, we mention some future directions which include the many interval scenario and a bispectral property of $\hat{K}$.

To my parents, for always putting me in a position to succeed.

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CHAPTER 1: BACKGROUND AND INTRODUCTION

One of the great advances of modern technology has been the use of various types of CT (computed tomography) scanners for medical diagnostics. We briefly mention how these machines operate and how they are related to this work. An in-depth introduction of the mathematics involved in medical imaging can be found in [9]. In Figure 1.1, the light-grey region represents the object to be scanned and L represents a generic path on which an x-ray will travel. Field of view and region of interest are abbreviated as FOV and ROI, respectively. The size of the FOV is determined by the detector size: a larger detector leads to a larger FOV (denoted by the dashed circle), see the left panel of Figure 1.1. A smaller detector leads to a smaller FOV, which can be contained strictly inside the object (see the right panel of Figure 1.1). We denote the function f as the attenuation coefficient of the object being scanned. In the right panel of Figure 1.1, the intervals $[a_1, a_2] \cup [a_5, a_6] =: I_e$ and $[a_3, a_4] =: I_i$ are called exterior and interior intervals, respectively. The interval I_e is outside the FOV (hence the name exterior intervals) and f is assumed to be known on I_i. CT scanners operate in the following way: as the source/detector (see Figure 1.1 left panel) rotates around the object, the data collected is line integrals of f. If the FOV is large enough to contain the support of f, stable reconstruction of f is possible (Figure 1.1, left panel). This is not the case in Figure 1.1, right panel.

Any time a patient gets a CT scan, they are exposed to radiation. Reducing the patients exposure to radiation while maintaining image quality is obviously desirable.

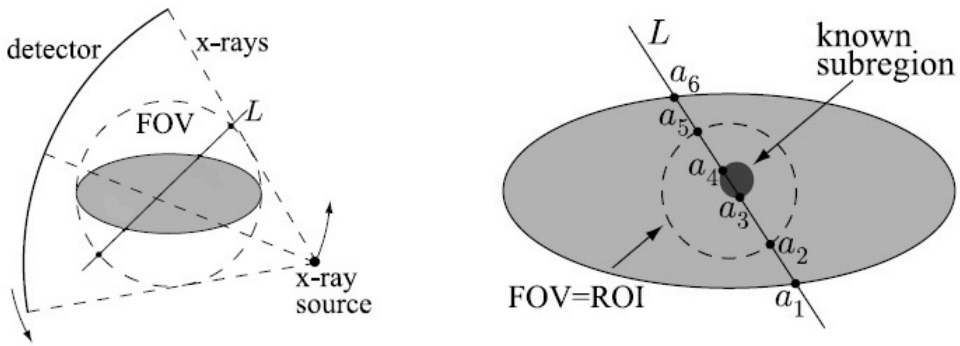

Figure 1.1: Left panel - a 2D setup for CT. Right panel - the interior problem with prior data. f is assumed to be known inside the "known subregion".

If we are only interested in imaging a ROI in the object, the whole object must be irradiated to produce an image using classical techniques. In order to reduce the radiation dosage, we shrink the x-ray beam and, hence, the FOV to include only the ROI, see right panel of Figure 1.1. Thus the goal is to recover f on the ROI from knowing line integrals of f on all lines intersecting the ROI (incomplete tomographic data). This is called the *interior problem of tomography*. A powerful tool when investigating this interior problem is called the Gelfand-Graev formula, which relates the tomographic data of an object with its one-dimensional Hilbert transform along lines (see [11]). With the help of this formula, the interior problem of tomography can be reduced to the problem of inverting the Hilbert transform from incomplete data. As we can see in Figure 1.1 right panel, f is supported on $[a_1, a_6]$ but the data $\mathcal{H}f$, where $\mathcal{H}$ is the Hilbert transform, is only available on $[a_2, a_5]$. So the equation to be solved is

$$\mathcal{H}f(x) = \frac{1}{\pi}\int_{a_1}^{a_6} \frac{f(y)}{y-x}dy = \varphi(x), \quad x \in [a_2, a_5], \tag{1.1}$$

where $\varphi(x)$ is the data. We want to recover f for $x \in [a_2, a_5]$ (where the data is available). When inverting operators with truncated data, both stability and uniqueness of this inversion is of concern. According to [15], unique recovery of f is impossible because $\mathcal{H}$ has a non-trivial kernel. Thus if one wants to recover f, some additional information is necessary. One way to guarantee unique recovery of f from its Hilbert transform is to assume some prior knowledge of f, meaning f is known on a subset of the ROI (see Figure 1.1, right panel). This is a reasonable assumption to make because this is often the case in practice. For example, $f \equiv 0$ inside the lung of a patient. There are also situations when there are several areas where f is known (e.g. two lungs). The problem is to study the stability of inversion for different configurations of I_e, I_i. It should be mentioned that the study of the interior problem of tomography with a known subregion began in [4].

In [3], the authors studied the interior problem of tomography in great detail and also address the ill-posedness of inverting certain Hilbert transforms. The context of [3] is nearly identical to this dissertation, so we wish to summarize their results and compare the differences. Let $g \in \mathbb{N}$ be fixed and choose real numbers a_i, $i = 1, 2, \ldots, 2g+2$, so that $a_i < a_{i+1}$ for $1 \leq i \leq 2g+1$. Define the exterior intervals (outside of FOV) $I_e = [a_1, a_2] \cup [a_{2g+1}, a_{2g+2}]$ and interior intervals (f is known on I_i) $I_i = [a_3, a_4] \cup [a_5, a_6] \cup \cdots \cup [a_{2g-1}, a_{2g}]$. See Figure 1.1, right panel for the scenario when $g = 2$. The authors analyzed the SVD system

$$\begin{aligned} \mathcal{H}_e^{-1}[h_n](y) &:= -\frac{w(y)}{\pi}\int_{I_e}\frac{h_n(x)}{w(x)(x-y)}dx = 2\lambda_n f_n(y),\ y\in I_i \\ \mathcal{H}_i[f_n](x) &:= \frac{1}{\pi}\int_{I_i}\frac{f_n(y)}{y-x}dy = 2\lambda_n h_n(x),\ x\in I_e \end{aligned} \tag{1.2}$$

where $w(x) = \sqrt{(a_{2g+2}-x)(x-a_1)}$ and $\mathcal{H}_e^{-1}$ is just notation. This SVD system is important because the rate at which $\lambda_n \to 0$ as $n \to \infty$ is related to the ill-posedness of inverting $\mathcal{H}_e^{-1}$. They reformulate this SVD system as an eigenvalue problem for an integral operator $\hat{K} : L^2(I_i \cup I_e) \to L^2(I_i \cup I_e)$ which has kernel

$$K(z,x) := \frac{w^{1/2}(x)w^{-1/2}(z)\chi_e(z)\chi_i(x) + w^{1/2}(z)w^{-1/2}(x)\chi_e(x)\chi_i(z)}{2\pi i(x-z)}, \tag{1.3}$$

where χ_e, χ_i are indicator functions on I_e, I_i, respectively. $\hat{K}$ is a self-adjoint, Hilbert-Schmidt operator and thus has a *discrete set of eigenvalues* that can accumulate only to $\lambda = 0$. Moreover, the eigenvalues of $\hat{K}$ coincide with the singular values of $\mathcal{H}_e^{-1}$. The authors define the resolvent integral operator $\hat{R}$ by the formula

$$\left(I + \hat{R}\right)\left(I + \frac{1}{\lambda}\hat{K}\right) = I \tag{1.4}$$

and show that the kernel of $\hat{R}$, using the method of A. Its et al. [14], is given by

$$R(x,z;\lambda) = \frac{\vec{g}^t(x)\Gamma^{-1}(x;\lambda)\Gamma(z;\lambda)\vec{f}(z)}{2\pi i\lambda(z-x)} \tag{1.5}$$

where

$$\vec{f}(z) = \begin{bmatrix} \frac{i\chi_e(z)}{\sqrt{w(z)}} \\ \sqrt{w(z)}\chi_i(x) \end{bmatrix}, \quad \vec{g}(x) = \begin{bmatrix} -i\sqrt{w(x)}\chi_i(x) \\ \frac{\chi_e(x)}{\sqrt{w(x)}} \end{bmatrix}, \tag{1.6}$$

and the 2×2 matrix $\Gamma(z;\lambda)$ is the solution to a particular Riemann-Hilbert problem (RHP), see RHP 2.2.1 for a specific case. The large n asymptotics of the eigenvalues and eigenfunctions of $\hat{K}$ are found via Deift-Zhou steepest descent method. These results are valid *provided that the intervals I_e, I_i are separated.*

This dissertation studies the scenario when $I_e = [b_L, 0]$ and $I_i = [0, b_R]$, where $b_L < 0 < b_R$. This was previously investigated in [16] and it was shown that a particular differential operator L (see (5.1)) corresponding to these two touching intervals has only continuous spectrum. The authors were also able to construct two isometric transformations U_1, U_2 such that $U_2 \mathcal{H}_L U_1^*$ is a multiplication operator with $\sigma(\omega)$, $\omega \geq (b_L^2 + b_R^2)/8$. Here ω is the spectral parameter of L and $\mathcal{H}_L$ is the finite Hilbert transform (FHT) mapping $L^2([b_L, 0]) \to L^2([0, b_R])$. It was also shown that $\sigma(\omega) \to 0$ as $\omega \to \infty$ exponentially fast which implies that the problem of finding f is severely ill-posed. The leading order asymptotic behavior of the kernels of U_1, U_2 as $\omega \to \infty$ was found asymptotically. These kernels were obtained explicitly only in the symmetric case where $b_L = -b_R$.

In this dissertation we construct U_1, U_2 explicitly in the non symmetric case, i.e. $b_L \neq -b_R$. We are able to express the kernel of $\mathcal{H}_L^* \mathcal{H}_L$ in terms of the matrix $\Gamma(z;\lambda)$, which is the solution of RHP 2.2.1. The jump of the resolvent (in the spectral variable, λ) is computed explicitly, thus classical spectral theory (see [2], [19]) describes how to construct U_1 and U_2. We also obtain the leading order asymptotics of $\Gamma(z;\lambda)$ when $\lambda \to 0$. The key idea is that this Riemann-Hilbert approach can be generalized to the scenario where there are multiple intervals with touching endpoints. Even in this general setting, the results from the two touching interval scenario are necessary, as we show in section 7.2. It is not known how to generalize the approach of [16], which relies on a commuting differential operator.

This dissertation is organized as follows: in chapter 2, we introduce a self-adjoint integral operator $\hat{K}$ which acts on $L^2([b_L, b_R])$ and describe its relation to the FHTs $\mathcal{H}_L, \mathcal{H}_R$. We then express the resolvent of $\hat{K}$ in terms of $\Gamma(z;\lambda)$, a solution to a particular RHP and express $\Gamma(z;\lambda)$ explicitly in terms of hypergeometric functions. Then, in chapter 3, we find the small λ asymptotics of $\Gamma(z;\lambda)$. In chapter 4 we briefly summarize the spectral theorem for self-adjoint operators with simple spectrum then diagonalize $\mathcal{H}_L^*\mathcal{H}_L$ and $\mathcal{H}_R^*\mathcal{H}_R$. In chapter 5, we obtain the results of [16] but explicitly, instead of asymptotically. Lastly, in chapter 6, we show that the diagonalizations obtained in chapters 4 and 5 are equivalent. We conclude with some future directions in chapter 7. The solution of RHP 2.2.1 is constructed in Appendix A and some auxiliary results are stated in Appendix B and C.

CHAPTER 2: INTEGRAL OPERATOR $\hat{K}$ AND RHP

Let us begin by defining the finite Hilbert transforms $\mathcal{H}_L : L^2([b_L, 0]) \to L^2([0, b_R])$ and $\mathcal{H}_R : L^2([0, b_R]) \to L^2([b_L, 0])$ by

$$\mathcal{H}_L[f](y) := \frac{1}{\pi} \int_{b_L}^{0} \frac{f(x)}{x-y} dx, \quad \mathcal{H}_R[g](x) := \frac{1}{\pi} \int_{0}^{b_R} \frac{g(y)}{y-x} dy. \tag{2.1}$$

Notice that the adjoint of $\mathcal{H}_L$ is $-\mathcal{H}_R$.

2.1 Definition and Properties of $\hat{K}$

We define the integral operator $\hat{K} : L^2([b_L, b_R]) \to L^2([b_L, b_R])$ by the requirements

$$\hat{K}\big|_{L^2([b_L,0])} = \frac{1}{2i}\mathcal{H}_L, \quad \hat{K}\big|_{L^2([0,b_R])} = \frac{1}{2i}\mathcal{H}_R. \tag{2.2}$$

Explicitly,

$$\hat{K}[f](z) := \int_{b_L}^{b_R} K(z,x) f(x)\, dx, \quad \text{where} \quad K(z,x) := \frac{\chi_L(x)\chi_R(z) + \chi_R(x)\chi_L(z)}{2\pi i(x-z)} \tag{2.3}$$

and χ_L, χ_R are indicator functions on $[b_L, 0], [0, b_R]$, respectively.

Proposition 2.1.1. *The integral operator $\hat{K} : L^2([b_L, b_R]) \to L^2([b_L, b_R])$ is self-adjoint and bounded, but not Hilbert-Schmidt.*

Proof. The boundedness of $\hat{K}$ follows from the boundedness of the Hilbert transform on $L^2(\mathbb{R})$. We can see that $\hat{K}$ is self-adjoint because $K(z,x) = \overline{K(x,z)}$. Lastly, the operator is not Hilbert-Schmidt because

$$\int_{b_L}^{b_R} \int_{b_L}^{b_R} |K(z,x)|^2 \, dx \, dz = \frac{1}{2\pi^2} \int_0^{b_R} \int_{b_L}^{0} \frac{dx \, dz}{(x-z)^2} \tag{2.4}$$

is not finite. □

2.2 Resolvent of $\hat{K}$ and the Riemann-Hilbert Problem

The operator $\hat{K}$ falls within the class of "integrable kernels" (see [14]) and it is known that its spectral properties are intimately related to a suitable Riemann-Hilbert problem. In particular, the kernel of the resolvent integral operator $\hat{R} = \hat{R}(\lambda) : L^2([b_L, b_R]) \to L^2([b_L, b_R])$, defined by

$$(I + \hat{R}(\lambda)) \left(I - \frac{1}{\lambda} \hat{K} \right) = I, \tag{2.5}$$

can be expressed through the solution $\Gamma(z;\lambda)$ of the following RHP.

Riemann-Hilbert Problem 2.2.1. *Find a 2×2 matrix-function $\Gamma(z;\lambda)$, $\lambda \in$*

$\overline{\mathbb{C}} \setminus [-1/2, 1/2]$, *analytic for* $z \in \overline{\mathbb{C}} \setminus [b_L, b_R]$ *and satisfying*

$$\Gamma(z_+;\lambda) = \Gamma(z_-;\lambda) \begin{bmatrix} 1 & -\frac{i}{\lambda} \\ 0 & 1 \end{bmatrix}, \quad z \in [b_L, 0], \tag{2.6}$$

$$\Gamma(z_+;\lambda) = \Gamma(z_-;\lambda) \begin{bmatrix} 1 & 0 \\ \frac{i}{\lambda} & 1 \end{bmatrix}, \quad z \in [0, b_R], \tag{2.7}$$

$$\Gamma(z;\lambda) = \begin{bmatrix} \mathrm{O}\,(1) & \mathrm{O}\,(\log(z-b_L)) \end{bmatrix}, \quad z \to b_L, \tag{2.8}$$

$$\Gamma(z;\lambda) = \begin{bmatrix} \mathrm{O}\,(\log(z-b_R)) & \mathrm{O}\,(1) \end{bmatrix}, \quad z \to b_R, \tag{2.9}$$

$$\Gamma(z;\lambda) \in L^2([b_L, b_R]), \tag{2.10}$$

$$\Gamma(z;\lambda) = I + \mathrm{O}\left(z^{-1}\right), \quad z \to \infty. \tag{2.11}$$

The endpoint behavior of $\Gamma(z;\lambda)$ *is described column-wise and the intervals* $(b_L, 0)$ *and* $(0, b_R)$ *are positively oriented.*

Introduce function

$$a(\lambda) = \frac{1}{i\pi} \ln\left(\frac{i + \sqrt{4\lambda^2 - 1}}{2\lambda}\right), \tag{2.12}$$

where the standard branch of the logarithm is taken. All relevant properties of $a(\lambda)$ are mentioned in Appendix B and we will often write a in place of $a(\lambda)$ for convenience. We are able to construct the solution of RHP 2.2.1 in terms of the hypergeometric functions

$$h_\infty(z) := e^{a\pi i} z^{-a} {}_2F_1\left(\begin{matrix} a, a+1 \\ 2a+2 \end{matrix} \,\middle|\, \frac{1}{z}\right) \implies h'_\infty(z) = -a e^{a\pi i} z^{-a-1} {}_2F_1\left(\begin{matrix} a+1, a+1 \\ 2a+2 \end{matrix} \,\middle|\, \frac{1}{z}\right), \tag{2.13}$$

$$s_\infty(z) := -\frac{z^{a+1}}{e^{a\pi i}} {}_2F_1\left(\begin{matrix} -a-1, -a \\ -2a \end{matrix} \,\middle|\, \frac{1}{z}\right) \implies s'_\infty(z) = \frac{a+1}{-e^{a\pi i}} z^a {}_2F_1\left(\begin{matrix} -a, -a \\ -2a \end{matrix} \,\middle|\, \frac{1}{z}\right), \tag{2.14}$$

where h_∞, s_∞ are linearly independent solutions of the ODE

$$z(1-z)w''(z) + a(\lambda)(a(\lambda)+1)w(z) = 0. \tag{2.15}$$

Recall that the standard Pauli matrices are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.16}$$

Theorem 2.2.2. *For $\lambda \in \overline{\mathbb{C}} \setminus [-1/2, 1/2]$, the unique solution to RHP 2.2.1 is*

$$\Gamma(z;\lambda) = \sigma_2 Q^{-1}(\lambda) \hat{\Gamma}^{-1}\left(M_1(\infty)\right) \begin{bmatrix} 1 & \frac{b_L b_R}{z(b_R - b_L)(a+1)} \\ 0 & 1 \end{bmatrix} \hat{\Gamma}\left(M_1(z)\right) Q(\lambda)\sigma_2, \tag{2.17}$$

where $M_1(z) = \frac{b_R(z-b_L)}{z(b_R-b_L)}$ and

$$Q = \begin{bmatrix} -\tan(a\pi) & 0 \\ 0 & 4^{2a+1} e^{a\pi i} \frac{\Gamma(a+3/2)\Gamma(a+1/2)}{\Gamma(a)\Gamma(a+2)} \end{bmatrix} \begin{bmatrix} 1 & e^{a\pi i} \\ -e^{a\pi i} & 1 \end{bmatrix}, \quad \hat{\Gamma}(z) = \begin{bmatrix} h_\infty(z) & s_\infty(z) \\ h'_\infty(z) & s'_\infty(z) \end{bmatrix}. \tag{2.18}$$

Here $a := a(\lambda)$ *which is defined in* (2.12) *and* h_∞, s_∞ *are defined in* (2.13),(2.14).

Proof. The matrix $\Gamma(z;\lambda)$ was constructed in Appendix A. We will show that this solution is unique. Assume that $\Gamma_1, \Gamma_2,\ \Gamma_1 \neq \Gamma_2$ are two solutions of RHP 2.2.1. Notice that any solution of RHP 2.2.1 has determinant 1, thus Γ_1, Γ_2 are invertible. It can now be verified that the matrix $\Gamma_2^{-1}\Gamma_1$ has no jump on $[b_L, b_R]$, no pole at $z = b_L, 0, b_R$, is analytic in $\mathbb{C}$ and tends to identity when $z \to \infty$. By Liouville's Theorem, $\Gamma_2^{-1}\Gamma_1 = I$. □

Remark 2.2.3. For any $\lambda \in \mathbb{C}\setminus[-1/2, 1/2]$, the function $\Gamma(z;\lambda)$ has the symmetries

$$\overline{\Gamma(\overline{z};\overline{\lambda})} = \Gamma(z;\lambda), \quad \sigma_3\Gamma(z;-\lambda)\sigma_3 = \Gamma(z;\lambda) \tag{2.19}$$

which both follow from the observation that $\sigma_3\Gamma(z;-\lambda)\sigma_3$ and $\overline{\Gamma(\overline{z};\overline{\lambda})}$ also solve RHP 2.2.1.

We now show the relation between $\hat{K}$ and $\Gamma(z;\lambda)$.

Theorem 2.2.4. *With the resolvent operator* $\hat{R}$ *defined by* (2.5), *let the kernel of* $\hat{R}$ *be denoted by* R. *Then,*

$$R(z,x;\lambda) = \frac{\vec{g}_1^t(x)\Gamma^{-1}(x;\lambda)\Gamma(z;\lambda)\vec{f}_1(z)}{2\pi i\lambda(z-x)} \ \textit{where}\ \vec{f}_1(z) = \begin{bmatrix} i\chi_L(z) \\ \chi_R(z) \end{bmatrix},\ \vec{g}_1(x) = \begin{bmatrix} -i\chi_R(x) \\ \chi_L(x) \end{bmatrix}. \tag{2.20}$$

The matrix $\Gamma(z;\lambda)$ *is defined in* (2.17) *and functions* χ_L, χ_R *are indicator functions on* $[b_L, 0], [0, b_R]$, *respectively.*

The proof is the same as in [3] (Lemma 3.16) so it will be omitted here. An important ingredient of the proof is the observation that the jump of $\Gamma(z;\lambda)$ can be compactly written as

$$\Gamma(z_+;\lambda) = \Gamma(z_-;\lambda)\left(I - \frac{1}{\lambda}\vec{f}_1(z)\vec{g}_1^t(z)\right) \tag{2.21}$$

for $z \in [b_L, b_R]$.

CHAPTER 3: SMALL λ ASYMPTOTICS OF $\Gamma(z;\lambda)$

In this section we only consider the symmetric scenario when $b_R = -b_L = 1$. The main result of this section is Theorem 3.3.14, which describes the small λ asymptotics of $\Gamma(z;\lambda)$ in various regions of the $z-$plane. This result follows from the application of the so-called Deift-Zhou steepest descent method, pioneered by Deift and Zhou in [6]. During this process, it will be necessary to find the small λ asymptotics of $\Gamma(z;\lambda)$ when z is a small, but fixed, distance from the origin. We address this obstacle by stating and proving a particular (linear) uniform steepest descent method and then applying it to the hypergeometric functions that appear in $\Gamma(z;\lambda)$.

3.1 A particular uniform steepest descent method

We can see in (2.17) that the hypergeometric functions that are present in $\Gamma(z;\lambda)$ are evaluated at $\frac{z+1}{2z}$, thus for convenience we introduce

$$\eta = \frac{z+1}{2z}. \tag{3.1}$$

Recall the definition of $h_\infty(\eta)$ (see (2.13)) and apply 15.3.1 of [1] to obtain

$$h_\infty(\eta) := e^{a\pi i}\eta^{-a}{}_2F_1\left(\begin{matrix} a, a+1 \\ 2a+2 \end{matrix} \,\middle|\, \frac{1}{\eta}\right) = e^{a\pi i}\eta^{-a}\frac{\Gamma(2a+2)}{\Gamma(a+1)^2}\int_0^1 \left(\frac{t(1-t)}{1-t/\eta}\right)^a dt. \tag{3.2}$$

See Appendix B for the definition and properties of the function $a(\lambda)$. For now we only consider $\Im\lambda \geq 0$ and it will be seen towards the end of this subsection that the results are similar when $\Im\lambda \leq 0$, see Remark 3.1.10. From Proposition B.0.1, we see that $\Im[a] \to -\infty$ as $\lambda \to 0$ with $\Im\lambda \geq 0$. With that in mind, define function

$$S_\eta(t) = S(t,\eta) := -i \ln\left(\frac{t(1-t)}{1-\frac{t}{\eta}}\right) \tag{3.3}$$

where the branch cuts of $S_\eta(t)$ in t variable are chosen to be $(-\infty, 0)$, $(1, \infty)$, and the ray from $t = \eta$ to $t = \infty$ with angle $\arg\eta$. Thus, we have

$$\int_0^1 \left(\frac{t(1-t)}{1-\frac{t}{\eta}}\right)^{a(\lambda)} dt = \int_0^1 e^{i\Re[a(\lambda)]S_\eta(t)} e^{-\Im[a(\lambda)]S_\eta(t)}\, dt, \tag{3.4}$$

so we can find the small λ asymptotics of this integral in a similar manner to the steepest descent method. Since we have four hypergeometric functions (see (2.13),(2.14)), we consider a slightly more general integrand. Define sets

$$\Omega := \left\{\eta = \frac{z+1}{2z} :\ M \leq \eta \leq 2M\right\}, \tag{3.5}$$

$$\Omega_+ := \left\{\eta = \frac{z+1}{2z} :\ M \leq \eta \leq 2M,\ 0 \leq \arg(\eta) \leq \pi\right\}, \tag{3.6}$$

where M is a large, positive, fixed number that is to be determined, see Remark 3.1.11. Notice that the set of all z such that $\frac{z+1}{2z} \in \Omega$ is a small annulus about the origin. The goal of this subsection is to prove the following theorem.

Theorem 3.1.1. *Let $\epsilon > 0$ be small, fixed and suppose $F(t,\eta,\lambda)$ satisfies the follow-*

ing properties:

1. *For every* $(\eta,\lambda)\in\Omega_+\times\overline{B}(0,\epsilon)$, $F(t,\eta,\lambda)$ *is analytic in* t *for* $t\in B(1/2,1/2)$,

2. *For every* $t\in B(1/2,1/2)$, $F(t,\eta,\lambda)$ *is continuous in* (η,λ) *for* $(\eta,\lambda)\in\Omega_+\times\overline{B}(0,\epsilon)$,

3. *For every* $(\eta,\lambda)\in\Omega_+\times\overline{B}(0,\epsilon)$, $F(t,\eta,\lambda)=\mathrm{O}\left(t^{c_0}\right)$ *as* $t\to 0$, *where* $c_0>-1$,

4. *For every* $(\eta,\lambda)\in\Omega_+\times\overline{B}(0,\epsilon)$, $F(t,\eta,\lambda)=\mathrm{O}\left((1-t)^{c_1}\right)$ *as* $t\to 1$, *where* $c_1>-1$,

5. *For every* $(\eta,\lambda)\in\Omega_+\times\overline{B}(0,\epsilon)$, $|F(t_-^*(\eta),\eta,\lambda)|>0$.

Then as $\lambda\to 0$, *provided that either* $\Im\lambda\geq 0$ *or* $\Im\lambda\leq 0$,

$$\begin{aligned}&\int_0^1 F(t,\eta,\lambda)e^{-\Im[a(\lambda)]S_\eta(t)}dt\\&=e^{-\Im[a(\lambda)]S_\eta(t_-^*(\eta))}F\left(t_-^*(\eta),\eta,\lambda\right)\sqrt{\frac{2\pi}{\Im[a(\lambda)]S''_\eta(t_-^*(\eta))}}\left[1+\mathrm{O}\left(\frac{1}{\Im[a(\lambda)]}\right)\right],\end{aligned}\tag{3.7}$$

where $S_\eta(t),a(\lambda),\Omega_+$ *are defined in* (3.3), (2.12), (3.6), *respectively. This approximation is uniform for* $\eta\in\Omega_+$.

The idea of the proof is as follows: we deform the contour of integration in (3.7) from $[0,1]$ to a path we call γ_η, which passes through a relevant saddle point of $S_\eta(t)$. We then show that the leading order contribution in Theorem 3.1.1 comes from a small neighborhood in the t plane centered at this previously mentioned saddle point of $S_\eta(t)$ and the contribution outside of this neighborhood is of lower order.

3.1.1 Deformation of $[0,1]$ to γ_η

We begin by locating the saddle points of $S_\eta(t)$.

Proposition 3.1.2. *For $\eta \in \Omega_+$, the function $S_\eta(t)$ has exactly two simple saddle points $t^*_\pm(\eta)$ defined by $S'_\eta(t^*_\pm(\eta)) = 0$. Explicitly,*

$$t^*_+(\eta) = \eta + \sqrt{\eta^2 - \eta} = 2\eta + \mathrm{O}\left(1\right) \quad \text{as } \eta \to \infty, \tag{3.8}$$

$$t^*_-(\eta) = \eta - \sqrt{\eta^2 - \eta} = \frac{1}{2} + \mathrm{O}\left(\eta^{-1}\right) \quad \text{as } \eta \to \infty, \tag{3.9}$$

where the branch for $t^*_\pm(\eta)$ is $[0,1]$. Moreover,

$$S_\eta(t^*_\pm(\eta)) = -2i \ln\left(t^*_\pm(\eta)\right) \quad \text{and} \quad S''_\eta(t^*_-(\eta)) = \frac{2i}{t^*_-(\eta)\left(1 - t^*_-(\eta)\right)}. \tag{3.10}$$

See (3.6), (3.3) for the definitions of $\Omega_+, S_\eta(t)$, respectively.

The proof is a simple exercise. For any $\eta \in \Omega_+$, we want the path γ_η to have the property

$$\Re\left[S_\eta(t) - S_\eta(t^*_-(\eta))\right] \le 0 \tag{3.11}$$

for all $t \in \gamma_\eta$ with equality only when $t = t^*_-(\eta)$, so the understanding of the level curve

$$\Re\left[S_\eta(t) - S_\eta(t^*_-(\eta))\right] = 0 \tag{3.12}$$

is paramount.

Lemma 3.1.3. *For each $\eta \in \Omega_+$ we have the following:*

1. *There is exactly one curve* l_0 *emitted from* $t = 0$ *so that* $\Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right] = 0$ *for all* $t \in l_0$*. Moreover, when* $|t|$ *is sufficiently small,* l_0 *lies in the sector* $|\arg(t)| \le \pi/4$*.*

2. *There is exactly one curve* l_1 *emitted from* $t = 1$ *so that* $\Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right] = 0$ *for all* $t \in l_1$*. Moreover, when* $|1-t|$ *is sufficiently small,* l_1 *lies in the sector* $|\arg(1-t)| \le \pi/4$*.*

3. *There exists exactly one* $\theta = \theta_{u,\eta} \in (0, \pi)$ *such that* $\Re\left[S_\eta(1/2 + 1/2e^{i\theta_{u,\eta}}) - S_\eta(t_-^*(\eta))\right] = 0$*. Moreover, for* M *sufficiently large,* $\theta_{u,\eta} \in (\pi/4, 3\pi/4)$*.*

4. *There exists exactly one* $\theta = \theta_{l,\eta} \in (-\pi, 0)$ *such that* $\Re\left[S_\eta(1/2 + 1/2e^{i\theta_{l,\eta}}) - S_\eta(t_-^*(\eta))\right] = 0$*. Moreover, for* M *sufficiently large,* $\theta_{l,\eta} \in (-3\pi/4, -\pi/4)$*.*

See (3.6) *for* Ω_+, M *and* (3.3) *for* $S_\eta(t)$*.*

Figure 3.1 is a visualization of Lemma 3.1.3. The blue lines are the branch cuts of $S_\eta(t)$, the red curves is the level set $\Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right] = 0$ and the black dashed circle has center and radius 1/2. The + denotes regions where $\Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right] > 0$ and $-$ denotes regions where $\Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right] < 0$.

Proof. For brevity, define

$$\alpha_\eta(t) := \Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right] = \arg\left(\frac{\eta t(1-t)}{\eta - t}\right) - 2\arg\left(t_-^*(\eta)\right). \tag{3.13}$$

The statements listed above are equivalent to

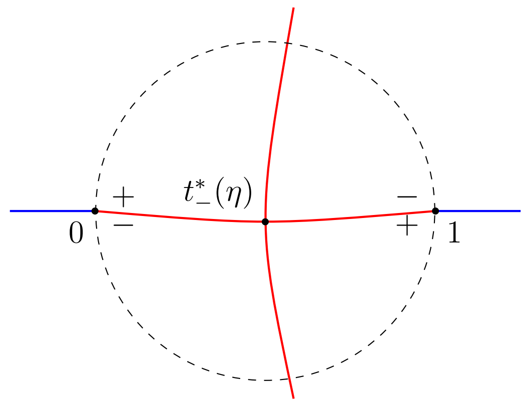

Figure 3.1: A visualization of Lemma 3.1.3.

1. For any $r > 0$ sufficiently small, $\alpha_\eta(re^{i\theta})$ is increasing for $-\pi < \theta < \pi$ and $\exists! \theta_{0,\eta,r} \in (-\pi/4, \pi/4)$ such that $\alpha_\eta(re^{i\theta_{0,\eta,r}}) = 0$,

2. For any $r > 0$ sufficiently small, $\alpha_\eta(1 - re^{i\theta})$ is increasing for $-\pi < \theta < \pi$ and $\exists! \theta_{1,\eta,r} \in (-\pi/4, \pi/4)$ such that $\alpha_\eta(1 - re^{i\theta_{1,\eta,r}}) = 0$,

3. $\alpha_\eta\left(\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)$ is increasing for $0 < \theta < \pi$ and $\exists! \theta_{u,\eta} \in (\pi/4, 3\pi/4)$ such that $\alpha_\eta\left(\frac{1}{2} + \frac{1}{2}e^{i\theta_{u,\eta}}\right) = 0$,

4. $\alpha_\eta\left(\frac{1}{2} + \frac{1}{2}e^{i\theta}\right)$ is increasing for $-\pi < \theta < 0$ and $\exists! \theta_{l,\eta} \in (-3\pi/4, -\pi/4)$ such that $\alpha_\eta\left(\frac{1}{2} + \frac{1}{2}e^{i\theta_{l,\eta}}\right) = 0$.

The proofs of each of the four claims above are similar so we prove the first.

$$\alpha_\eta(re^{i\theta}) = \arg(re^{i\theta}) + \arg(1 - re^{i\theta}) - \arg(\eta - re^{i\theta}) + \arg(\eta) - 2\arg\left(t_-^*(\eta)\right) \quad (3.14)$$

$$\equiv \theta + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - 1}\right) - \tan^{-1}\left(\frac{r\sin\theta - \Im\eta}{r\cos\theta - \Re\eta}\right)$$

$$+ \arg(\eta) - 2\arg\left(t_-^*(\eta)\right) \pmod{\pi} \quad (3.15)$$

Differentiating with respect to θ, we have

$$\frac{d}{d\theta}\left[\alpha_\eta(re^{i\theta})\right] = 1 + \frac{r^2 - r\cos\theta}{(r\cos\theta - 1)^2 + r^2\sin^2\theta} - \frac{r^2 - r(\Re(\eta)\cos\theta + \Im(\eta)\sin\theta)}{(r\cos\theta - \Re\eta)^2 + (r\sin\theta - \Im\eta)^2} \to 1 \quad (3.16)$$

as $r \to 0$. So with r small enough, $\frac{d}{d\theta}\left[\alpha_\eta(re^{i\theta})\right] > 0$ for any θ and for any $\eta \in \Omega_+$. Take M sufficiently large so that

$$|\arg\left(t_-^*(\eta)\right)| < \pi/8 \quad (3.17)$$

for all $\eta \in \Omega_+$. Thus when r is sufficiently small,

$$\alpha_\eta(re^{i\pi/4}) = \frac{\pi}{4} + \tan^{-1}\left(\frac{r/\sqrt{2}}{r/\sqrt{2} - 1}\right) - \arg\left(1 - \frac{re^{i\pi/4}}{\eta}\right) - 2\arg\left(t_-^*(\eta)\right) > 0 \quad (3.18)$$

and similarly it can be shown that $\alpha_\eta(re^{-i\pi/4}) < 0$. Intermediate Value Theorem can now be applied to show $\exists!\theta_{0,\eta,r} \in (-\pi/4, \pi/4)$ such that $\alpha_\eta(re^{i\theta_{0,\eta,r}}) = 0$.

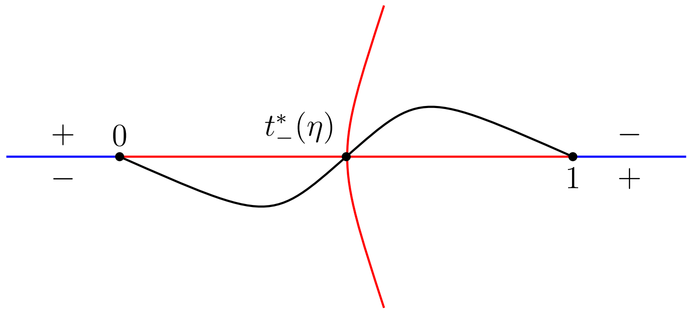

Figure 3.2: The path γ_η (black).

□

Remark 3.1.4. According to Lemma 3.1.3, for every $\eta \in \Omega_+$, the set $t \in B(1/2, 1/2)$ is split into 4 sectors by the level curve $\Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right] = 0$, see Figure 3.1.

We have now proven that we can deform $[0,1]$ to an 'appropriate' path γ_η, as described in the following Theorem.

Theorem 3.1.5. *For each $\eta \in \Omega_+$, the path $[0,1]$ can be continuously deformed to a path γ_η which begins at $t = 0$ (with $\arg(t) < -\pi/4$ for small $|t|$), passes through $t = t_-^*(\eta)$, ends at $t = 1$ (with $\arg(t-1) > 3\pi/4$ for small $|t-1|$), and $\gamma_\eta \subset B(1/2, 1/2)$. Moreover, $\Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right] \leq 0$ for all $t \in \gamma_\eta$ with equality only when $t = t_-^*(\eta)$. See (3.6), (3.3) for $\Omega_+, S_\eta(t)$, respectively.*

The new path of integration, γ_η, is the black curve in Figure 3.2.

3.1.2 Estimates

Here we will split γ_η into 3 pieces (see Figure 3.3) and show that the leading order contribution in Theorem 3.1.1 comes from the piece containing $t_-^*(\eta)$. Define radius

$$r := r_M = \max_{\eta \in \Omega_+} \left| \frac{1}{2} - t_-^*(\eta) \right| \tag{3.19}$$

so we have that $t_-^*(\eta) \in \overline{B}(1/2, r)$ for all $\eta \in \Omega_+$ where Ω_+ is defined in (3.6). Also define function

$$v_\eta(t) := v\,(t, \eta) = \sqrt{S_\eta(t_-^*(\eta)) - S_\eta(t)} \tag{3.20}$$

where $S_\eta(t)$ was defined in (3.3) and the square root is defined so that

$$v_\eta(t) = \big(t - t_-^*(\eta)\big)\sqrt{-\frac{1}{2}S_\eta''(t_-^*(\eta)) + \mathrm{O}\left(t - t_-^*(\eta)\right)} \quad \text{as } t \to t_-^*(\eta). \tag{3.21}$$

The function $v_\eta(t)$ will be an essential change of variables in the integral (3.7) so we mention its important properties.

Lemma 3.1.6. *The function $v_\eta(t)$, defined in (3.20), has the following properties:*

1. *For every $\eta \in \Omega_+$, $v_\eta'(t_-^*(\eta)) = \sqrt{-\frac{1}{2}S_\eta''(t_-^*(\eta))}$ and $2 \leq \left|S_\eta''(t_-^*(\eta))\right| \leq 32$, (here $'$ denotes differentiation with respect to t)*

2. *For every $\eta \in \Omega_+$, $v_\eta(t)$ is biholomorphic for $t \in B(1/2, 2r)$,*

3. *For every $\eta \in \Omega_+$ and any $t_1, t_2 \in B(1/2, 2r)$, $v_\eta(t)$ satisfies the bi-Lipschitz*

condition

$$\frac{2}{9}|t_1 - t_2| \leq |v_\eta(t_1) - v_\eta(t_2)| \leq \frac{9}{2}|t_1 - t_2|, \tag{3.22}$$

4. *Let I_η be the image of $B(1/2, 2r)$ under the map $v_\eta(t)$. Then there exists a $\delta_* > 0$ such that $B(0, \delta_*) \subset \bigcap_{\eta \in \Omega_+} I_\eta$.*

See (3.6)*,* (3.3)*,* (3.19) *for $\Omega_+, S_\eta(t), r$, respectively.*

Both the statement and proof of item 4 are based on Lemma 2.2 in [17].

Proof. 1. The evaluation of $v'_\eta(t)$ when $t = t^*_-(\eta)$ is a simple calculation. From Proposition 3.1.2 we have that

$$S''_\eta(t^*_-(\eta)) = \frac{2i}{t^*_-(\eta)(1 - t^*_-(\eta))} \tag{3.23}$$

and

$$32 \geq \left| \frac{2i}{t^*_-(\eta)(1 - t^*_-(\eta))} \right| \geq 2 \tag{3.24}$$

because M can be made sufficiently large so that

$$1/4 \leq |t^*_-(\eta)| \leq 1 \text{ and } 1/4 \leq |1 - t^*_-(\eta)| \leq 1 \tag{3.25}$$

for all $\eta \in \Omega_+$.

2. For any fixed $\eta \in \Omega_+$, the number $\min\left\{|t^*_-(\eta)|, |t^*_-(\eta) - 1|\right\}$ is the distance

from $t = t_-^*(\eta)$ to the nearest singularity of $v_\eta(t)$. Define d as

$$d := d_M = \frac{1}{2} \min_{\eta \in \Omega_+} \min \left\{ |t_-^*(\eta)|, |t_-^*(\eta) - 1| \right\}. \tag{3.26}$$

Notice that $d_M \to 1/4$ as $M \to \infty$. Thus, $v_\eta(t)$ is analytic in $B(1/2, d)$ for all $\eta \in \Omega_+$ and we can write

$$v_\eta(t) = \sum_{n=1}^{\infty} c_n(\eta) \left(t - t_-^*(\eta) \right)^n, \tag{3.27}$$

where

$$c_1(\eta) = \sqrt{-\frac{1}{2} S_\eta''(t_-^*(\eta))}, \quad |c_n(\eta)| \leq \frac{N}{d^n} \quad \text{with} \quad N := N_M = \max_{\eta \in \Omega_+} \max_{t \in \partial B(1/2, d_M)} |v_\eta(t)|, \tag{3.28}$$

by use of Cauchy's estimate. Notice that N_M tends to a finite, non-zero constant as $M \to \infty$. Choose M sufficiently large so that

$$r_M \leq \frac{d_M}{3} \left(1 - \sqrt{\frac{2N_M}{2N_M + d_M}} \right). \tag{3.29}$$

Notice that this implies that $r_M < \frac{d_M}{3}$. Now to show that $v_\eta(t)$ is one-to-one in $B(1/2, 2r)$, let $t_1, t_2 \in B(1/2, 2r)$.

$$\begin{aligned}
\left|\frac{v_\eta(t_1) - v_\eta(t_2)}{t_1 - t_2}\right| &= \left|\frac{\sum_{n=1}^{\infty} c_n(\eta)\left[(t_1 - t_-^*(\eta))^n - (t_2 - t_-^*(\eta))^n\right]}{t_1 - t_2}\right| \\
&= \left|c_1(\eta) + \sum_{n=2}^{\infty} c_n(\eta) \sum_{j=0}^{n-1} (t_1 - t_-^*(\eta))^j (t_2 - t_-^*(\eta))^{n-1-j}\right| \\
&\geq |c_1(\eta)| - \left|\sum_{n=2}^{\infty}\sum_{j=0}^{n-1} c_n(\eta)(t_1 - t_-^*(\eta))^j (t_2 - t_-^*(\eta))^{n-1-j}\right| \\
&\geq 1 - \sum_{n=2}^{\infty}\sum_{j=0}^{n-1} \frac{N}{d^n}(3r)^{n-1} \\
&= 1 - \frac{N}{d}\left[-1 + \sum_{n=1}^{\infty} n\left(\frac{3r}{d}\right)^{n-1}\right] \\
&= 1 - \frac{N}{d}\left[\frac{d^2}{(d-3r)^2} - 1\right] \\
&\geq \frac{1}{2}
\end{aligned}$$

where in the last inequality we have used (3.29). So we have shown that $v_\eta(t)$ is one-to-one and $v_\eta'(t) \neq 0$ (since $|v_\eta'(t)| \geq 1/2$) for $t \in B(1/2, 2r)$.

3. We have immediately from part 2 that $\frac{2}{9}|t_1 - t_2| \leq |v_\eta(t_1) - v_\eta(t_2)|$. We can show $|v_\eta(t_1) - v_\eta(t_2)| \leq \frac{9}{2}|t_1 - t_2|$ in a similar fashion to part 2.

4. Lastly, notice that $0 \in \bigcap_{\eta \in \Omega_+} I_\eta$ since $t_-^*(\eta) \in B(1/2, 2r)$ for every $\eta \in \Omega_+$ and $v_\eta(t_-^*(\eta)) = 0$. We show that 0 is an interior point of $\bigcap_{\eta \in \Omega_+} I_\eta$ so via contradiction, assume 0 is not an interior point. Then we can find a sequence $\{w_n\}$ such that $w_n \notin \bigcap_{\eta \in \Omega_+} I_\eta$ and $w_n \to 0$. Also we can find a sequence $\{\eta_n\}$ so that $w_n \notin I_{\eta_n}$. Let $\tilde{w}_n$ be the point on the line connecting 0 and w_n so that

$\tilde{w}_n \in \partial I_{\eta_n}$. Since $w_n \to 0$, $\tilde{w}_n \to 0$ as well. Since $\tilde{w}_n \in \partial I_{\eta_n}$, there exists $t_n \in \partial B(1/2, 2r)$ so that $v_{\eta_n}(t_n) = \tilde{w}_n$. For any $n \in \mathbb{N}$,

$$|\tilde{w}_n| = \left|v_{\eta_n}(t_n) - v_{\eta_n}(t^*_-(\eta_n))\right| \tag{3.30}$$

$$\geq \frac{1}{2}\left|t_n - t^*_-(\eta_n)\right| \tag{3.31}$$

$$\geq \frac{1}{2}\left(\left|t_n - \frac{1}{2}\right| - \left|t^*_-(\eta_n) - \frac{1}{2}\right|\right) \tag{3.32}$$

$$\geq \frac{1}{2}(2r - r) = \frac{r}{2} \tag{3.33}$$

which contradicts our observation that $\tilde{w}_n \to 0$, thus 0 is an interior point. Therefore, there exists a $\delta_* > 0$ so that $B(0, \delta_*) \subset \bigcap_{\eta \in \Omega_+} I_\eta$.

□

Let $\delta > 0$ be fixed so that $\delta_* > \delta$ and define

$$t^*_L(\eta) := v_\eta^{-1}(-\delta), \quad t^*_R(\eta) := v_\eta^{-1}(\delta). \tag{3.34}$$

With the previous Lemma in mind, we now write $\gamma_\eta = \gamma_{L,\eta} + \gamma_{C,\eta} + \gamma_{R,\eta}$, where $\gamma_{C,\eta}$ is the image of $[-\delta, \delta]$ under the map v_η^{-1}, $\gamma_{L,\eta}$ is the portion of γ beginning at $t = 0$ and ending at $t = t^*_L(\eta)$, and $\gamma_{L,\eta}$ is the portion of γ beginning at $t = t^*_R(\eta)$ and ending at $t = 1$. The curves $\gamma_{L,\eta}, \gamma_{C,\eta}, \gamma_{R,\eta}$ are pictured in Figure 3.3.

Remark 3.1.7. Notice that

$$\left|t^*_-(\eta) - t^*_R(\eta)\right| = \left|v_\eta^{-1}(0) - v_\eta^{-1}(\delta)\right| \geq \frac{2}{9}|\delta - 0| > 0 \tag{3.35}$$

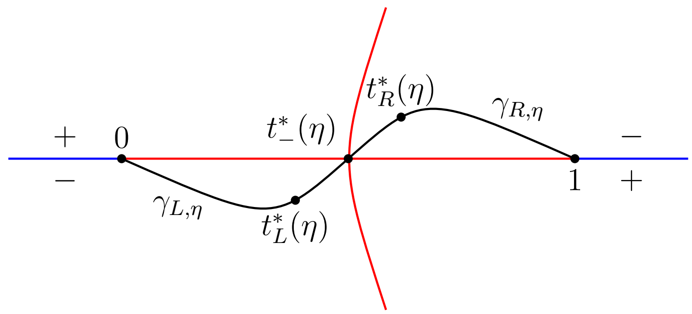

Figure 3.3: $\gamma_\eta = \gamma_{L,\eta} + \gamma_{C,\eta} + \gamma_{R,\eta}$

for all $\eta \in \Omega_+$ by use of (3.22). An identical statement holds for $t^*_L(\eta)$.

Lemma 3.1.8. *Assume $F(t,\eta,\lambda)$ satisfies the hypothesis of Theorem 3.1.1. Then as $\lambda \to 0$, provided $\Im\lambda \geq 0$,*

$$\int_{\gamma_{C,\eta}} F(t,\eta,\lambda) e^{-\Im[a(\lambda)]S_\eta(t)} dt$$
$$= e^{-\Im[a(\lambda)]S_\eta(t^*_-(\eta))} F\left(t^*_-(\eta),\eta,\lambda\right) \sqrt{\frac{2\pi}{\Im[a(\lambda)]S''_\eta(t^*_-(\eta))}} \left[1 + \mathrm{O}\left(\frac{1}{\Im[a(\lambda)]}\right)\right], \quad (3.36)$$

where $a(\lambda), S_\eta(t)$ are defined in (2.12), (3.3), *respectively. This approximation is uniform for $\eta \in \Omega_+$.*

Proof. Recall that $\gamma_{C,\eta}$ is the image of $[-\delta,\delta]$ under the map v_η^{-1}, where v_η is defined in (3.20). In other words, $\gamma_{C,\eta}$ is the path of steepest descent for $S_\eta(t) - S_\eta(t^*_-(\eta))$ and thus $\Im[S_\eta(t) - S_\eta(t^*_-(\eta))] = 0$ and $\Re[S_\eta(t) - S_\eta(t^*_-(\eta))] \leq 0$ on $\gamma_{C,\eta}$ with equality only when $t = t_-^*(\eta)$. Using change of variables

$$v = v_\eta(t) = \sqrt{S_\eta(t_-^*(\eta)) - S_\eta(t)}, \tag{3.37}$$

we see that the new bounds of integration are

$$v|_{t=t_L^*(\eta)} = v_\eta(t_L^*(\eta)) = -\delta, \quad v|_{t=t_R^*(\eta)} = v_\eta(t_R^*(\eta)) = \delta \tag{3.38}$$

and

$$dt = \frac{-2v_\eta(t)}{S'_\eta(t)} dv =: f_\eta(v) dv. \tag{3.39}$$

So we have

$$\int_{\gamma_{C,\eta}} F(t,\eta,\lambda) e^{-\Im[a] S_\eta(t)} dt = e^{-\Im[a] S_\eta(t_-^*)} \int_{\gamma_{C,\eta}} F(t,\eta,\lambda) e^{-\Im[a][S_\eta(t) - S_\eta(t_-^*)]} dt \tag{3.40}$$

$$= e^{-\Im[a] S_\eta(t_-^*)} \int_0^\delta e^{\Im[a] v^2} F(t(v),\eta,\lambda) \left[f_\eta(v) + f_\eta(-v) \right] dv \tag{3.41}$$

because t is an even function of v. Since $v_\eta(t)$ is biholomorphic in $B(1/2, 2r)$ for every $\eta \in \Omega_+$, both $f_\eta(v) = \frac{dt}{dv}$ and $F(t(v), \eta, \lambda)$ are analytic for $v \in B(0, \delta^*)$ by Lemma 3.1.6. So we have

$$F(t(v),\eta,\lambda) \left[f_\eta(v) + f_\eta(-v) \right] = \sum_{n=0}^{\infty} b_{2n}(\eta, \Re[a]) v^{2n} =: b_0(\eta,\lambda) + v^2 R_2(v,\eta,\lambda) \tag{3.42}$$

where

$$b_0(\eta,\lambda) = 2F(t_-^*(\eta),\eta,\lambda)\sqrt{\frac{2}{-S_\eta''(t_-^*(\eta))}}, \tag{3.43}$$

$$b_{2n}(\eta,\lambda) = \frac{1}{2\pi i}\int_{|w|=\delta} \frac{F(t(w),\eta,\lambda)\,[f_\eta(w)+f_\eta(-w)]}{w^{2n+1}}dw \tag{3.44}$$

for $n \geq 1$, where $\delta < \delta_*$. Define

$$K = \max_{\eta\in\Omega_+}\max_{v\in\partial B(0,\delta_*)} |F(t(v),\eta,\lambda)f_\eta(v)| \tag{3.45}$$

and notice that K is finite since both F, f are analytic in $B(1/2, 2r)$ and the preimage of $B(0,\delta^*)$ is a subset of $B(1/2,2r)$ for all $(\eta,\lambda)\in\Omega_+\times\overline{B}(0,\epsilon)$. By Cauchy's estimate we now have

$$|b_{2n}(\eta,\lambda)| \leq \frac{2K}{\delta_*^{2n}} \tag{3.46}$$

and also for $v\in[0,\delta]$,

$$|R_2(v,\eta,\lambda)| = \left|\sum_{n=1}^{\infty} b_{2n}(\eta,\lambda)v^{2(n-1)}\right| \leq \frac{2K}{\delta_*^2}\sum_{n=1}^{\infty}\left(\frac{\delta}{\delta_*}\right)^{2(n-1)} \leq \frac{2K}{\delta_*^2-\delta^2} =: R. \tag{3.47}$$

Returning to (3.41), using a second change of variable $-\tau = \Im[a]v^2$, we have

$$e^{-\Im[a]S_\eta(t_-^*)}\int_0^\delta e^{\Im[a]v^2}F(t(v),\eta,\lambda)\,[f_\eta(v)+f_\eta(-v)]\,dv \tag{3.48}$$

$$= e^{-\Im[a]S_\eta(t_-^*)}\int_0^\delta e^{\Im[a]v^2}\left[b_0(\eta,\lambda)+v^2R_2(v,\eta,\lambda)\right]dv \tag{3.49}$$

$$= \frac{e^{-\Im[a]S_\eta(t_-^*)}}{2\sqrt{-\Im[a]}}\int_0^{-\Im[a]\delta^2} e^{-\tau}\left[\frac{b_0(\eta,\lambda)}{\sqrt{\tau}} - \frac{\sqrt{\tau}}{\Im[a]}R_2\left(t\left(\sqrt{\frac{-\tau}{\Im[a]}}\right),\eta,\lambda\right)\right]d\tau \tag{3.50}$$

Using the definition and asymptotics of the Incomplete Gamma function (see [7] 8.2.2, 8.2.11), we have

$$\int_0^{-\Im[a]\delta^2} e^{-\tau}\tau^{1/2-1}d\tau = \Gamma(1/2) - \Gamma(1/2, -\Im[a]\delta^2) = \sqrt{\pi} + \mathrm{O}\left(\sqrt{-\Im[a]}e^{\Im[a]\delta^2}\right) \quad (3.51)$$

and

$$\left|\int_0^{-\Im[a]\delta^2} e^{-\tau}\tau^{3/2-1}R_2\left(t\left(\sqrt{\frac{-\tau}{\Im[a]}}\right), \eta, \lambda\right)d\tau\right| \quad (3.52)$$

$$\leq R\int_0^{-\Im[a]\delta^2} e^{-\tau}\tau^{3/2-1}d\tau \quad (3.53)$$

$$= R\cdot\Gamma(3/2) + \mathrm{O}\left((-\Im[a])^{3/2}e^{\Im[a]\delta^2}\right) \quad (3.54)$$

with the error term being uniform with respect to η, λ. Now we have

$$\int_{\gamma_{C,\eta}} F(t,\eta,\lambda)e^{-\Im[a(\lambda)]S_\eta(t)}dt = \frac{e^{-\Im[a]S_\eta(t_-^*)}}{2\sqrt{-\Im[a]}}\left[b_0(\eta,\lambda)\sqrt{\pi} + \mathrm{O}\left(\frac{1}{\Im[a]}\right)\right], \quad (3.55)$$

which is equivalent to the statement of the Lemma since $b_0(\eta,\lambda)$ is uniformly bounded away from 0, due to the assumptions on $F(t_-^*(\eta),\eta,\lambda)$.

□

Next we show that the contribution away from the saddle point is negligible.

Lemma 3.1.9. *Assume $F(t,\eta,\lambda)$ satisfies the hypothesis of Theorem 3.1.1. Then,* *as $\lambda \to 0$ with $\Im\lambda \geq 0$, we have the bound*

$$\left| \int_{\gamma_{L+R,\eta}} F(t,\eta,\lambda) e^{-\Im[a(\lambda)]S_\eta(t)}\ dt \right| \leq e^{-\Im[a(\lambda)]\Re[S_\eta(t_-^*(\eta))]} \cdot e^{-\Im[a(\lambda)]c_*} K_{L+R}, \tag{3.56}$$

where $S_\eta(t)$ is defined in (3.3) *and constants c_*, K_{L+R} are η, λ independent, finite, and $c_* < 0$.*

Proof. The constant c_* is defined as

$$c_* = \max_{\eta \in \Omega_+} \max_{t \in \gamma_{L+R,\eta}} \Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right]. \tag{3.57}$$

For any fixed $\eta \in \Omega_+$, $\max_{t \in \gamma_{L+R,\eta}} \Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right] < 0$ because this was how the path $\gamma_{L+R,\eta}$ was constructed. Since Ω_+ is a compact set, $c_* < 0$. Now to prove the inequality,

$$\left| \int_{\gamma_{L+R,\eta}} F(t,\eta,\lambda) e^{-\Im[a(\lambda)]S_\eta(t)}\ dt \right| \tag{3.58}$$

$$\leq e^{-\Im[a(\lambda)]\Re[S_\eta(t_-^*(\eta))]} \int_{\gamma_{L+R,\eta}} |F(t,\eta,\lambda)|\, e^{-\Im[a(\lambda)]\Re\left[S_\eta(t) - S_\eta(t_-^*(\eta))\right]}\ dt \tag{3.59}$$

$$\leq e^{-\Im[a(\lambda)]\Re[S_\eta(t_-^*(\eta))]} e^{-\Im[a(\lambda)]c_*} \int_{\gamma_{L+R,\eta}} |F(t,\eta,\lambda)|\ dt \tag{3.60}$$

$$\leq e^{-\Im[a(\lambda)]\Re[S_\eta(t_-^*(\eta))]} \cdot e^{-\Im[a(\lambda)]c_*} K_{L+R} \tag{3.61}$$

where

$$K_{L+R} := \max_{\eta \in \Omega_+} \max_{\lambda \in \overline{B}(0,\epsilon)} \int_{\gamma_{L+R,\eta}} |F(t,\eta,\lambda)|\ dt. \tag{3.62}$$

This constant is finite since for every $(\eta,\lambda) \in \Omega_+ \times \overline{B}(0,\epsilon)$, $F(t,\eta,\lambda)$ is analytic for

$t \in B(1/2, 1/2)$, has L^1 behavior at endpoints $t = 0, 1$, and $\gamma_{L+R,\eta}$ has finite arc length.

□

The combination of Lemmas 3.1.8 and 3.1.9 proves Theorem 3.1.1 provided that λ is in the upper half plane.

Remark 3.1.10. When $\lambda \to 0$ in the lower half plane, the key difference is that $\Im[a(\lambda)] \to \infty$ (see Proposition B.0.1). With that in mind, rewrite the integrand of (3.7) as

$$F(t, \eta, \lambda) e^{-\Im[a(\lambda)] S_\eta(t)} = F(t, \eta, \lambda) e^{\Im[a(\lambda)][-S_\eta(t)]}. \tag{3.63}$$

We can replace S with $-S$ and carry out the same analysis as before. The only difference will be that the regions in the t plane where $\Re[S_\eta(t) - S_\eta(t^*_-(\eta))] < 0$ and $\Re[S_\eta(t) - S_\eta(t^*_-(\eta))] > 0$ will swap, so we deform $[0, 1]$ to a different contour, see Figure 3.4. All previous ideas from this section can now be applied using the new contour γ.

Remark 3.1.11. Throughout this subsection we have placed a number of restrictions on the constant M, which is used to describe the region Ω, Ω_+, see (3.5), (3.6). We fix $M < \infty$ so that (3.17), (3.25), (3.29), and Lemma 3.1.3 are satisfied.

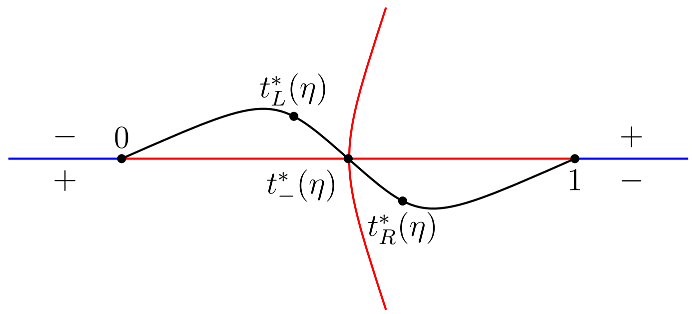

Figure 3.4: Path of integration when λ is in the lower half plane.

3.2 Small λ asymptotics of $\Gamma(z;\lambda)$ for $z \in \tilde{\Omega}$

Using our steepest descent method (Theorem 3.1.1), we can now calculate the leading order asymptotics of $\Gamma(z;\lambda)$. We will calculate the leading order behavior of each factor of $\Gamma(z;\lambda)$ (see (2.17)) and state the small λ leading order behavior of $\Gamma(z;\lambda)$ for $\frac{z+1}{2z} \in \Omega$, see (3.5) for Ω. Define sets

$$\tilde{\Omega} := \left\{ z : \ \frac{z+1}{2z} \in \Omega \right\}, \tag{3.64}$$

$$\tilde{\Omega}_+ := \left\{ z : \ \frac{z+1}{2z} \in \Omega_+ \right\}, \tag{3.65}$$

where Ω, Ω_+ are defined in (3.5), (3.6). Notice that $\tilde{\Omega}$ is a small annulus about the origin. We begin with applying Theorem 3.1.1 to the hypergeometric functions which are present in $\Gamma(z;\lambda)$. We will frequently use the change of variables

$$\varkappa = -\ln \lambda. \tag{3.66}$$

Corollary 3.2.1. *As $\lambda = e^{-\varkappa} \to 0$, provided that either $\Im\lambda \geq 0$ or $\Im\lambda \leq 0$,*

$$h_\infty\left(\frac{z+1}{2z}\right) = i4^a \frac{\sqrt{2z}(1-z^2)^{1/4}}{1+\sqrt{1-z^2}} e^{\mp\varkappa g(z)\pm\frac{\varkappa}{2}} \left(1 + \mathrm{O}\left(\varkappa^{-1}\right)\right), \tag{3.67}$$

$$h'_\infty\left(\frac{z+1}{2z}\right) = i4^a \frac{\sqrt{2z}(1-z^2)^{1/4}}{1+\sqrt{1-z^2}} e^{\mp\varkappa g(z)\pm\frac{\varkappa}{2}} \frac{\mp 2z\varkappa}{i\pi\sqrt{1-z^2}} \left(1 + \mathrm{O}\left(\varkappa^{-1}\right)\right), \tag{3.68}$$

$$s_\infty\left(\frac{z+1}{2z}\right) = i4^{-a} \frac{1+\sqrt{1-z^2}}{\sqrt{2z}} \frac{(1-z^2)^{1/4}}{2z} e^{\pm\varkappa g(z)\mp\frac{\varkappa}{2}} \left(1 + \mathrm{O}\left(\varkappa^{-1}\right)\right), \tag{3.69}$$

$$s'_\infty\left(\frac{z+1}{2z}\right) = i4^{-a} \frac{1+\sqrt{1-z^2}}{\sqrt{2z}} \frac{(1-z^2)^{1/4}}{2z} e^{\pm\varkappa g(z)\mp\frac{\varkappa}{2}} \frac{\pm 2z\varkappa}{i\pi\sqrt{1-z^2}} \left(1 + \mathrm{O}\left(\varkappa^{-1}\right)\right), \tag{3.70}$$

where $a := a(\lambda)$, see (2.12)*,* $\operatorname{sgn}(\Im\varkappa) = \mp 1$ *and each approximation is uniform for $z \in \tilde{\Omega}_+$. The functions h_∞, h'_∞ and s_∞, s'_∞ are defined in* (2.13)*,*(2.14)*, respectively. See* (3.65)*,* (3.106) *for $\tilde{\Omega}_+, g(z)$, respectively. The functions $\sqrt{1-z^2}$ and $(1-z^2)^{1/4}$ have a branch cuts on $[-1,1]$ and $(-\infty,1)$, respectively.*

Proof. The result is a consequence of Theorem 3.1.1 with particular selections of the function $F(t,\eta,\lambda)$ and the following calculations. Using the integral representation of hypergeometric functions in [1] 15.3.1, we see that

$$h_\infty(\eta) = e^{a\pi i}\eta^{-a}{}_2F_1\left(\begin{matrix} a, a+1 \\ 2a+2 \end{matrix} \,\middle|\, \frac{1}{\eta}\right) \tag{3.71}$$

$$= e^{a\pi i}\eta^{-a}\frac{\Gamma(2a+2)}{\Gamma(a+1)^2}\int_0^1 \left(\frac{t(1-t)}{1-t/\eta}\right)^a dt \tag{3.72}$$

$$= e^{a\pi i}\eta^{-a}\frac{\Gamma(2a+2)}{\Gamma(a+1)^2}\sqrt{\frac{\pi}{a(\lambda)}}\left(1-\frac{1}{\eta}\right)^{1/4}\left(1+\sqrt{1-\frac{1}{\eta}}\right)^{-1-2a(\lambda)}\left[1+\mathrm{O}\left(\frac{1}{a(\lambda)}\right)\right], \tag{3.73}$$

where we have used Theorem 3.1.1 with $F(t,\eta,\lambda) = e^{i\Re[a(\lambda)]S_\eta(t)}$. Taking $\eta = \frac{z+1}{2z}$, we have

$$h_\infty\left(\frac{z+1}{2z}\right) = \frac{e^{a\pi i}\Gamma(2a+2)}{\Gamma(a+1)^2}\sqrt{\frac{\pi}{a}}\left(\frac{1-z}{1+z}\right)^{\frac{1}{4}}\left(\frac{2z}{z+1}\right)^a\left(1+\sqrt{\frac{1-z}{1+z}}\right)^{-1-2a}\left[1+\mathrm{O}\left(\frac{1}{a}\right)\right]. \tag{3.74}$$

We calculate that

$$\left(\frac{z+1}{2z}\right)^{-a}\left(1+\sqrt{\frac{1-z}{1+z}}\right)^{-2a} = \exp\left[-a\pi i g(z) - \frac{a\pi i}{2}\right] \tag{3.75}$$

$$= \frac{\sqrt{z(1+z)}}{\sqrt{2}(1+\sqrt{1-z^2})}e^{-\varkappa g(z)-\varkappa/2}\left(1+\mathrm{O}\left(\lambda^2\right)\right) \tag{3.76}$$

as $\lambda \to 0$ with $\Im\lambda \geq 0$ and $z \in \tilde{\Omega}_+$, where we have used Proposition B.0.1 and $g(z)$ is defined in (3.106). From [7] 5.5.5 and 5.11.13, we have

$$\frac{\Gamma(2a+2)}{\Gamma(a+1)^2} = \frac{4^{a+1/2}}{\sqrt{\pi}}\cdot\frac{\Gamma\left(a+\frac{3}{2}\right)}{\Gamma(a+1)} = 4^{a+1/2}\sqrt{\frac{a}{\pi}}\left(1+\mathrm{O}\left(\frac{1}{a}\right)\right). \tag{3.77}$$

Combining the previous two equations gives the result for $h_\infty\left(\frac{z+1}{2z}\right)$ when $\Im\lambda \geq 0$. The approximation when $\Im\lambda \leq 0$ can be found in an similar manner. Now for $h'_\infty(\eta)$ (see (2.13) for definition), we use [1] 15.3.1 to obtain the following integral representation.

$$h'_\infty(\eta) = -e^{a\pi i}\frac{a\Gamma(2a+2)}{\eta^{a+1}\Gamma(a+1)^2}\int_0^1 \frac{1}{1-t/\eta}\left(\frac{t(1-t)}{1-t/\eta}\right)^a dt \tag{3.78}$$

$$= -2ae^{a\pi i}4^a\eta^{-a-1}\left(1-\frac{1}{\eta}\right)^{-1/4}\left(1+\sqrt{1-\frac{1}{\eta}}\right)^{-1-2a(\lambda)}\left\{1+\mathrm{O}\left(\frac{1}{a}\right)\right\} \tag{3.79}$$

We have used Theorem 3.1.1 with $F(t,\eta,\lambda) = \frac{e^{i\Re[a(\lambda)]S_\eta(t)}}{1-t/\eta}$. Notice that $h_\infty(\eta)\big|_{a\to -a-1} = s_\infty(\eta)$. The functions $s_\infty(\eta), s'_\infty(\eta)$, as written, only have the integral representation [1] 15.3.1 for $-1/2 \le \Re[a(\lambda)] < 0$. To remedy this, use [7] 15.5.19 with $z \to 1/\eta$, $a \to -a$, $b \to -a-1$, and $c \to -2a$ to obtain

$$\begin{aligned}
&\frac{a(a-1)}{\eta}\left(1-\frac{1}{\eta}\right){}_2F_1\left(\begin{matrix}-a+2,-a+1\\-2a+2\end{matrix}\,\middle|\,\frac{1}{\eta}\right)\\
&+2a(2a-1)\left(1-\frac{1}{\eta}\right){}_2F_1\left(\begin{matrix}-a+1,-a\\-2a+1\end{matrix}\,\middle|\,\frac{1}{\eta}\right)\\
&=2a(2a-1){}_2F_1\left(\begin{matrix}-a,-a-1\\-2a\end{matrix}\,\middle|\,\frac{1}{\eta}\right).
\end{aligned} \tag{3.80}$$

Now with $z \to 1/\eta$, $a \to -a+1$, $b \to -a$, and $c \to -2a+1$, we have

$$\begin{aligned}
&\frac{a(a-2)}{\eta}\left(1-\frac{1}{\eta}\right){}_2F_1\left(\begin{matrix}-a+3,-a+2\\-2a+3\end{matrix}\,\middle|\,\frac{1}{\eta}\right)\\
&+2(1-a)\left(1-2a+\frac{2(a-1)}{\eta}\right){}_2F_1\left(\begin{matrix}-a+2,-a+1\\-2a+2\end{matrix}\,\middle|\,\frac{1}{\eta}\right)\\
&=2(1-a)(1-2a){}_2F_1\left(\begin{matrix}-a+1,-a\\-2a+1\end{matrix}\,\middle|\,\frac{1}{\eta}\right).
\end{aligned} \tag{3.81}$$

Combining the two previous equations, we see that

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}-a,-a-1\\-2a\end{matrix}\,\middle|\,\frac{1}{\eta}\right) &= \frac{a(a-2)}{2\eta(a-1)(2a-1)}\left(1-\frac{1}{\eta}\right)^2{}_2F_1\left(\begin{matrix}-a+3,-a+2\\-2a+3\end{matrix}\,\middle|\,\frac{1}{\eta}\right)\\
&\quad+a\left(1-\frac{1}{\eta}\right)\left[2(2a-1)+\frac{3(1-a)}{\eta}\right]{}_2F_1\left(\begin{matrix}-a+2,-a+1\\-2a+2\end{matrix}\,\middle|\,\frac{1}{\eta}\right).
\end{aligned} \tag{3.82}$$

The perk of this equation is that the right hand side has an integral representation for

$-1/2 \leq \Re[a] \leq 1/2$. Thus we can apply Theorem 3.1.1 twice and obtain the leading order asymptotics. So we have shown

$$s_\infty(\eta) = -e^{-a\pi i}\eta^{a+1}{}_2F_1\left(\begin{matrix}-a-1,-a\\-2a\end{matrix}\,\middle|\,\frac{1}{\eta}\right) \tag{3.83}$$

$$= -\frac{1}{2}e^{-a\pi i}4^{-a}\eta^{a+1}\left(1+\sqrt{1-\frac{1}{\eta}}\right)^{1+2a}\left(1-\frac{1}{\eta}\right)^{1/4}\left[1+\mathrm{O}\left(\frac{1}{a}\right)\right]. \tag{3.84}$$

A similar process can be repeated for $s'_\infty(\eta)$ and we obtain

$$s'_\infty(\eta) = \frac{a+1}{-e^{a\pi i}}\eta^a{}_2F_1\left(\begin{matrix}-a,-a\\-2a\end{matrix}\,\middle|\,\frac{1}{\eta}\right) \tag{3.85}$$

$$= -2(a+1)e^{-a\pi i}4^{-a-1}\eta^a\left(1-\frac{1}{\eta}\right)^{-1/4}\left(1+\sqrt{1-\frac{1}{\eta}}\right)^{1+2a(\lambda)}\left[1+\mathrm{O}\left(\frac{1}{a}\right)\right]. \tag{3.86}$$

□

We have an immediate Corollary.

Corollary 3.2.2. *As $\lambda = e^{-\varkappa} \to 0$, provided that either $\Im\lambda \geq 0$ or $\Im\lambda \leq 0$,*

$$\hat{\Gamma}\left(\frac{z+1}{2z}\right) = i(1-z^2)^{\sigma_3/4}\begin{bmatrix}1 & 0\\0 & \pm\frac{2z\varkappa}{i\pi}\end{bmatrix}(I+i\sigma_2)\left(\frac{\sqrt{2z}}{1+\sqrt{1-z^2}}\right)^{\sigma_3}\begin{bmatrix}1 & 0\\0 & \frac{1}{2z}\end{bmatrix}\times$$

$$\times\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)4^{a\sigma_3}e^{\mp\varkappa g(z)\sigma_3}e^{\pm\frac{\varkappa}{2}\sigma_3}, \quad \mathrm{sgn}[\Im\varkappa] = \mp 1 \tag{3.87}$$

which is uniform for $z \in \tilde{\Omega}_+$. See (3.65), (2.18), (3.106) *for $\tilde{\Omega}_+, \hat{\Gamma}, g$, respectively.*

It remains to find the small λ leading order asymptotics of the remaining factors of

$\Gamma(z;\lambda)$, see (2.17). This is a tedious, but straightforward exercise.

Lemma 3.2.3. *We are able to evaluate* $\hat{\Gamma}\left(\frac{1}{2}\right)$ *(see* (2.18) *for definition of* $\hat{\Gamma}$*) explicitly in terms of Gamma functions as*

$$\hat{\Gamma}^{-1}\left(\frac{1}{2}\right)=e^{-\frac{a\pi i}{2}\sigma_3}4^{-a\sigma_3}\begin{bmatrix}\frac{\Gamma\left(-\frac{a}{2}\right)\Gamma\left(-\frac{1}{2}-a\right)}{4\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)\Gamma(-1-a)} & \frac{i\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)\Gamma\left(-\frac{1}{2}-a\right)}{8\Gamma\left(1-\frac{a}{2}\right)\Gamma(-a)}\\ \frac{i\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(a+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+1\right)\Gamma(a)} & \frac{-\Gamma\left(\frac{a}{2}+1\right)\Gamma\left(a+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma(a+2)}\end{bmatrix},\tag{3.88}$$

where $a(\lambda)$ *is defined in* (2.12)*. Moreover, as* $\lambda=e^{-\varkappa}\to 0$*, provided that either* $\Im\lambda\geq 0$ *or* $\Im\lambda\leq 0$*,*

$$\hat{\Gamma}^{-1}\left(\frac{1}{2}\right)=\sqrt{\frac{2}{i}}e^{\mp\frac{\varkappa}{2}\sigma_3}4^{-a\sigma_3}\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)\begin{bmatrix}\frac{1}{4} & 0\\ 0 & 1\end{bmatrix}(I+i\sigma_2)\begin{bmatrix}1 & 0\\ 0 & \pm\frac{\pi}{2\varkappa}\end{bmatrix},\quad \operatorname{sgn}[\Im\varkappa]=\mp 1.\tag{3.89}$$

Proof. Recall, from (2.13), that

$$h_\infty\left(\frac{z+1}{2z}\right)=e^{a\pi i}\left(\frac{z+1}{2z}\right)^{-a}{}_2F_1\left(\begin{matrix}a,a+1\\ 2a+2\end{matrix}\,\middle|\,\frac{2z}{z+1}\right).\tag{3.90}$$

Using [1] 15.3.15 then 15.1.20,

$${}_2F_1\left(\begin{matrix}a,a+1\\ 2a+2\end{matrix}\,\middle|\,2\right)=(-1)^{-a/2}{}_2F_1\left(\begin{matrix}\frac{a}{2},\frac{a}{2}+1\\ a+\frac{3}{2}\end{matrix}\,\middle|\,1\right)=e^{-a\pi i/2}\frac{\sqrt{\pi}\Gamma\left(a+\frac{3}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{3}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)},\tag{3.91}$$

where we have taken $(-1)^{-a/2}=e^{-a\pi i/2}$. Repeating this process for $h'_\infty, s_\infty, s'_\infty$, we obtain our explicit result. The asymptotics directly follow from the use of Stirling's formula and Proposition B.0.1.

□

Recall from (2.17) that $Q(\lambda)$ is defined as

$$Q(\lambda) = D(I + ie^{a\pi i}\sigma_2), \tag{3.92}$$

where

$$D := \begin{bmatrix} -\tan(a\pi) & 0 \\ 0 & 4^{2a+1}e^{a\pi i}\frac{\Gamma(a+3/2)\Gamma(a+1/2)}{\Gamma(a)\Gamma(a+2)} \end{bmatrix}. \tag{3.93}$$

Lemma 3.2.4. *As $\lambda = e^{-\varkappa} \to 0$ provided that either $\Im\lambda \geq 0$ or $\Im\lambda \leq 0$,*

$$D = i \begin{bmatrix} \pm 1 & 0 \\ 0 & 4^{2a+1}e^{\pm\varkappa} \end{bmatrix} \left(I + \mathrm{O}\left(\varkappa^{-1}\right)\right), \quad \mathrm{sgn}[\Im\varkappa] = \mp 1. \tag{3.94}$$

Proof. Asymptotics of ratios of Gamma functions can be found in [7] 5.11.13. This fact combined Proposition B.0.1 gives the result. □

We are ready to put the pieces from this section together and obtain the asymptotics of $\Gamma(z;\lambda)$ as $\lambda \to 0$ for $z \in \tilde{\Omega}$. Define the matrix

$$\Phi(z) := \frac{1}{2\sqrt{z}(z^2-1)^{1/4}} \begin{bmatrix} i + z + \sqrt{z^2-1} & -i - z + \sqrt{z^2-1} \\ i - z + \sqrt{z^2-1} & -i + z + \sqrt{z^2-1} \end{bmatrix} \tag{3.95}$$

$$= \left(I + \frac{i}{2z}(\sigma_3 - i\sigma_2)\right)\left(\frac{z^2-1}{z^2}\right)^{\sigma_1/4} \tag{3.96}$$

which is a solution to the so-called model RHP (see RHP 3.3.4 and take $x = y = i/2$).

Lemma 3.2.5. *As $\lambda = e^{-\varkappa} \to 0$, provided that either $\Im\lambda \geq 0$ or $\Im\lambda \leq 0$,*

$$D^{-1}\hat{\Gamma}^{-1}\left(\frac{1}{2}\right)\begin{bmatrix}1 & \frac{-1}{2z(a+1)}\\ 0 & 1\end{bmatrix}\hat{\Gamma}\left(\frac{z+1}{2z}\right)e^{\varkappa g(z)\sigma_3}D = \begin{cases}\Phi(z)\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right), & \Im\lambda \geq 0\\ \sigma_3\Phi(z)\sigma_3\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right), & \Im\lambda \leq 0\end{cases} \tag{3.97}$$

which is uniform for $z \in \tilde{\Omega}_+$. See (3.65), (3.26), (2.18), (3.95), (3.106) *for definitions of $\tilde{\Omega}_+$, D, $\hat{\Gamma}$, Φ, $g(z)$, respectively.*

Proof. First take $\Im\lambda \geq 0$; we begin with a few observations and preparatory calculations:

$$\left(\frac{\sqrt{2z}}{1+\sqrt{1-z^2}}\right)^{\sigma_3}\begin{bmatrix}1 & 0\\ 0 & \frac{2}{z}\end{bmatrix} = \frac{\sqrt{2}}{z^{3/2}}\begin{bmatrix}1-\sqrt{1-z^2} & 0\\ 0 & 1+\sqrt{1-z^2}\end{bmatrix} = \frac{\sqrt{2}}{z^{3/2}}\left[I-\sigma_3\sqrt{1-z^2}\right], \tag{3.98}$$

$$(I+i\sigma_2)\begin{bmatrix}1 & -1\\ 0 & -iz\end{bmatrix}(1-z^2)^{\sigma_3/4}(I+i\sigma_2) = \frac{(I-\sigma_1)+iz\left(\sigma_3-i\sigma_2\right)+\sqrt{1-z^2}\left(\sigma_3+i\sigma_2\right)}{(1-z^2)^{1/4}}. \tag{3.99}$$

Recall that the radicals were defined so that $\sqrt{1-z^2} = i\sqrt{z^2-1}$ and $(1-z^2)^{1/4} = \sqrt{i}(z^2-1)^{1/4}$. Now, taking the leading order term of $D^{-1}\hat{\Gamma}^{-1}\left(\frac{1}{2}\right)\begin{bmatrix}1 & \frac{-1}{2z(a+1)}\\ 0 & 1\end{bmatrix}\hat{\Gamma}\left(\frac{z+1}{2z}\right)e^{\varkappa g\sigma_3}D$

from Lemmas and Corollaries 3.2.4, 3.2.3, 3.2.2, we have

$$\begin{aligned}
&\frac{\sqrt{2i}}{4}(I+i\sigma_2)\begin{bmatrix}1 & -1\\ 0 & -iz\end{bmatrix}(1-z^2)^{\sigma_3/4}(I+i\sigma_2)\left(\frac{\sqrt{2z}}{1+\sqrt{1-z^2}}\right)^{\sigma_3}\begin{bmatrix}1 & 0\\ 0 & \frac{2}{z}\end{bmatrix}\\
&=\frac{(z^2-1)^{-1/4}}{2z^{3/2}}\left[(I-\sigma_1)+iz\left(\sigma_3-i\sigma_2\right)+\sqrt{1-z^2}\left(\sigma_3+i\sigma_2\right)\right]\cdot\left[I-\sigma_3\sqrt{1-z^2}\right]\\
&=\frac{(z^2-1)^{-1/4}}{2z^{3/2}}\left[(I-\sigma_1)-\sqrt{1-z^2}\left(\sigma_3+i\sigma_2\right)iz\left(\sigma_3-i\sigma_2\right)-iz\sqrt{1-z^2}\left(I+\sigma_1\right)\right.\\
&\qquad\left.+\sqrt{1-z^2}\left(\sigma_3+i\sigma_2\right)-\left(1-z^2\right)\left(I-\sigma_1\right)\right]\\
&=\frac{(z^2-1)^{-1/4}}{2\sqrt{z}}\left[(i\sigma_3+\sigma_2)+z\left(I-\sigma_1\right)+\sqrt{z^2-1}\left(I+\sigma_1\right)\right]\\
&=\Phi(z).
\end{aligned}$$

When $\Im\lambda \le 0$, observe that the leading order term of $D^{-1}\hat{\Gamma}^{-1}\left(\frac{1}{2}\right)\begin{bmatrix}1 & \frac{-1}{2z(a+1)}\\ 0 & 1\end{bmatrix}\hat{\Gamma}\left(\frac{z+1}{2z}\right)e^{\varkappa g\sigma_3}D$ is now

$$\sigma_3\cdot\frac{\sqrt{2i}}{4}(I+i\sigma_2)\begin{bmatrix}1 & -1\\ 0 & -iz\end{bmatrix}(1-z^2)^{\sigma_3/4}(I+i\sigma_2)\left(\frac{\sqrt{2z}}{1+\sqrt{1-z^2}}\right)^{\sigma_3}\begin{bmatrix}1 & 0\\ 0 & \frac{2}{z}\end{bmatrix}\cdot\sigma_3 \quad (3.100)$$

and thus we have the leading order term. Now adding in the lower order terms, we obtain the result with careful calculation.

□

Define matrix

$$\Psi(z;\varkappa)=\begin{cases}\Phi(z), & \text{for } \Im[\varkappa]\le 0,\\ \sigma_1\Phi(z)\sigma_1, & \text{for } \Im[\varkappa]\ge 0.\end{cases} \tag{3.101}$$

where $\Phi(z)$ was defined in (3.95). The matrix $\sigma_1\Phi(z)\sigma_1$ is also a solution to RHP 3.3.4, take $x=y=-i/2$. Now we are ready to prove one of the main results of this chapter.

Theorem 3.2.6. *We have the following approximation*

$$\Gamma(z;\lambda)=\begin{cases}\Psi(z;\varkappa)\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)e^{-\varkappa g(z)\sigma_3}, & z\in\tilde{\Omega}\setminus\mathcal{L}_{L,R}^{(\pm)},\\ \Psi(z;\varkappa)\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)\begin{bmatrix}1 & 0\\ \pm ie^{\varkappa(2g(z)-1)} & 1\end{bmatrix}e^{-\varkappa g(z)\sigma_3}, & z\in\tilde{\Omega}\cap\mathcal{L}_{L}^{(\pm)},\\ \Psi(z;\varkappa)\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)\begin{bmatrix}1 & \mp ie^{-\varkappa(2g(z)+1)}\\ 0 & 1\end{bmatrix}e^{-\varkappa g(z)\sigma_3}, & z\in\tilde{\Omega}\cap\mathcal{L}_{R}^{(\pm)},\end{cases}$$

as $\lambda=e^{-\varkappa}\to 0$*, provided that either* $\Im\lambda\ge 0$ *or* $\Im\lambda\le 0$*, which is uniform for* $z\in\tilde{\Omega}$*. See* (3.64)*,* (3.101)*,* (3.106)*,* (2.17) *for* $\tilde{\Omega},\Psi,g,\Gamma$*, respectively and Figure 3.5 for the sets* $\mathcal{L}_L^{(\pm)}$ *and* $\mathcal{L}_R^{(\pm)}$*.*

Proof. First assume $\Im\lambda\ge 0$ and $z\in\tilde{\Omega}_+$. The following calculation

$$e^{-\varkappa g\sigma_3}Q=e^{-\varkappa g\sigma_3}D\left(I+ie^{a\pi i}\sigma_2\right)e^{-\varkappa g\sigma_3}e^{\varkappa g\sigma_3}=D\left(e^{-2\varkappa g\sigma_3}+ie^{a\pi i}\sigma_2\right)e^{\varkappa g\sigma_3} \tag{3.102}$$

and use of Lemma 3.2.5 gives us

$$
\begin{aligned}
\Gamma(z;\lambda) &= \sigma_2 Q^{-1}\hat{\Gamma}^{-1}(\infty)\begin{bmatrix}1 & \frac{-1}{2z(a+1)}\\ 0 & 1\end{bmatrix}\hat{\Gamma}\left(\frac{z+1}{2z}\right)Q\sigma_2\\
&= \frac{\sigma_2}{1+e^{2a\pi i}}(I - ie^{a\pi i}\sigma_2)D^{-1}\hat{\Gamma}^{-1}(\infty)\begin{bmatrix}1 & \frac{-1}{2z(a+1)}\\ 0 & 1\end{bmatrix}\hat{\Gamma}\left(\frac{z+1}{2z}\right)e^{\varkappa g\sigma_3}D\times\\
&\times(\sigma_2 e^{2\varkappa g\sigma_3} + ie^{a\pi i}I)e^{-\varkappa g\sigma_3}\\
&= \frac{\sigma_2}{1+e^{2a\pi i}}(I - ie^{a\pi i}\sigma_2)\Phi(z)\left(I + \mathrm{O}\left(\varkappa^{-1}\right)\right)(\sigma_2 e^{2\varkappa g\sigma_3} + ie^{a\pi i}I)e^{-\varkappa g\sigma_3}\\
&= \frac{\Phi(z)}{1+e^{2a\pi i}}(\Phi^{-1}(z)\sigma_2\Phi(z) - ie^{a\pi i}I)\left(I + \mathrm{O}\left(\varkappa^{-1}\right)\right)(\sigma_2 e^{2\varkappa g\sigma_3} + ie^{a\pi i}I)e^{-\varkappa g\sigma_3}\\
&= \frac{\Phi(z)}{1+e^{2a\pi i}}\left(I + \mathrm{O}\left(\varkappa^{-1}\right)\right)(\Phi^{-1}(z)\sigma_2\Phi(z) - ie^{a\pi i}I)(\sigma_2 e^{2\varkappa g\sigma_3} + ie^{a\pi i}I)e^{-\varkappa g\sigma_3}\\
&= \Phi(z)\left(I + \mathrm{O}\left(\varkappa^{-1}\right)\right)(I - \frac{i\sigma_2}{e^{a\pi i}}e^{2\varkappa g\sigma_3})e^{-\varkappa g\sigma_3}\\
&= \Phi(z)\left(I + \mathrm{O}\left(\varkappa^{-1}\right)\right)\begin{bmatrix}1 & ie^{-\varkappa(2g+1)}\\ -ie^{\varkappa(2g-1)} & 1\end{bmatrix}e^{-\varkappa g\sigma_3},
\end{aligned}
\tag{3.103}
$$

as desired. Now take $\Im\lambda \leq 0$ and proceed similar to above. We use the calculation

$$
e^{\varkappa g\sigma_3}Q = e^{\varkappa g\sigma_3}D(I + ie^{a\pi i}\sigma_2)e^{-\varkappa g\sigma_3}e^{\varkappa g\sigma_3} = D(I + ie^{a\pi i}e^{2\varkappa g\sigma_3}\sigma_2)e^{\varkappa g\sigma_3} \tag{3.104}
$$

and Lemma 3.2.5 to obtain

$$
\begin{aligned}
\Gamma(z;\lambda) &= \sigma_2 Q^{-1}\hat{\Gamma}^{-1}(\infty)\begin{bmatrix}1 & \frac{-1}{2z(a+1)}\\ 0 & 1\end{bmatrix}\hat{\Gamma}(z)Q\sigma_2\\
&= \frac{\sigma_2}{1+e^{2a\pi i}}(I-ie^{a\pi i}\sigma_2)D^{-1}\hat{\Gamma}^{-1}(\infty)\begin{bmatrix}1 & \frac{-1}{2z(a+1)}\\ 0 & 1\end{bmatrix}\hat{\Gamma}(z)e^{-\varkappa g\sigma_3}\times\\
&\times D(\sigma_2+ie^{a\pi i}e^{2\varkappa g\sigma_3})e^{-\varkappa g\sigma_3}\\
&= \frac{\sigma_2}{1+e^{2a\pi i}}(I-ie^{a\pi i}\sigma_2)\sigma_3\Phi(z)\sigma_3\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)(\sigma_2+ie^{a\pi i}e^{2\varkappa g\sigma_3})e^{-\varkappa g\sigma_3}\\
&= i\sigma_1\Phi(z)(I+e^{a\pi i}\Phi^{-1}(z)\sigma_2\Phi(z))\sigma_3\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)(\sigma_2+ie^{a\pi i}e^{2\varkappa g\sigma_3})e^{-\varkappa g\sigma_3}\\
&= i\sigma_1\Phi(z)\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)(I+e^{a\pi i}\Phi^{-1}(z)\sigma_2\Phi(z))\sigma_3(\sigma_2+ie^{a\pi i}e^{2\varkappa g\sigma_3})e^{-\varkappa g\sigma_3}\\
&= i\sigma_1\Phi(z)\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)(-i\sigma_1+ie^{a\pi i}\sigma_3e^{2\varkappa g\sigma_3})e^{-\varkappa g\sigma_3}\\
&= \sigma_1\Phi(z)\sigma_1\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)\begin{bmatrix}1 & ie^{-\varkappa(2g+1)}\\ -ie^{\varkappa(2g-1)} & 1\end{bmatrix}e^{-\varkappa g\sigma_3}. \qquad (3.105)
\end{aligned}
$$

The results for $\Im z \geq 0$ are immediate via use of the symmetry $\overline{\Gamma(\bar{z};\bar{\lambda})} = \Gamma(z;\lambda)$. Recall that $\overline{\Phi(\bar{z})} = \sigma_1\Phi(z)\sigma_1$ from Remark 3.3.6, $\overline{g(\bar{z})} = g(z)$ and $\varkappa(\lambda) = \overline{\varkappa(\bar{\lambda})}$.

□

3.3 Deift-Zhou steepest descent method

Let us begin with the definition of the g-function

$$
g(z) = \frac{1}{i\pi}\ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) - \frac{1}{2}, \qquad (3.106)
$$

where the branch cut of $\sqrt{1-z^2}$ is $[-1,1]$, $\sqrt{1-z^2} = iz + \mathrm{O}(1)$ as $z \to \infty$ and the principle branch of the logarithm is taken. This g-function will play an important role so we list its relevant properties, all of which are simple calculations.

Proposition 3.3.1. *$g(z)$ has the following properties:*

1. *$g(z)$ is analytic on $\mathbb{C} \setminus [-1,1]$,*
2. *$g_+(z) + g_-(z) = 1$ for $z \in [-1,0]$,*
3. *$g_+(z) + g_-(z) = -1$ for $z \in [0,1]$,*
4. *$\Re(2g(z)-1) = 0$ for $z \in [-1,0]$ and $\Re(2g(z)-1) < 0$ for $z \in \overline{\mathbb{C}} \setminus [-1,0]$,*
5. *$\Re(2g(z)+1) = 0$ for $z \in [0,1]$ and $\Re(2g(z)+1) > 0$ for $z \in \overline{\mathbb{C}} \setminus [0,1]$,*
6. *$g(\infty) = 0$,*
7. *$g(z)$ is Schwarz symmetric.*

As in the previous section, we wish to work with a large parameter when $\lambda \to 0$ so define

$$\varkappa := -\ln \lambda. \tag{3.107}$$

3.3.1 Transformation $\Gamma(z;\lambda) \to Z(z;\varkappa)$

Our first transformation will be

$$Y(z;\varkappa) := \Gamma(z; e^{-\varkappa}) e^{\varkappa g(z) \sigma_3}, \tag{3.108}$$

where $\Gamma(z;\lambda)$ was defined in (2.17). Since $\Gamma(z;\lambda)$ is the solution of RHP 2.2.1, it is easy to show that $Y(z;\varkappa)$ solves the following RHP.

Riemann-Hilbert Problem 3.3.2. *Find a matrix* $Y(z;\varkappa)$, $e^{-\varkappa} = \lambda \in \mathbb{C} \setminus \{0\}$, *analytic for* $z \in \bar{\mathbb{C}} \setminus [-1,1]$ *and satisfying the following conditions:*

$$Y(z_+;\varkappa) = Y(z_-;\varkappa) \begin{bmatrix} e^{\varkappa(g_+-g_-)} & -ie^{-\varkappa(g_++g_--1)} \\ 0 & e^{-\varkappa(g_+-g_-)} \end{bmatrix}, \quad z \in (-1,0) \tag{3.109}$$

$$Y(z_+;\varkappa) = Y(z_-;\varkappa) \begin{bmatrix} e^{\varkappa(g_+-g_-)} & 0 \\ ie^{\varkappa(g_++g_-+1)} & e^{-\varkappa(g_+-g_-)} \end{bmatrix}, \quad z \in (0,1) \tag{3.110}$$

$$Y(z;\varkappa) = 1 + \mathrm{O}\left(z^{-1}\right) \quad \textit{as } z \to \infty, \tag{3.111}$$

$$Y(z;\varkappa) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(\log(z+1)) \end{bmatrix} \quad \textit{as } z \to -1, \tag{3.112}$$

$$Y(z;\varkappa) = \begin{bmatrix} \mathrm{O}(\log(z-1)) & \mathrm{O}(1) \end{bmatrix} \quad \textit{as } z \to 1, \tag{3.113}$$

$$Y(z;\varkappa) \in L^2_{loc} \quad \textit{as } z \to 0. \tag{3.114}$$

The endpoint behavior is listed column-wise.

The jumps for $Y(z;\varkappa)$ on $(-1,0)$ and $(0,1)$ can be written as

$$Y(z_+;\varkappa) = Y(z_-;\varkappa) \begin{bmatrix} 1 & 0 \\ ie^{\varkappa(2g_-(z)-1)} & 1 \end{bmatrix} (-i\sigma_1) \begin{bmatrix} 1 & 0 \\ ie^{\varkappa(2g_+(z)-1)} & 1 \end{bmatrix}, \quad z \in (-1,0) \tag{3.115}$$

$$Y(z_+;\varkappa) = Y(z_-;\varkappa) \begin{bmatrix} 1 & \frac{1}{i}e^{-\varkappa(2g_-(z)+1)} \\ 0 & 1 \end{bmatrix} (i\sigma_1) \begin{bmatrix} 1 & \frac{1}{i}e^{-\varkappa(2g_+(z)+1)} \\ 0 & 1 \end{bmatrix}, \quad z \in (0,1). \tag{3.116}$$

This decomposition can be verified by direct matrix multiplication and by using the

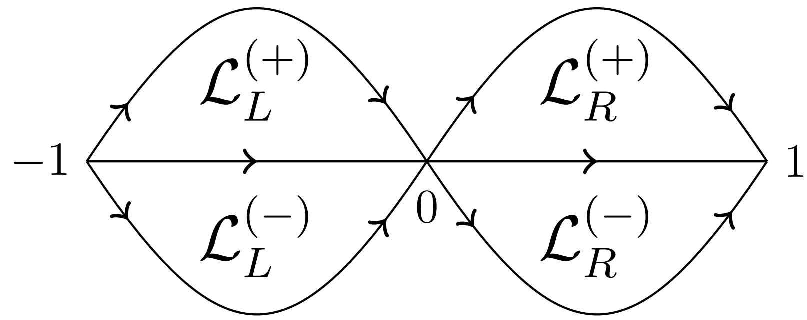

Figure 3.5: Lense regions $\mathcal{L}_{L,R}^{(\pm)}$.

jump properties of $g(z)$ in Proposition 3.3.1. We define the 'lense' regions $\mathcal{L}_{L,R}^{(\pm)}$ as in Figure 3.5.

Recall, from Proposition 3.3.1, that $\Re[2g(z)+1] \geq 0$ with equality only for $z \in (0,1)$ and $\Re[2g(z)-1] \leq 0$ with equality only for $z \in (-1,0)$. Our second and final transformation is

$$Z(z;\varkappa) := \begin{cases} Y(z;\varkappa), & z \text{ outside the lenses} \\ Y(z;\varkappa)\begin{bmatrix} 1 & 0 \\ \mp i e^{\varkappa(2g(z)-1)} & 1 \end{bmatrix}, & z \in \mathcal{L}_L^{(\pm)} \\ Y(z;\varkappa)\begin{bmatrix} 1 & \mp \frac{1}{i} e^{-\varkappa(2g(z)+1)} \\ 0 & 1 \end{bmatrix}, & z \in \mathcal{L}_R^{(\pm)}. \end{cases} \tag{3.117}$$

Since $Y(z;\varkappa)$ solves RHP 3.3.2, it is a direct calculation to show $Z(z;\varkappa)$ solves the following RHP.

Riemann-Hilbert Problem 3.3.3. *Find a matrix $Z(z;\varkappa)$, analytic on the complement of the arcs of Figure 3.5, satisfying the jump conditions*

$$Z(z_+;\varkappa) = Z(z_-;\varkappa) \begin{cases} \begin{bmatrix} 1 & 0 \\ ie^{\varkappa(2g-1)} & 1 \end{bmatrix} & z \in \partial\mathcal{L}_L^{(\pm)} \setminus \mathbb{R}, \\ \begin{bmatrix} 1 & \frac{1}{i}e^{-\varkappa(2g+1)} \\ 0 & 1 \end{bmatrix} & z \in \partial\mathcal{L}_R^{(\pm)} \setminus \mathbb{R}, \\ -i\sigma_1 & z \in (-1,0), \\ i\sigma_1 & z \in (0,1), \end{cases} \tag{3.118}$$

normalized by

$$Z(z;\varkappa) = 1 + \mathrm{O}\left(z^{-1}\right), \text{ as } z \to \infty, \tag{3.119}$$

and with the same endpoint behavior as $Y(z;\varkappa)$ near the endpoints $z = 0, \pm 1$, see (3.112).

The jumps for $Z(z;\varkappa)$ on $\partial\mathcal{L}_{L,R}^{(\pm)}$ will be exponentially small as long as z is a fixed distance away from $0, \pm 1$ due to Proposition 3.3.1. If we 'ignore' the jumps on the lenses of the RHP for $Z(z;\varkappa)$, we obtain the so-called model RHP.

Riemann-Hilbert Problem 3.3.4. *Find a matrix $\Psi(z)$, analytic on $\overline{\mathbb{C}} \setminus [-1,1]$,* *and satisfying*

$$\Psi_+(z) = \Psi_-(z)(-i\sigma_1), \ \textit{for } z \in [-1,0],$$

$$\Psi_+(z) = \Psi_-(z)(i\sigma_1), \ \textit{for } z \in [0,1],$$

$$\Psi(z) = \mathrm{O}\left(|z \mp 1|^{-\frac{1}{4}}\right), \ \textit{as } z \to \pm 1,$$

$$\Psi(z) = \mathrm{O}\left(|z|^{-\frac{1}{2}}\right), \ \textit{as } z \to 0,$$

$$\Psi(z) = 1 + \mathrm{O}\left(z^{-1}\right) \ \textit{as } z \to \infty.$$

We can see that RHP 3.3.4 will not have a unique solution because of the non-L^2 behavior at $z = 0$ but we can classify the 'degree' of non-uniqueness.

Theorem 3.3.5. *If $\Psi(z)$ is a solution to RHP 3.3.4, then there exists $x, y \in \mathbb{C}$ so that*

$$\Psi(z) = \left(I + \frac{1}{z}\begin{bmatrix} x & -x \\ y & -y \end{bmatrix}\right)\left(\frac{z^2-1}{z^2}\right)^{\sigma_1/4}. \tag{3.120}$$

Proof. The Sokhotski-Plemelj formula (see [12]) can be applied to this problem to obtain the solution

$$\Psi_1(z) = \beta(z)^{\sigma_1}, \quad \text{where} \ \ \beta(z) = \left(\frac{z^2-1}{z^2}\right)^{1/4}. \tag{3.121}$$

Take any solution to RHP 3.3.4 (different from $\Psi_1(z)$) and call it $\Psi_2(z)$. Then it can be seen that the matrix $\Psi_2(z)\Psi_1^{-1}(z)$ has no jumps in the complex plane, $\Psi_2(z)\Psi_1^{-1}(z) = I + \mathrm{O}\left(z^{-1}\right)$ as $z \to \infty$ and $\Psi_2(z)\Psi_1^{-1}(z) = \mathrm{O}\left(z^{-1}\right)$ as $z \to 0$. Then it must be that

$$\Psi_2(z)\Psi_1^{-1}(z) = I + \frac{A}{z}, \tag{3.122}$$

where A is a constant matrix. Notice that

$$\Psi_1(z) = \frac{\beta(z)}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \frac{1}{2\beta(z)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}, \tag{3.123}$$

so we have

$$\Psi_2(z) = \left(I + \frac{A}{z}\right)\Psi_1(z) = \left(I + \frac{A}{z}\right)\left(\frac{\beta(z)}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \frac{1}{2\beta(z)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right). \tag{3.124}$$

Since $\Psi_2(z)$ is a solution of RHP 3.3.4, it must be true that $\Psi_2(z) = \mathrm{O}\left(z^{-1/2}\right)$ as $z \to 0$. Thus the matrix A must satisfy

$$A \cdot \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \implies A = \begin{bmatrix}x & -x\\ y & -y\end{bmatrix} \tag{3.125}$$

where $x, y \in \mathbb{C}$, as desired.

□

Remark 3.3.6. For any $x \in \mathbb{C}$ so that $\Re[x] = 0$, then

$$\Psi(z) = \left(I + \frac{1}{z}\begin{bmatrix}x & -x\\ x & -x\end{bmatrix}\right)\left(\frac{z^2-1}{z^2}\right)^{\sigma_1/4} \tag{3.126}$$

has the symmetry

$$\overline{\Psi(\overline{z})} = \sigma_1\Psi(z)\sigma_1. \tag{3.127}$$

This follows from the fact that

$$\overline{\begin{bmatrix} x & -x \\ x & -x \end{bmatrix}} = \begin{bmatrix} -x & x \\ -x & x \end{bmatrix} = \sigma_1 \begin{bmatrix} x & -x \\ x & -x \end{bmatrix} \sigma_1 \tag{3.128}$$

and $\left(\frac{z^2-1}{z^2}\right)^{\sigma_1/4}$ commutes with σ_1.

3.3.2 Approximation of $Z(z;\varkappa)$ and Main Result

We will construct a piecewise (in z) approximation of $Z(z;\varkappa)$ when $\varkappa \to \infty$. Our approach is very similar to that in [3]. Define sets $\mathbb{D}_j = B(j,l)$, (the disc with center j and radius l) $j = 0, \pm 1$, and l is chosen so that $\partial\mathbb{D}_0 \subset \tilde{\Omega}$, see (3.64) for definition of $\tilde{\Omega}$. The idea is as follows: on the lenses $\mathcal{L}_{L,R}^{(\pm)}$ (see Figure 3.5) outside the sets $\mathbb{D}_j$, $j = 0, \pm 1$, the jumps of $Z(z;\varkappa)$ are uniformly close to the identity matrix thus $\Psi(z;\varkappa)$ (a solution to model RHP, see (3.101)) is a 'good' approximation of $Z(z;\varkappa)$. Inside $\mathbb{D}_j$, $j = 0, \pm 1$, we construct local approximations that are commonly called 'parametrices'. The solution of the so-called Bessel RHP is necessary.

Riemann-Hilbert Problem 3.3.7. *Let $\nu \in (0,\pi)$ be any fixed number. Find a matrix $\mathcal{B}_\nu(\zeta)$ that is analytic off the rays $\mathbb{R}_-$, $e^{\pm i\theta}\mathbb{R}^+$ and satisfies the following* *conditions.*

$$\mathcal{B}_{\nu+}(\zeta) = \mathcal{B}_{\nu-}(\zeta) \begin{bmatrix} 1 & 0 \\ e^{-4\sqrt{\zeta} \pm i\pi\nu} & 1 \end{bmatrix}, \ \zeta \in e^{\pm i\theta}\mathbb{R}_+ \tag{3.129}$$

$$\mathcal{B}_{\nu+}(\zeta) = \mathcal{B}_{\nu-}(\zeta) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \zeta \in \mathbb{R}_- \tag{3.130}$$

$$\mathcal{B}_{\nu}(\zeta) = \mathrm{O}\left(\zeta^{-\frac{|\nu|}{2}}\right) \ \textit{for } \nu \neq 0 \ \textit{or} \ \mathrm{O}\left(\log\zeta\right) \ \textit{for } \nu = 0 \ \textit{as } \zeta \to 0, \tag{3.131}$$

$$\mathcal{B}_{\nu}(\zeta) = F(\zeta)\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{\zeta}}\right)\right) \ \textit{as } \zeta \to \infty, \quad F(\zeta) = (2\pi)^{-\sigma_3/2}\zeta^{-\frac{\sigma_3}{4}}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}. \tag{3.132}$$

This RHP has an explicit solution in terms of Bessel functions and can be found in [18]. Define local coordinates at points $z = \pm 1$ as

$$-4\sqrt{\xi_{-1}(z)} = \varkappa(2g(z) - 1), \quad \text{for } z \in \mathbb{D}_{-1}, \tag{3.133}$$

$$4\sqrt{\xi_1(z)} = \varkappa(2g(z) + 1), \quad \text{for } z \in \mathbb{D}_1. \tag{3.134}$$

We call $\tilde{Z}(z;\varkappa)$ our approximation of $Z(z;\varkappa)$ and define

$$\tilde{Z}(z;\varkappa) := \begin{cases} \Psi(z;\varkappa), & z \in \mathbb{C} \setminus \bigcup_{j=-1}^{1} \mathbb{D}_j, \\ \Psi(z;\varkappa) i^{-\frac{\sigma_3}{2}} F^{-1}(\xi_{-1}) B_0(\xi_{-1}) i^{\frac{\sigma_3}{2}}, & z \in \mathbb{D}_{-1}, \\ Z(z;\varkappa), & z \in \mathbb{D}_0, \\ \Psi(z;\varkappa) i^{-\frac{\sigma_3}{2}} \sigma_1 F^{-1}(\xi_1) B_0(\xi_1) \sigma_1 i^{\frac{\sigma_3}{2}}, & z \in \mathbb{D}_1, \end{cases} \tag{3.135}$$

where $Z(z;\varkappa)$ is the solution of RHP 3.3.3, $B_0(\xi_j)$ is the solution of RHP 3.3.7, and $\Psi(z;\varkappa)$ is a solution of the model RHP 3.3.4 defined in (3.101).

Remark 3.3.8. The matrix $\tilde{Z}(z;\varkappa)$ was constructed to have the same jumps (exact) as $Z(z;\varkappa)$ when $z \in \mathbb{D}_{0,\pm 1} \cup [-1,1]$. For an in depth construction, we refer the reader to [3], section 4.3.

Define the error matrix as

$$\mathcal{E}(z;\varkappa) := Z(z;\varkappa)\tilde{Z}^{-1}(z;\varkappa), \tag{3.136}$$

where $Z(z;\varkappa)$ is the solution of RHP 3.3.3 and $\tilde{Z}(z;\varkappa)$ was defined in (3.135). It is clear that $\mathcal{E}(z;\varkappa) = I + \mathrm{O}\left(z^{-1}\right)$ as $z \to \infty$ since both $Z(z;\varkappa), \tilde{Z}(z;\varkappa)$ have this behavior. $\mathcal{E}(z;\varkappa)$ has no jumps inside $\mathbb{D}_{-1,0,1}$ because $\tilde{Z}(z;\varkappa)$ was constructed to have the same jumps as $Z(z;\varkappa)$ inside $\mathbb{D}_{-1,0,1}$, see Remark 3.3.8. Thus $\mathcal{E}(z;\varkappa)$ will

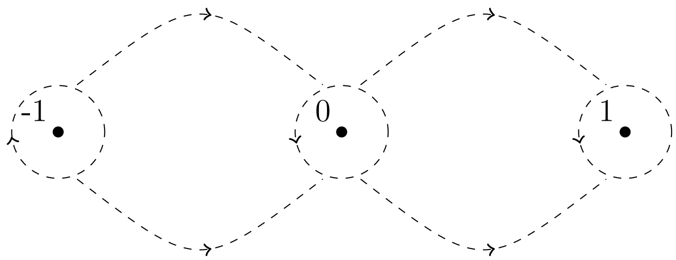

Figure 3.6: The contour Σ, where $\mathcal{E}(z;\varkappa)$ has jumps.

have jumps on $\partial\mathbb{D}_{-1,0,1}$, $\partial\mathcal{L}_{L,R}^{(\pm)}\setminus\mathbb{D}_{0,\pm1}$, and be analytic elsewhere. Explicitly,

$$\mathcal{E}(z_+;\varkappa)=\mathcal{E}(z_-;\varkappa)\begin{cases}\Psi(z;\varkappa)i^{-\frac{\sigma_3}{2}}F^{-1}(\xi_{-1})B_0(\xi_{-1})i^{\frac{\sigma_3}{2}}\Psi^{-1}(z;\varkappa), & z\in\partial\mathbb{D}_{-1},\\ I+\Psi(z;\varkappa)\begin{bmatrix}0 & 0\\ ie^{\varkappa(2g(z)-1)} & 0\end{bmatrix}\Psi^{-1}(z;\varkappa), & z\in\partial\mathcal{L}_L^{(\pm)}\setminus\mathbb{D}_{-1,0},\\ Z(z;\varkappa)\Psi^{-1}(z;\varkappa), & z\in\partial\mathbb{D}_0,\\ I+\Psi(z;\varkappa)\begin{bmatrix}0 & -ie^{-\varkappa(2g(z)+1)}\\ 0 & 0\end{bmatrix}\Psi^{-1}(z;\varkappa), & z\in\partial\mathcal{L}_R^{(\pm)}\setminus\mathbb{D}_{0,1},\\ \Psi(z;\varkappa)i^{-\frac{\sigma_3}{2}}\sigma_1F^{-1}(\xi_1)B_0(\xi_1)\sigma_1i^{\frac{\sigma_3}{2}}\Psi^{-1}(z;\varkappa), & z\in\partial\mathbb{D}_1.\end{cases} \tag{3.137}$$

The matrix $\Psi(z;\varkappa)$ was defined in (3.101) and F^{-1},B_0 can be found in RHP 3.3.7. Call Σ the collection of arcs where $\mathcal{E}(z;\varkappa)$ has a jump, as described in Figure 3.6.

Remark 3.3.9. In Theorem 3.2.6, we obtained the leading order behavior of $\Gamma(z;\lambda)$ for $z\in\tilde{\Omega}$ as $\lambda\to 0$. This Theorem can easily be written in terms of $Z(z;\varkappa)$ instead of $\Gamma(z;\lambda)$ by applying the transformations (see section 3.3.1) $\Gamma \to Y \to Z$.

We have now proven the following corollary.

Corollary 3.3.10. *As $\varkappa \to \infty$, provided that either $\Im\varkappa \geq 0$ or $\Im\varkappa \leq 0$,*

$$Z(z;\varkappa) = \Psi(z;\varkappa)\left(I + \mathrm{O}\left(\varkappa^{-1}\right)\right), \tag{3.138}$$

which is uniform for $z \in \tilde{\Omega}$. $Z(z;\varkappa)$ is the solution of RHP 3.3.3 and $\Psi(z;\varkappa)$ is defined in (3.101)*.*

Now revisiting the jumps of $\mathcal{E}(z;\varkappa)$ in (3.137), we have another Corollary.

Corollary 3.3.11. *As $\varkappa \to \infty$, provided that $\Im\varkappa \geq 0$ or $\Im\varkappa \leq 0$,*

$$\mathcal{E}(z_+;\varkappa) = \mathcal{E}(z_-;\varkappa)\begin{cases} I + \mathrm{O}\left(\varkappa^{-1}\right), & z \in \partial\mathbb{D}_{-1,0,1}, \\ I + \mathrm{O}\left(e^{\varkappa(2g(z)-1)}\right), & z \in \partial\mathcal{L}_L^{(\pm)} \setminus \mathbb{D}_{-1,0}, \\ I + \mathrm{O}\left(e^{-\varkappa(2g(z)+1)}\right), & z \in \partial\mathcal{L}_R^{(\pm)} \setminus \mathbb{D}_{0,1}, \end{cases} \tag{3.139}$$

which is uniform for $z \in \Sigma$, see Figure 3.6 for Σ. Functions $\mathcal{E}(z;\varkappa)$, $g(z)$ are defined in (3.136)*,* (3.106)*, respectively.*

Remark 3.3.12. The factors $e^{-\varkappa(2g(z)+1)}$ and $e^{\varkappa(2g(z)-1)}$ in (3.139) are exponentially small for all z in the corresponding set, in light of Proposition 3.3.1.

Proof. The behavior for $z \in \partial\mathbb{D}_{\pm 1}, \partial\mathbb{D}_0$ is a direct consequence of (3.132), Corollary 3.3.10, respectively. The behavior on the lenses is clear via inspection of (3.137).

□

Corollary 3.3.13. *Let $\varkappa \to \infty$, provided that $\Im\varkappa \geq 0$ or $\Im\varkappa \leq 0$. Then for z in simply connected compact subsets of $\mathbb{C} \setminus \{0, \pm 1\}$, we have the uniform approximation*

$$Z(z;\varkappa) = \Psi(z;\varkappa)\left(I + \mathrm{O}\left(\varkappa^{-1}\right)\right). \tag{3.140}$$

See RHP 3.3.3 for $Z(z;\varkappa)$ and (3.101) *for $\Psi(z;\varkappa)$.*

Proof. Choose any simply connected (informally this means that the set consists of one 'piece' has no 'holes') compact subset of $\mathbb{C} \setminus \{0, \pm 1\}$ and call it J; then the disks $\mathbb{D}_{0,\pm 1}$ can be taken sufficiently small in order to not intersect J and lenses $\mathcal{L}_{L,R}^{(\pm)}$ can be formed while still retaining all of their necessary properties. Corollaries 3.3.10, 3.3.11 and the so-called small norm theorem, see [5] (Theorem 7.171) can now be applied to conclude that $\mathcal{E}(z;\varkappa) = I + \mathrm{O}\left(\varkappa^{-1}\right)$ uniformly for $z \in J$. This is equivalent to the stated result.

□

We are now ready to prove the main result of this chapter.

Theorem 3.3.14. *Let $\lambda = e^{-\varkappa} \to 0$, provided that either $\Im\lambda \geq 0$ or $\Im\lambda \leq 0$. Then,*

1. *For z in compact subsets of $\mathbb{C} \setminus [-1, 1]$ we have the uniform approximation*

$$\Gamma(z;\lambda) = \Psi(z;\varkappa)\left(I + \mathrm{O}\left(\varkappa^{-1}\right)\right)e^{-\varkappa g(z)\sigma_3}. \tag{3.141}$$

2. For z in compact subsets of $(-1,0)\cup(0,1)$ we have the uniform approximation

$$\Gamma(z_\pm;\lambda) = \begin{cases} \Psi(z_\pm;\varkappa)\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)\begin{bmatrix} 1 & 0 \\ \pm i e^{\varkappa(2g_\pm(z)-1)} & 1 \end{bmatrix} e^{-\varkappa g_\pm(z)\sigma_3}, & z\in(-1,0), \\ \Psi(z_\pm;\varkappa)\left(I+\mathrm{O}\left(\varkappa^{-1}\right)\right)\begin{bmatrix} 1 & \mp i e^{-\varkappa(2g_\pm(z)+1)} \\ 0 & 1 \end{bmatrix} e^{-\varkappa g_\pm(z)\sigma_3}, & z\in(0,1), \end{cases} \tag{3.142}$$

where $\pm$ denotes the upper/lower shore of the real axis in the z-plane. See (2.17), (3.101), (3.106) *for the definitions of Γ,Ψ,g, respectively.*

Proof. This Theorem is a direct consequence of Corollary 3.3.13. We simply need to revert the transforms that took $\Gamma\to Z$. Doing so, we find that

$$\Gamma(z;\lambda) = \begin{cases} Z(z;\varkappa)e^{-\varkappa g(z)\sigma_3}, & z \text{ outside lenses} \\ Z(z;\varkappa)\begin{bmatrix} 1 & 0 \\ \pm i e^{\varkappa(2g(z)-1)} & 1 \end{bmatrix} e^{-\varkappa g(z)\sigma_3}, & z\in\mathcal{L}_L^{(\pm)} \\ Z(z;\varkappa)\begin{bmatrix} 1 & \mp i e^{-\varkappa(2g(z)+1)} \\ 0 & 1 \end{bmatrix} e^{-\varkappa g(z)\sigma_3}, & z\in\mathcal{L}_R^{(\pm)}. \end{cases} \tag{3.143}$$

If z is in a compact subset of $\mathbb{C}\setminus[-1,1]$ or $(-1,0)\cup(0,1)$, we can construct the lenses $\mathcal{L}_{L,R}^{(\pm)}$ so that they do not intersect this compact set, so simply connected is not necessary here. Applying Corollary 3.3.13 gives the result.

□

CHAPTER 4: SPECTRAL PROPERTIES AND DIAGONALIZATION OF $\mathcal{H}_R^*\mathcal{H}_R$ AND $\mathcal{H}_L^*\mathcal{H}_L$

The goal of this chapter is to construct unitary operators $U_R : L^2([0, b_R]) \to L^2(J, \sigma_R)$ and $U_L : L^2([0, b_R]) \to L^2(J, \sigma_L)$ such that

$$U_R^*\mathcal{H}_R^*\mathcal{H}_R U_R = \lambda^2, \quad U_L^*\mathcal{H}_L^*\mathcal{H}_L U_L = \lambda^2 \tag{4.1}$$

where λ^2 is a multiplication operator (the space is clear by context) and the spectral measure σ_L, σ_R are to be determined. This is to be understood in the sense of operator equality on $L^2(J, \sigma_R), L^2(J, \sigma_L)$, respectively. We will begin this section with a brief summary of the spectral theory for a self-adjoint operator with simple spectrum.

4.1 Basic Facts About Diagonalizing a Self-Adjoint Operator with Simple Spectrum

For an in-depth review of the spectral theorem for self-adjoint operators, see [2], [19], [8]. We present a short summary of this topic which is directly related to the needs of this dissertation. Let $\mathcal{K}$ be a Hilbert space and let A be a self-adjoint operator with simple spectrum acting on $\mathcal{K}$. Recall from [2], that a self-adjoint operator has simple spectrum if there is a vector $g \in \mathcal{K}$ so that the span of $\hat{E}_\Delta[g]$, where Δ runs through the set of all subintervals of the real line, is dense in $\mathcal{K}$. Here the operator

$\hat{E}_t$ denotes the so-called *resolution of the identity* for the operator A, which we will define shortly. Define $\hat{R}$, the resolvent of A, via the formula

$$\hat{R}(t) = (tI - A)^{-1}. \tag{4.2}$$

Then, according to [8] p.921, the resolution of the identity is defined by the formula

$$\hat{E}_{(\alpha,\beta)} := \hat{E}_\beta - \hat{E}_\alpha = \lim_{\epsilon \to 0^+} \lim_{\delta \to 0^+} \int_{\alpha+\delta}^{\beta-\delta} \frac{-1}{2\pi i} \left[\hat{R}(t + i\epsilon) - \hat{R}(t - i\epsilon) \right] \, dt, \tag{4.3}$$

where $\alpha < \beta$. Once we obtain $\hat{E}_t$, we can construct the unitary operators which will diagonalize A, as described in the following Theorem from [2] p.279.

Theorem 4.1.1. *If A is a self-adjoint operator with simple spectrum, if g is any generating element, and if $\sigma(t) = \langle \hat{E}_t[g], g \rangle$, then the formula*

$$f = \int_{\mathbb{R}} f(t) \, d\hat{E}_t[g] \tag{4.4}$$

associates with each function $f(t) \in L^2(\mathbb{R}, \sigma)$ a vector $f \in \mathcal{K}$, and this correspondence is an isometric mapping of $L^2(\mathbb{R}, \sigma)$ onto $\mathcal{K}$. It maps the domain $D(Q)$ of the multiplication operator Q in $L^2(\mathbb{R}, \sigma)$ into the domain $D(A)$ of the operator A, and if the element $f \in D(A)$ corresponds to the function $f(t) \in L^2(\mathbb{R}, \sigma)$, then the element Af corresponds to the function $tf(t)$.

Thus our immediate goal moving forward is to construct the resolution of the identity for $\mathcal{H}_R^* \mathcal{H}_R$ and $\mathcal{H}_L^* \mathcal{H}_L$.

Remark 4.1.2. In the remaining chapters and sections of this dissertation we will frequently encounter the following Möbius transforms:

$$M_1(x) = \frac{b_R(x-b_L)}{x(b_R-b_L)}, \quad M_2(x) = \frac{b_R b_L x}{x(b_R+b_L)-b_R b_L}, \tag{4.5}$$

$$M_3(x) = M_1(M_2(x)) = \frac{-b_L(x-b_R)}{x(b_R-b_L)}, \quad M_4(x) = \frac{x(b_R-b_L)}{x(b_R+b_L)-2b_R b_L}. \tag{4.6}$$

4.2 Resolution of the Identity for $\mathcal{H}_R^*\mathcal{H}_R$ and $\mathcal{H}_L^*\mathcal{H}_L$

From (4.3), knowledge of the resolvent operator is paramount. We are able to express the resolvents of $\mathcal{H}_R^*\mathcal{H}_R, \mathcal{H}_L^*\mathcal{H}_L$ in terms of the resolvent of $\hat{K}$.

Proposition 4.2.1. *The resolvent of $\mathcal{H}_R^*\mathcal{H}_R$ and $\mathcal{H}_L^*\mathcal{H}_L$ is*

$$\mathcal{R}_R(\lambda^2) := \left(I - \frac{1}{\lambda^2}\mathcal{H}_R^*\mathcal{H}_R\right)^{-1} = I + \pi_R \hat{R}(\lambda/2)\pi_R, \tag{4.7}$$

$$\mathcal{R}_L(\lambda^2) := \left(I - \frac{1}{\lambda^2}\mathcal{H}_L^*\mathcal{H}_L\right)^{-1} = I + \pi_L \hat{R}(\lambda/2)\pi_L, \tag{4.8}$$

where $\pi_R : L^2(\mathbb{R}) \to L^2([0, b_R])$, $\pi_L : L^2(\mathbb{R}) \to L^2([b_L, 0])$ are orthogonal projections (i.e. restrictions), $\hat{R}(\lambda)$ is defined by the relation (2.5) *and the kernel of $\hat{R}(\lambda)$ is computed in Theorem 2.2.4.*

Remark 4.2.2. The resolvents of $\mathcal{H}_R^*\mathcal{H}_R$ and $\mathcal{H}_L^*\mathcal{H}_L$, defined in (4.7), (4.8), are not in the same form of the resolvent defined in (4.3) in the previous subsection. Some elementary algebra shows that

$$\frac{\mathcal{R}_R(\lambda^2)}{\lambda^2} = \left(\lambda^2 I - \mathcal{H}_R^* \mathcal{H}_R\right)^{-1}, \tag{4.9}$$

which is of the form (4.3). An identical statement can be made for the resolvent of $\mathcal{H}_L^* \mathcal{H}_L$.

Proof. In the direct sum decomposition $L^2([b_L, b_R]) = L^2([b_L, 0]) \oplus L^2([0, b_R])$, $\hat{K}$ has the block structure

$$\hat{K} = \begin{bmatrix} 0 & -\frac{i}{2}\mathcal{H}_L \\ -\frac{i}{2}\mathcal{H}_R & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{i}{2}\mathcal{H}_R^* \\ -\frac{i}{2}\mathcal{H}_R & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{i}{2}\mathcal{H}_L \\ \frac{i}{2}\mathcal{H}_L^* & 0 \end{bmatrix}. \tag{4.10}$$

We can write (recall that $\hat{K}$ is bounded, see Proposition 2.1.1)

$$I + \hat{R}(\lambda) = \left(I - \frac{1}{\lambda}\hat{K}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{\hat{K}}{\lambda}\right)^n \tag{4.11}$$

where all the even powers in the right hand side of (4.11) are block diagonal and all the odd powers in (4.11) are block off-diagonal. Similarly, we can write

$$\left(I - \frac{1}{\lambda^2}\mathcal{H}_R^* \mathcal{H}_R\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{\mathcal{H}_R^* \mathcal{H}_R}{\lambda^2}\right)^n \tag{4.12}$$

and comparing with the series in (4.11) gives our result for the resolvent of $\mathcal{H}_R^* \mathcal{H}_R$. The proof for the resolvent of $\mathcal{H}_L^* \mathcal{H}_L$ is nearly identical.

□

To construct the resolution of the identity (see (4.3)), we need to compute the jump of the resolvent of $\mathcal{H}_R^*\mathcal{H}_R$ and $\mathcal{H}_L^*\mathcal{H}_L$. The kernel of the resolvent is expressed in terms of $\Gamma(z;\lambda)$ (see Theorem 2.2.4), so we need to compute the jump of $\Gamma(z;\lambda)$ in the $\lambda-$plane. We begin with an auxiliary Proposition.

Proposition 4.2.3. *For* $\lambda \in (-1/2, 0)$,

$$\hat{\Gamma}(z, \lambda_+) = \hat{\Gamma}(z, \lambda_-)\,\sigma_1, \tag{4.13}$$

$$Q_+(\lambda) = \left[\frac{-\tan(a\pi)\Gamma(a)\Gamma(a+2)}{e^{2\pi i a}4^{2a+1}\Gamma(a+3/2)\Gamma(a+1/2)}\right]_- \sigma_1 Q_-(\lambda). \tag{4.14}$$

The matrices $\hat{\Gamma}, Q$ *are defined in* (2.18)*.*

Proof. Recall from Proposition B.0.1 that $a_+(\lambda)+a_-(\lambda) = -1$ for $\lambda \in (-1/2, 0)$ and h_∞, s_∞ are defined in (2.13),(2.14), respectively. Then by inspection, we see that

$$h_\infty(z, a_+(\lambda)) = h_\infty(z, -1-a_-(\lambda)) = s_\infty(z, a_-(\lambda)) \tag{4.15}$$

so the jump of $\hat{\Gamma}$ follows. Computing the jump of Q is a straightforward, perhaps tedious, exercise.

□

We are now ready to compute the jump of $\Gamma(z;\lambda)$ in the $\lambda-$plane.

Theorem 4.2.4. *For* $\lambda \in (-1/2, 0) \cup (0, 1/2)$,

$$\Gamma(z;\lambda_+) = \Gamma(z;\lambda_-)\left[I - \frac{1}{z}\vec{f}(z,\lambda_-)\vec{g}^t(z,\lambda_-)\right], \tag{4.16}$$

where

$$\vec{f}(z,\lambda) := \frac{-2b_R b_L|\lambda|(2a(-|\lambda|)+1)}{b_R - b_L}\begin{bmatrix} d_R(z;-|\lambda|) \\ \operatorname{sgn}(\lambda) d_L(z;-|\lambda|)\end{bmatrix}, \quad \vec{g}(z,\lambda) := \begin{bmatrix} -\operatorname{sgn}(\lambda) d_L(z;-|\lambda|) \\ d_R(z;-|\lambda|)\end{bmatrix}. \tag{4.17}$$

Functions $d_R(z;\lambda), d_L(z;\lambda)$ *are defined as*

$$d_R(z;\lambda) := \alpha(\lambda) h'_\infty\left(M_1(z)\right) + \beta(\lambda) s'_\infty\left(M_1(z)\right), \tag{4.18}$$

$$d_L(z;\lambda) := -e^{a\pi i}\alpha(\lambda) h'_\infty\left(M_1(z)\right) + e^{-a\pi i}\beta(\lambda) s'_\infty\left(M_1(z)\right), \tag{4.19}$$

where $M_1(z)$, h'_∞, s'_∞ *are defined in Remark 4.1.2,* (2.13), (2.14), *respectively,* $a := a(\lambda)$ *is defined in Appendix B, and coefficients* $\alpha(\lambda), \beta(\lambda)$ *are*

$$\alpha(\lambda) := \frac{e^{-a\pi i}\tan(a\pi)\Gamma(a)}{4^{a+1}\Gamma(a+3/2)}, \quad \beta(\lambda) := \frac{4^a e^{a\pi i}\Gamma(a+1/2)}{\Gamma(a+2)}. \tag{4.20}$$

In (4.16), *we understand that when* λ *is on the lower shore of* $(0, 1/2)$, $-\lambda$ *is on the lower shore of* $(-1/2, 0)$.

Proof. The proof is straightforward; we have

$$\Gamma(z;\lambda_+) = \Gamma(z;\lambda_-)\Gamma(z;\lambda_-)^{-1}\Gamma(z;\lambda_+) \tag{4.21}$$

and thus we compute $\Gamma^{-1}(z;\lambda_-)\Gamma(z;\lambda_+)$ to obtain the stated result. We begin with $\lambda \in (-1/2, 0)$. Using the definition of $\Gamma(z;\lambda)$ (eq. (2.17)) and Proposition 4.2.3 we obtain

$$\Gamma^{-1}(z;\lambda_-)\Gamma(z;\lambda_+) = I - \frac{b_L b_R}{z(b_R - b_L)} \left(\frac{2a+1}{a(a+1)}\right)_- M^{-1}(z,\lambda_-) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M(z,\lambda_-) \tag{4.22}$$

where $M(z,\lambda) = \hat{\Gamma}\left(M_1(z)\right) Q\sigma_2$. Let m_{21}, m_{22} denote the (2,1), (2,2) elements of the matrix M, respectively. Then,

$$M^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M = |M|^{-1} \begin{bmatrix} m_{22}m_{21} & m_{22}^2 \\ -m_{21}^2 & -m_{21}m_{22} \end{bmatrix} = |M|^{-1} \begin{bmatrix} m_{22} \\ -m_{21} \end{bmatrix} \begin{bmatrix} m_{21} & m_{22} \end{bmatrix} \tag{4.23}$$

and, explicitly,

$$m_{21}(z;\lambda) = \frac{ie^{a\pi i}\Gamma(a+3/2)}{4^{-a-1}\Gamma(a)}\left[\frac{-\tan(a\pi)\Gamma(a)}{4^{a+1}\Gamma(a+3/2)}h'_\infty\left(M_1(z)\right) + \frac{4^a\Gamma(a+1/2)}{\Gamma(a+2)}s'_\infty\left(M_1(z)\right)\right] \tag{4.24}$$

$$=: \frac{ie^{a\pi i}\Gamma(a+3/2)}{4^{-a-1}\Gamma(a)}d_L(z;\lambda), \tag{4.25}$$

$$m_{22}(z;\lambda) = \frac{ie^{i\pi a}\Gamma(a+3/2)}{4^{-a-1}\Gamma(a)}\left[\frac{\tan(a\pi)\Gamma(a)}{e^{a\pi i}4^{a+1}\Gamma(a+3/2)}h'_\infty\left(M_1(z)\right) + \frac{4^a e^{a\pi i}\Gamma(a+1/2)}{\Gamma(a+2)}s'_\infty\left(M_1(z)\right)\right] \tag{4.26}$$

$$=: \frac{ie^{a\pi i}\Gamma(a+3/2)}{4^{-a-1}\Gamma(a)}d_R(z;\lambda), \tag{4.27}$$

$$|M(z;\lambda)| = -\frac{2e^{2\pi ia}4^{2a+1}\Gamma^2\left(a+\frac{3}{2}\right)}{\lambda a(a+1)\Gamma^2(a)}. \tag{4.28}$$

To compute $|M|$ we have used (A.15). Finally, we calculate

$$\left(\frac{2a+1}{a(a+1)}\right)_- M^{-1}(z,\lambda_-)\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} M(z,\lambda_-) = \left(\frac{2a+1}{a(a+1)|M|}\right)_- \begin{bmatrix}m_{22}\\ -m_{21}\end{bmatrix}_- \begin{bmatrix}m_{21} & m_{22}\end{bmatrix}_- \tag{4.29}$$

$$= \left(\frac{2a+1}{a(a+1)|M|}\right)_- \left(\frac{ie^{a\pi i}\Gamma(a+\frac{3}{2})}{4^{-a-1}\Gamma(a)}\right)^2_+ \begin{bmatrix}d_R(z;\lambda)\\ -d_L(z;\lambda)\end{bmatrix}_+ \begin{bmatrix}d_L(z;\lambda) & d_R(z;\lambda)\end{bmatrix}_- \tag{4.30}$$

$$= 2\lambda\left(2a_- + 1\right)\begin{bmatrix}d_R(z;\lambda)\\ -d_L(z;\lambda)\end{bmatrix}_- \begin{bmatrix}d_L(z;\lambda) & d_R(z;\lambda)\end{bmatrix}_- \tag{4.31}$$

which can be used to obtain the desired result. To calculate the jump of $\Gamma(z;\lambda)$ when $\lambda \in (0,1/2)$, we take advantage of the symmetry $\Gamma(z;\lambda) = \sigma_3\Gamma(z;-\lambda)\sigma_3$, see

Remark 2.2.3. For brevity, let

$$J(z;\lambda) := I - \frac{1}{z}\vec{f}(z;\lambda)\vec{g}^t(z;\lambda). \tag{4.32}$$

So we have proven that

$$\Gamma(z;\lambda_+) = \Gamma(z;\lambda_-)J(z;\lambda_-) \tag{4.33}$$

only for $\lambda \in (-1/2, 0)$. For $\lambda \in (0, 1/2)$,

$$\Gamma(z;\lambda_+) = \sigma_3\Gamma(z,(-\lambda)_-)\sigma_3 \tag{4.34}$$

$$= \sigma_3\Gamma(z,(-\lambda)_+)J^{-1}(z;(-\lambda)_-)\sigma_3 \tag{4.35}$$

$$= \Gamma(z;\lambda_-)\sigma_3 J^{-1}(z;(-\lambda)_-)\sigma_3. \tag{4.36}$$

It can now be verified that

$$\sigma_3 J^{-1}(z;(-\lambda)_-)\sigma_3 = I - \frac{1}{z}\vec{f}(z;\lambda_-)\vec{g}^t(z;\lambda_-) \tag{4.37}$$

for $\lambda \in (0, 1/2)$.

□

Recall from Proposition 4.2.1 and Theorem 2.2.4 that the resolvent of $\mathcal{H}_L^*\mathcal{H}_L$, $\mathcal{H}_R^*\mathcal{H}_R$ is expressed in terms of $\Gamma(z;\lambda)$. In light of the previous Theorem, we can now compute the jump of the resolvent of $\mathcal{H}_L^*\mathcal{H}_L$, $\mathcal{H}_R^*\mathcal{H}_R$ in the λ plane, which is required to construct the resolution of the identity, see (4.3).

Theorem 4.2.5. *The kernel of* $\hat{R}(x;\lambda)$ *is single valued for* $\lambda \in \mathbb{C} \setminus [-1/2, 1/2]$ *and satisfies the jump property*

$$R(z,x;\lambda_+) - R(z,x;\lambda_-) = \frac{\begin{bmatrix} -i\chi_R(x) & \chi_L(x) \end{bmatrix} \vec{f}(x;\lambda_-)\vec{g}^t(z;\lambda_-) \begin{bmatrix} i\chi_L(z) \\ \chi_R(z) \end{bmatrix}}{2\pi i \lambda x z} \tag{4.38}$$

for $\lambda \in (-1/2, 0) \cup (0, 1/2)$*, where* χ_L, χ_R *are characteristic functions on* $(b_L, 0), (0, b_R)$ *and* $R(z,x;\lambda)$*,* $\vec{f}(x;\lambda), \vec{g}(z;\lambda)$ *are defined in* (2.20),(4.17)*, respectively. When* $x, z \in (0, b_R)$*,*

$$R(z,x;\lambda_+) - R(z,x;\lambda_-) = \frac{b_L b_R (2a_-(-|\lambda|) + 1)}{\operatorname{sgn}(\lambda)\pi x z (b_R - b_L)} d_R(x; -|\lambda|) d_R(z; -|\lambda|) \tag{4.39}$$

and when $x, z \in (b_L, 0)$*,*

$$R(z,x;\lambda_+) - R(z,x;\lambda_-) = \frac{b_L b_R (2a_-(-|\lambda|) + 1)}{\operatorname{sgn}(\lambda)\pi x z (b_R - b_L)} d_L(x; -|\lambda|) d_L(z; -|\lambda|), \tag{4.40}$$

where d_R, d_L *are defined in* (4.18)*.*

Proof. Since $\Gamma(z;\lambda)$ is single valued for $\lambda \in \mathbb{C} \setminus [-1/2, 1/2]$, the same is true for $R(x,z;\lambda)$. Recall, from (2.2.4), that

$$R(z,x;\lambda) = \frac{\vec{g}_1^t(x)\Gamma^{-1}(x;\lambda)\Gamma(z;\lambda)\vec{f}_1(z)}{2\pi i \lambda (z - x)} \tag{4.41}$$

and let $\Delta_\lambda F(\lambda) := F(\lambda_+) - F(\lambda_-)$ for any F. To prove the result we need to compute

$\Delta_\lambda[\Gamma^{-1}(x;\lambda)\Gamma(z;\lambda)]$. For $\lambda \in (-1/2, 0)$, we calculate (see proof of Theorem 4.2.4)

$$\Gamma^{-1}(x;\lambda_-)\Gamma(z;\lambda_-) = M^{-1}(x,\lambda_-)\begin{bmatrix} 1 & \frac{b_L b_R(x-z)}{xz(b_R-b_L)(a+1)} \\ 0 & 1 \end{bmatrix}_- M(z,\lambda_-) \tag{4.42}$$

and

$$\Gamma^{-1}(x;\lambda_+)\Gamma(z;\lambda_+) = M^{-1}(x,\lambda_+)\begin{bmatrix} 1 & \frac{b_L b_R(x-z)}{xz(b_R-b_L)(a+1)} \\ 0 & 1 \end{bmatrix}_+ M(z,\lambda_+) \tag{4.43}$$

$$= M^{-1}(x,\lambda_-)\begin{bmatrix} 1 & \frac{-b_L b_R(x-z)}{axz(b_R-b_L)} \\ 0 & 1 \end{bmatrix}_- M(z,\lambda_-) \tag{4.44}$$

so we have that

$$\Delta_\lambda[\Gamma^{-1}(x;\lambda)\Gamma(z;\lambda)] = \left(\frac{-b_L b_R(x-z)(2a+1)}{xza(a+1)(b_R-b_L)}\right)_- M^{-1}(x,\lambda_-)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M(z,\lambda_-) \tag{4.45}$$

$$= \frac{z-x}{xz}\vec{f}(x;\lambda_-)\vec{g}^t(z;\lambda_-) \tag{4.46}$$

Now for $\lambda \in (0, 1/2)$, we again take advantage of the symmetry $\Gamma(z;\lambda) = \sigma_3\Gamma(z;-\lambda)\sigma_3$, see Remark 2.2.3. The process is the same as in the proof of Theorem 4.16. Directly from (2.20) we have

$$\Delta_\lambda R(z,x;\lambda) = \frac{\vec{g}_1^t(x)\Delta_\lambda\Gamma^{-1}(x;\lambda)\Gamma(z;\lambda)\vec{f}_1(z)}{2\pi i\lambda(z-x)} \tag{4.47}$$

and plugging in our calculation of $\Delta_\lambda\Gamma^{-1}(x;\lambda)\Gamma(z;\lambda)$ gives the result.

□

For convenience we define

$$D_R(z;\lambda) := d_R(z;-|\lambda|/2), \quad D_L(z;\lambda) := d_L(z;-|\lambda|/2), \tag{4.48}$$

where d_L, d_R are defined in (4.18). With (4.3) in mind, we can now prove the following theorem.

Theorem 4.2.6. *For $0 \le \lambda^2 \le 1$, $g \in L^2([b_L, 0])$, $f \in L^2([0, b_R])$, the operators*

$$\hat{E}_{R,\lambda^2}[f](x) = \int_0^{\lambda^2}\int_0^{b_R} \varphi_R(x,\mu^2)\varphi_R(z,\mu^2)f(z)\ dzd\mu^2, \tag{4.49}$$

$$\hat{E}_{L,\lambda^2}[g](x) = \int_0^{\lambda^2}\int_{b_L}^{0} \varphi_L(x,\mu^2)\varphi_L(z,\mu^2)g(z)\ dzd\mu^2, \tag{4.50}$$

where $a := a_-(-|\mu|/2)$, are the resolution of the identity for $\mathcal{H}_R^*\mathcal{H}_R$, $\mathcal{H}_L^*\mathcal{H}_L$, respectively. The kernels φ_L, φ_R are

$$\varphi_R(x,\lambda^2) := \frac{D_R(x;\lambda)}{x\pi|\lambda|}\sqrt{\frac{|b_L|b_R(2a+1)}{2i(b_R-b_L)}}, \quad \varphi_L(x,\lambda^2) := \frac{D_L(x;\lambda)}{x\pi|\lambda|}\sqrt{\frac{|b_L|b_R(2a+1)}{2i(b_R-b_L)}}, \tag{4.51}$$

where D_L, D_R are defined in (4.48).

Proof. We construct $\hat{E}_{R,\lambda^2}$ only; $\hat{E}_{L,\lambda^2}$ can be constructed in an identical manner. The definition of the resolution of the identity can be found in (4.3). From the proof of Theorem 4.3.2, we can see that there are no eigenvalues and thus by [2] (see sec. 82), $\hat{E}_{R,\lambda^2}$ has no points of discontinuity so we can take $\delta = 0$. Moreover, we can move the ϵ limit inside the integral as the kernel of $\hat{R}$ has analytic continuation above and below the interval $(0,1)$. It is clear that the spectrum of $\mathcal{H}_R^*\mathcal{H}_R$ is $\lambda^2 \in [0,1]$ in view of Theorem 4.2.5, which shows that

$$\mathcal{R}_{R+}(\lambda^2) - \mathcal{R}_{R-}(\lambda^2) \begin{cases} = 0, & \lambda^2 \in \overline{\mathbb{R}} \setminus [0,1] \\ \neq 0, & \lambda^2 \in (0,1), \end{cases} \tag{4.52}$$

where $\mathcal{R}_R$ is the resolvent of $\mathcal{H}_R^*\mathcal{H}_R$, see (4.7). So from (4.3), (4.7), and Remark 4.2.2 we know that

$$\hat{E}_{R,\lambda^2} = \frac{-1}{2\pi i}\int_0^{\lambda^2} \frac{1}{\mu^2}\left[\mathcal{R}_+(\mu^2) - \mathcal{R}_-(\mu^2)\right]\, d\mu^2 \tag{4.53}$$

$$= \int_0^{\lambda^2} \frac{-\mathrm{sgn}(\mu)}{2\pi i \mu^2}\pi_R\left[\hat{R}_+(\mu/2) - \hat{R}_-(\mu/2)\right]\pi_R\, d\mu^2, \tag{4.54}$$

and now plugging in (4.39) gives the result. Note that $\Delta_{\lambda^2}\mathcal{R}(\lambda^2) = \mathrm{sgn}(\lambda)\Delta_\lambda \hat{R}(\lambda/2)$ (here $\Delta F := F_+ - F_-$), because when λ is on the upper shore of $(-1,0)$, λ^2 is on the lower shore of $(0,1)$.

□

4.3 Nature of the Spectrum of $\mathcal{H}_R^*\mathcal{H}_R$ and $\mathcal{H}_L^*\mathcal{H}_L$

In this section, we show that the spectrum of $\mathcal{H}_R^*\mathcal{H}_R$ and $\mathcal{H}_L^*\mathcal{H}_L$ is simple and purely absolutely continuous. We will prove statements in this section for $\mathcal{H}_R^*\mathcal{H}_R$ only because the statements and ideas for proofs are nearly identical for $\mathcal{H}_L^*\mathcal{H}_L$. Notice that the resolution of the identity of $\mathcal{H}_R^*\mathcal{H}_R$ (see (4.49)) can be compactly written as

$$\hat{E}_{R,\lambda^2}[f](x) = \int_0^{\lambda^2} \varphi_R(x,\mu^2)\tilde{f}(\mu^2)\ d\mu^2, \tag{4.55}$$

where the operator $f \mapsto \tilde{f}$ is defined as

$$\tilde{f}(\mu^2) := \int_0^{b_R} \varphi_R(z,\mu^2) f(z)\ dz \tag{4.56}$$

and the kernel φ_R is defined in (4.51). We begin with an important Lemma.

Lemma 4.3.1. *The map $f \mapsto \tilde{f}$, where $\tilde{f}$ is defined in* (4.56)*, is an isometry from $L^2([0,b_R])$ to $L^2([0,1],\lambda^2)$.*

Proof. Let $f \in C_0^\infty((0,b_R))$, $\Delta = (\alpha,\beta)$ where $0 < \alpha < \beta < 1$ and denote

$$f_\Delta(x) := \int_\Delta d\hat{E}_{R,\lambda^2}[f](x) = \int_\Delta \varphi(x,\lambda^2)\tilde{f}(\lambda^2)\ d\lambda^2, \tag{4.57}$$

where $\hat{E}_{R,\lambda^2}$, defined in (4.49), is the resolution of the identity of $\mathcal{H}_R^*\mathcal{H}_R$. Notice that $\varphi(x,\lambda^2)$ (defined in (4.51)) is smooth and real-valued for $(x,\lambda^2) \in (0,b_R) \times (0,1)$ (see Appendix C for properties of D_R). Since $\hat{E}_{R,\lambda^2}$ is the resolution of the identity, $f_\Delta \to f$ in $L^2([0,b_R])$ as $\Delta \to [0,1]$, so

$$\langle f, f_\Delta\rangle \to \langle f, f\rangle = \|f\|_{L^2([0,b_R])}. \tag{4.58}$$

Also,

$$\langle f, f_\Delta\rangle = \int_0^{b_R} f(x) \int_\Delta \varphi(x,\lambda^2)\overline{\tilde{f}}(\lambda^2)\ d\lambda^2 dx = \int_\Delta \left|\tilde{f}(\lambda)\right|^2\ d\lambda^2 \tag{4.59}$$

which is clearly increasing as $\Delta \to [0,1]$. So we have an isometry provided $f \in C_0^\infty((0,b_R))$. But $C_0^\infty((0,b_R))$ is dense in $L^2([0,b_R])$ so this isometry extends to all of $L^2([0,b_R])$ by continuity.

□

We are now ready to conclude this section.

Theorem 4.3.2. *The spectrum of* $\mathcal{H}_R^*\mathcal{H}_R$, $\mathcal{H}_L^*\mathcal{H}_L$ *is* $\lambda^2 \in [0,1]$ *and is simple and purely absolutely continuous.*

Proof. We have shown previously in (4.52) that the spectrum in $[0,1]$.

Now for simple spectrum. According to [2], the spectrum of $\mathcal{H}_R^*\mathcal{H}_R$ is *simple* if there is a vector $g \in L^2([0,b_R])$ so that the span of $\hat{E}_{R,\Delta}[g](x)$, where Δ runs through the set of all intervals of $[0,1]$, is dense in $L^2([0,b_R])$. Such a vector g is called a *generating vector*; we will show that

$$g(x) := \chi_{[0,b_R]}(x), \tag{4.60}$$

the characteristic function on $[0,b_R]$, is a generating vector. So for any $f \in L^2([0,b_R])$ we want to show

$$\left\| f - \sum_{j=1}^{n} \alpha_{jn} \hat{E}_{R,I_{jn}}[g](x) \right\|_{L^2([0,b_R])} \to 0 \tag{4.61}$$

as $n \to \infty$, where α_{jn} and I_{jn} are to be determined. Using the properties of $\hat{E}_{R,\lambda^2}$, we calculate

$$f(x) - \sum_{j=1}^{n} \alpha_{jn} \hat{E}_{R,I_{jn}}[g](x) = \int_0^1 \varphi(x,\mu^2) \left\{ \tilde{f}(\mu^2) - \tilde{\phi}_n(\mu^2)\tilde{g}(\mu^2) \right\} \, d\mu^2 \qquad (4.62)$$

where $\varphi(x,\mu^2), \tilde{f}$ are defined in (4.51), (4.56), respectively, and $\tilde{\phi}_n$ is the simple function

$$\tilde{\phi}_n(\mu^2) = \sum_{j=1}^{n} \alpha_{jn} \chi_{I_{jn}}(\mu^2). \qquad (4.63)$$

Let Δ be any interval subset of $[0,1]$; then for any $f \in L^2([0,b_R])$,

$$\langle \hat{E}_{R,\Delta}[f](\cdot), \hat{E}_{R,\Delta}[f](\cdot) \rangle = \langle \hat{E}^2_{R,\Delta}[f](\cdot), f \rangle = \langle \hat{E}_{R,\Delta}[f](\cdot), f \rangle = \left\| \tilde{f} \right\|^2_{L^2(\Delta,\lambda^2)}, \qquad (4.64)$$

where we have used the fact that $\hat{E}_{R,\lambda^2}$ is a projection operator, see [2] p.214. Using the properties of $\hat{E}_{R,\lambda^2}$, we can write the left hand side of (4.62) as

$$\sum_{j=1}^{n} \hat{E}_{R,I_{jn}}[f - \alpha_{jn} g](x). \qquad (4.65)$$

Now using (4.64), (4.65), and the fact that $\hat{E}_{R,\Delta_j} \hat{E}_{R,\Delta_k} = 0$ whenever $\Delta_j \cap \Delta_k = \emptyset$

(see [2]), we see that

$$\left\| f - \sum_{j=1}^{n} \alpha_{jn} \hat{E}_{R,I_{jn}}[g](x) \right\|^2_{L^2([0,b_R])} = \left\| \sum_{j=1}^{n} \hat{E}_{R,I_{jn}}[f - \alpha_{jn} g](x) \right\|^2_{L^2([0,b_R])} \tag{4.66}$$

$$= \sum_{j=1}^{n} \left\| \hat{E}_{R,I_{jn}}[f - \alpha_{jn} g](x) \right\|^2_{L^2([0,b_R])} \tag{4.67}$$

$$= \sum_{j=1}^{n} \left\| \tilde{f} - \alpha_{jn} \tilde{g} \right\|^2_{L^2(I_{jn},\lambda^2)} \tag{4.68}$$

$$= \left\| \tilde{f} - \tilde{\phi}_n \tilde{g} \right\|^2_{L^2([0,1],\lambda^2)}, \tag{4.69}$$

since the intervals I_{jn} are disjoint and $\tilde{\phi}_n$ was defined in (4.63). Now our goal is to show that any $\tilde{f} \in L^2([0,1], \lambda^2)$ can be approximated by $\tilde{\phi}_n \tilde{g}$. Using the properties of D_L, D_R in Appendix C, it can be shown that

$$\tilde{g}(\lambda^2) = \int_0^{b_R} \varphi(x, \lambda^2) g(x)\ dx = -D_L(\infty; \lambda) \sqrt{\frac{|b_L| b_R (2a+1)}{2i(b_R - b_L)}} \tag{4.70}$$

and $\tilde{g}(\lambda^2)$ is real analytic for $\lambda^2 \in (0,1)$. It is clear that any $\tilde{f}$ can be approximated by pieces of the smooth function $\tilde{g}$, so we have

$$\left\| f - \sum_{j=1}^{n} \alpha_{jn} \hat{E}_{R,I_{jn}}[g](x) \right\|^2_{L^2([0,b_R])} = \left\| \tilde{f} - \tilde{\phi}_n \tilde{g} \right\|^2_{L^2([0,1],\lambda^2)} \to 0, \tag{4.71}$$

as desired. Thus, the spectrum of $\mathcal{H}_R^* \mathcal{H}$ is simple and $g = \chi_{[0,b_R]}$ is a generating vector.

Lastly, to show that the spectrum of $\mathcal{H}_R^* \mathcal{H}_R$ is purely absolutely continuous, we begin by showing that the resolvent of $\mathcal{H}_R^*\mathcal{H}_R$ (see Proposition 4.2.1 and Theorem 2.2.4) does not have any poles in the λ plane (thus $\mathcal{H}_R^*\mathcal{H}_R$ has no eigenvalues). Recall from Proposition 4.2.1 that the resolvent of $\mathcal{H}_R^*\mathcal{H}_R$ is

$$\mathcal{R}_R(\lambda^2) := \left(I - \frac{1}{\lambda^2}\mathcal{H}_R^*\mathcal{H}_R\right)^{-1} = I + \pi_R\hat{R}(\lambda/2)\pi_R \tag{4.72}$$

and the kernel (see (2.20)) of $\hat{R}(\lambda)$ is

$$R(z,x;\lambda) = \frac{\vec{g}_1^t(x)\Gamma^{-1}(x;\lambda)\Gamma(z;\lambda)\vec{f}_1(z)}{2\pi i\lambda(z-x)}. \tag{4.73}$$

Since $\Gamma(z;\lambda)$ is the solution of RHP 2.2.1, it does not have any poles in the λ plane and is singled valued for $\lambda \in \overline{\mathbb{C}} \setminus [-1/2, 1/2]$. so it is left to verify that $\lambda = 0$ is not a pole of $\hat{R}(\lambda)$. But this is the case since if $f \in L^2([0, b_R])$ and $\mathcal{H}_R^*\mathcal{H}_R[f] = 0$ it must be true that $f \equiv 0$ because the null space of $\mathcal{H}_R$ is $\{0\}$. Thus, $\mathcal{H}_R^*\mathcal{H}_R$ has no eigenvalues and its spectrum is continuous. To show that the spectrum of $\mathcal{H}_R^*\mathcal{H}_R$ is purely absolutely continuous, we need to show that

$$\sigma_f(\lambda^2) := \langle \hat{E}_{R,\lambda^2}[f], f\rangle, \tag{4.74}$$

where $\hat{E}_{R,\lambda^2}$ is the resolution of the identity of $\mathcal{H}_R^*\mathcal{H}_R$, is absolutely continuous function for all $f \in L^2([0, b_R])$ which are real-valued, smooth and vanish at $0, b_R$ (the set of all such f form a dense set in $L^2([0, b_R])$). The kernel of $d\hat{E}_{R,\lambda^2}/d\lambda^2$ is computed in (4.79), then it is a simple calculation to show that

$$\sigma'_f(\lambda^2) = \frac{-b_L b_R(2a+1)}{2\pi^2 i\lambda^2(b_R - b_L)} \left(\int_0^{b_R} \frac{D_R(\xi;\lambda)}{\xi} f(\xi)\, d\xi \right)^2, \tag{4.75}$$

where $'$ denotes differentiation with respect to λ^2. The integral in $\sigma'_f(\lambda^2)$ is real analytic because the integrand is smooth for $\xi \in (0, b_R)$ and $D_R(\xi, \lambda)$ is real analytic for $\lambda^2 \in (0,1)$, so $\sigma'_f(\lambda^2)$ is real analytic for $\lambda^2 \in (0,1)$ and thus $\sigma_f(\lambda^2)$ is also real analytic for $\lambda^2 \in (0,1)$. Since $\hat{E}_{R,\lambda^2}$ is the resolution of the identity, it must be true that $\sigma_f(0) = 0$ and $\sigma_f(1) = ||f||^2$. Thus $\sigma_f(\lambda^2)$ is absolutely continuous for $\lambda^2 \in [0,1]$ for any f which is a real-valued, smooth function vanishing at $0, b_R$, so the spectrum of $\mathcal{H}_R^*\mathcal{H}_R$ is purely absolutely continuous.

□

4.4 Diagonalization of $\mathcal{H}_R^*\mathcal{H}_R$ and $\mathcal{H}_L^*\mathcal{H}_L$

We are now ready to use Theorem 4.1.1 and build the unitary operators which will diagonalize $\mathcal{H}_R^*\mathcal{H}_R$ and $\mathcal{H}_L^*\mathcal{H}_L$. Recall from Theorem 4.3.2 that $\chi_{[0,b_R]}, \chi_{[b_L,0]}$ are generating vectors for $\mathcal{H}_R^*\mathcal{H}_R, \mathcal{H}_L^*\mathcal{H}_L$, respectively. Following Theorem 4.1.1, we define $U_L : L^2([b_L, 0]) \to L^2([0,1], \sigma_L)$ and $U_R : L^2([0, b_R]) \to L^2([0,1], \sigma_R)$ by (here

$\tilde{f}, \tilde{g}$ are generic L^2 functions, not to be confused with (4.56))

$$U_L^*[\tilde{g}](y) := \int_0^1 \tilde{g}(\lambda^2)\ d\hat{E}_{L,\lambda^2}[\chi_{[b_L,0]}](y), \tag{4.76}$$

$$U_R^*[\tilde{f}](x) := \int_0^1 \tilde{f}(\lambda^2)\ d\hat{E}_{R,\lambda^2}[\chi_{[0,b_R]}](x), \tag{4.77}$$

where $\hat{E}_{L,\lambda^2}, \hat{E}_{R,\lambda^2}$ are the resolutions of the identity for $\mathcal{H}_L^*\mathcal{H}_L, \mathcal{H}_R^*\mathcal{H}_R$, respectively and are defined in (4.50), (4.49), respectively. The spectral measures σ_L, σ_R are defined as

$$\sigma_L(\lambda^2) := \langle \hat{E}_{L,\lambda^2}[\chi_{[b_L,0]}](x), \chi_{[b_L,0]}(x)\rangle, \quad \sigma_R(\lambda^2) := \langle \hat{E}_{R,\lambda^2}[\chi_{[0,b_R]}](x), \chi_{[0,b_R]}(x)\rangle. \tag{4.78}$$

We are able to compute $U_L^*, U_R^*, \sigma_L', \sigma_R'$ explicitly (here $'$ denotes differentitation with respect to λ^2) by using property 6 of Proposition C.0.1. We find that

$$\frac{d\hat{E}_{R,\lambda^2}}{d\lambda^2}[\chi_{[0,b_R]}](x) = \frac{b_L b_R(2a+1)}{2\pi i x|\lambda|(b_R - b_L)} D_R(x;\lambda) D_L(\infty;\lambda), \tag{4.79}$$

$$\frac{d\hat{E}_{L,\lambda^2}}{d\lambda^2}[\chi_{[b_L,0]}](y) = \frac{-b_L b_R(2a+1)}{2\pi i y|\lambda|(b_R - b_L)} D_L(y;\lambda) D_R(\infty;\lambda), \tag{4.80}$$

and the derivatives of the spectral measures $\sigma_R(\lambda^2), \sigma_L(\lambda^2)$ (defined in (4.78)) are

$$\frac{d\sigma_R(\lambda^2)}{d\lambda^2} = \frac{|b_L| b_R(a+1/2)}{i(b_R - b_L)} D_L^2(\infty;\lambda), \tag{4.81}$$

$$\frac{d\sigma_L(\lambda^2)}{d\lambda^2} = \frac{|b_L| b_R(a+1/2)}{i(b_R - b_L)} D_R^2(\infty;\lambda). \tag{4.82}$$

Notice that both quantities above are non-negative a.e. since $a + 1/2$ has zero real part and non-negative imaginary part for $\lambda^2 \in (0,1)$, and $D_R(\infty;\lambda)$ is real analytic for $\lambda^2 \in (0,1)$, see Appendix C. From the definitions of U_L^*, U_R^* in (4.76), (4.77), we plug in our calculations in (4.81), (4.82), (4.79), (4.80). Now some simple algebra shows that

$$U_L^*[\tilde{g}](y) = \int_0^1 \phi_L(y,\lambda)\tilde{g}(\lambda^2)\ d\sigma_L(\lambda^2), \quad U_R^*[\tilde{f}](x) = \int_0^1 \phi_R(x,\lambda)\tilde{f}(\lambda^2)\ d\sigma_R(\lambda^2). \tag{4.83}$$

where σ_L, σ_R were defined in (4.82), (4.81), respectively, and the kernels ϕ_L, ϕ_R are

$$\phi_L(y,\lambda) := \frac{D_L(y;\lambda)}{\pi y|\lambda| D_R(\infty;\lambda)}, \quad \phi_R(x,\lambda) := \frac{-D_R(x;\lambda)}{\pi x|\lambda| D_L(\infty;\lambda)}. \tag{4.84}$$

So for any $g \in L^2([b_L,0]), \tilde{g} \in L^2([0,1],\sigma_L), f \in L^2([0,b_R]), \tilde{f} \in L^2([0,1],\sigma_R)$ we have

$$U_L[g](\lambda^2) = \int_{b_L}^0 \phi_L(y,\lambda)g(y)\ dy, \quad U_L^*[\tilde{g}](y) = \int_0^1 \phi_L(y,\lambda)\tilde{g}(\lambda^2)\ d\sigma_L(\lambda^2), \tag{4.85}$$

$$U_R[f](\lambda^2) = \int_0^{b_R} \phi_R(x,\lambda)f(x)\ dx, \quad U_R^*[\tilde{f}](x) = \int_0^1 \phi_R(x,\lambda)\tilde{f}(\lambda^2)\ d\sigma_R(\lambda^2), \tag{4.86}$$

We have now proven the main result of this chapter.

Theorem 4.4.1. *The operators* $U_R : L^2([0,b_R]) \to L^2([0,1],\sigma_R)$*,* $U_L : L^2([b_L,0]) \to L^2([0,1],\sigma_L)$*, defined in* (4.86), (4.85)*, respectively, are unitary and*

$$U_R \mathcal{H}_R^* \mathcal{H}_R U_R^* = \lambda^2, \quad U_L \mathcal{H}_L^* \mathcal{H}_L U_L^* = \lambda^2 \tag{4.87}$$

in the sense of operator equality on $L^2([0,1],\sigma_R)$*,* $L^2([0,1],\sigma_L)$*, respectively, where*

λ^2 *is to be understood as a multiplication operator.*

Remark 4.4.2. The kernel of U_R and U_R^* is related to the kernel of the operator $f \to \tilde{f}$, defined in (4.56), by

$$\varphi(x, \lambda^2) = \phi_R(x, \lambda)\sqrt{\sigma_R'(\lambda^2)}, \tag{4.88}$$

where $\varphi, \phi_R, \sigma_R'$ are defined in (4.51), (4.84), (4.81), respectively. From this relation we can immediately see that (here $Tf = \tilde{f}$ and T^* is the adjoint of T)

$$\left\|T^* f\right\|_{L^2([0,b_R])} = \left\|U_R^* \frac{f}{\sqrt{\sigma_R'}}\right\|_{L^2([0,b_R])} = \left\|\frac{f}{\sqrt{\sigma_R'}}\right\|_{L^2([0,1],\sigma_R)} = \left\|f\right\|_{L^2([0,1],\lambda^2)} \tag{4.89}$$

for any $f \in L^2([0,1], \lambda^2)$, since U_R is unitary, by Theorem 4.4.1.

CHAPTER 5: DIAGONALIZATION OF $\mathcal{H}_R, \mathcal{H}_L$ VIA TITCHMARSH-WEYL THEORY

Using recent developments in the Titchmarsh-Weyl theory obtained in [10], it was shown in [16] that the operator

$$Lf(x) := [P(x)f'(x)]' + 2\left(x - \frac{b_R + b_L}{4}\right)^2 f(x), \quad P(x) := x^2(x - b_L)(x - b_R) \quad (5.1)$$

has only continuous spectrum and commutes with the FHTs $\mathcal{H}_L, \mathcal{H}_R$, defined in (2.1). We now state the main result of [16] and refer the reader to this paper for more details.

Theorem 5.0.1. *The operators $U_1 : L^2([b_L, 0]) \to L^2(J, \rho_1)$ and $U_2 : L^2([0, b_R]) \to L^2(J, \rho_2)$, where $J = [(b_L^2 + b_R^2)/8, \infty)$, are isometric transformations. Moreover, in the sense of operator equality on $L^2(J, \rho_2)$ one has*

$$U_2 \mathcal{H}_L U_1^* = \sigma(\omega), \quad (5.2)$$

where

$$\sigma(\omega) = \frac{-b_R}{b_L \cosh(\mu(\omega)\pi)}\left(1 + \mathrm{O}\left(\epsilon^{\frac{1}{2}-\delta}\right)\right), \quad \omega \to \infty \quad (5.3)$$

$\epsilon = \omega^{-1/2}$, $\mu(\omega) = \sqrt{\frac{\omega - (b_L + b_R)^2/8}{-b_L b_R} - \frac{1}{4}}$, *and* $0 < \delta << 1$ *is fixed.*

There is a minor typo in this theorem in [16]; when describing $\sigma(\lambda)$, the factor $\frac{a_2^3}{a_1}$ is incorrect and has been fixed here. The operators U_1, U_2 in Theorem 5.0.1 were obtained asymptotically when $\omega \to \infty$. Here we obtain these operators explicitly. Throughout this dissertation the spectral parameter of operators $\mathcal{H}_R^*\mathcal{H}_R, \mathcal{H}_L^*\mathcal{H}_L$ is λ^2, but in [16] the spectral parameter (also named λ) is for the differential operator L so we rename the spectral parameter of L to ω. Thus it is important to establish the relation between λ and ω. It is shown below in Remark 5.1.2 that, for $\lambda \in [-1, 1]$, the relation is

$$\omega(\lambda) = \frac{(b_L + b_R)^2}{8} + b_L b_R \cdot a_-(-|\lambda|/2)(a_-(-|\lambda|/2) + 1), \tag{5.4}$$

where $a(\lambda)$ is defined in Appendix B, which implies $\omega \in [(b_L^2 + b_R^2)/8, \infty)$. This relation between ω and λ also immediately implies

$$\lambda^2 = \operatorname{sech}^2(\mu(\omega)\pi) \iff i\mu(\omega) = a_-(-|\lambda|/2) + \frac{1}{2} \tag{5.5}$$

for $\lambda \in [-1, 1]$, where

$$\mu(\omega) = \sqrt{\frac{\omega - \frac{(b_L+b_R)^2}{8}}{-b_L b_R} - \frac{1}{4}}. \tag{5.6}$$

First we construct explicit solutions of the equation $Lf = \omega f$ in terms of the hypergeometric functions that appear in $\Gamma(z; \lambda)$. Then we follow the same process as described in [16] to create the unitary operators U_1, U_2 which will diagonalize $\mathcal{H}_L, \mathcal{H}_R$.

5.1 Explicit solutions to $Lf = \omega f$

In this section we construct a particular linearly independent solution set of $Lf = \omega f$ on $[b_L, 0]$ and $[0, b_R]$ which satisfy the properties required in [10]. Once these pairs of solutions are obtained, [10] tells us how to construct the spectral measure and unitary operators which will diagonalize $\mathcal{H}_L, \mathcal{H}_R$ (this will be done in section 5.2).

5.1.1 Right Interval

The goal of this subsection is to construct functions φ_2, ϑ_2 that have the following properties:

1. For $x \in [0, b_R]$ and $\omega \in [(b_L^2 + b_R^2)/8, \infty)$, $\varphi_2(x,\omega), \vartheta_2(x,\omega)$ are linearly independent solutions of the ODE

$$\omega g(x) = [P(x)g'(x)]' + 2\left(x - \frac{b_R + b_L}{4}\right)^2 g(x), \tag{5.7}$$

   where $P(x) := x^2(x - b_L)(x - b_R)$,

2. $\varphi_2(x,\omega), \vartheta_2(x,\omega) \in \mathbb{R}$, for all $x \in [0, b_R]$, $\omega \in \mathbb{R}$,

3. $P(x)\varphi_2'(x,\omega) \to 0$ as $x \to b_R^-$,

4. $P(x)W_x(\vartheta_2(x,\omega), \varphi_2(x,\omega)) = 1$ for all $x \in [0, b_R]$, $\omega \in \mathbb{C}$,

5. $\lim\limits_{x \to b_R^-} P(x)W_x(\vartheta_2(x,\omega), \varphi_2(x,\omega')) = 1$ for all $\omega, \omega' \in \mathbb{C}$,

which are necessary in order to use the results of [10]. We will build φ_2, ϑ_2 from the functions $x^{-1}h'_\infty$ and $x^{-1}s'_\infty$, where h'_∞, s'_∞ are defined in (2.13), (2.14). We know from [16] that the kernel of the unitary operator which will diagonalize $\mathcal{H}_R$ is expressed in terms of φ_2. But from Theorem 4.4.1 we know that the kernel of the unitary operator which diagonalizes $\mathcal{H}_R^*\mathcal{H}_R$ is expressed in terms of $x^{-1}D_R(x;\lambda)$, which is a linear combination of $x^{-1}h'_\infty$ and $x^{-1}s'_\infty$. Thus it is reasonable to think that both φ_2 and $x^{-1}h'_\infty, x^{-1}s'_\infty$ are solutions of the same ODE.

Theorem 5.1.1. *The functions*

$$g_{h,R}(x,\lambda) := \frac{1}{x}h'_\infty\left(M_1(x)\right), \; g_{s,R}(x,\lambda) := \frac{1}{x}s'_\infty\left(M_1(x)\right), \tag{5.8}$$

where $M_1(x), h'_\infty, s'_\infty$ are defined in Remark 4.1.2, (2.13), (2.14), *respectively, are linearly independent solutions of*

$$L[g](x) = \left[\frac{(b_L+b_R)^2}{8} + b_Lb_R \cdot a_-(-|\lambda|/2)(a_-(-|\lambda|/2)+1)\right]g(x), \tag{5.9}$$

where L is defined in (5.1)*. Moreover,*

$$W_x[g_{h,R}, g_{s,R}] = \frac{a(a+1)(2a+1)(b_R-b_L)}{P(x)}, \tag{5.10}$$

where $a := a_-(-|\lambda|/2)$ is defined in Appendix B.

Proof. We understand that h'_∞ and s'_∞ are functions of $-|\lambda|/2$, as in (4.48). Recall from (A.1) that $h_\infty(\eta), s_\infty(\eta)$ are linearly independent solutions to the ODE

$$\eta(1-\eta)w''(\eta) + a(a+1)w(\eta) = 0, \tag{5.11}$$

thus $h'_\infty(\eta), s'_\infty(\eta)$ are linearly independent solutions to the ODE

$$\eta(1-\eta)w'''(\eta) + (1-2\eta)w''(\eta) + a(a+1)w'(\eta) = 0. \tag{5.12}$$

Since

$$g_{h,R}(x) = \frac{1}{x}h'_\infty\left(M_1(x)\right), \tag{5.13}$$

it is easy to verify that

$$g'_{h,R}(x) = \frac{b_R b_L}{x^3(b_R - b_L)}h''_\infty\left(M_1(x)\right) - \frac{1}{x^2}h'_\infty\left(M_1(x)\right), \tag{5.14}$$

$$g''_{h,R}(x) = \frac{b_R^2 b_L^2}{x^5(b_R - b_L)^2}h'''_\infty\left(M_1(x)\right) - \frac{4b_R b_L}{x^4(b_R - b_L)}h''_\infty\left(M_1(x)\right) + \frac{2}{x^3}h'_\infty\left(M_1(x)\right). \tag{5.15}$$

Solving for h_∞ in terms of g, we have

$$h'_\infty\left(M_1(x)\right) = x g_{h,R}(x), \tag{5.16}$$

$$h''_\infty\left(M_1(x)\right) = \frac{x^3(b_R - b_L)}{b_R b_L}\left(g'_{h,R}(x) + \frac{g_{h,R}(x)}{x}\right), \tag{5.17}$$

$$h'''_\infty\left(M_1(x)\right) = \frac{x^5(b_R - b_L)^2}{b_R^2 b_L^2}\left(g''_{h,R}(x) + \frac{4}{x}g'_{h,R}(x) + \frac{2}{x^2}g_{h,R}(x)\right). \tag{5.18}$$

So taking $\eta = M_1(x)$ in (5.12), we have

$$0 = \frac{-b_R b_L (x-b_R)(x-b_L)}{x^2(b_R-b_L)^2} h_\infty'''\left(M_1(x)\right) - \frac{x(b_R+b_L)-2b_R b_L}{x(b_R-b_L)} h_\infty''\left(M_1(x)\right) + a(a+1)h_\infty'\left(M_1(x)\right). \tag{5.19}$$

Now writing h_∞ in terms of $g_{h,R}$, we obtain our result. To show linear independence, we compute the Wronskian. The Wronskian matrix is (here h_∞, s_∞ are to be evaluated at $M_1(x)$)

$$\begin{bmatrix} \frac{h_\infty'}{x} & \frac{s_\infty'}{x} \\ \frac{b_R b_L h_\infty''}{x^3(b_R-b_L)} - \frac{h_\infty'}{x^2} & \frac{b_R b_L s_\infty''}{x^3(b_R-b_L)} - \frac{s_\infty'}{x^2} \end{bmatrix} = \frac{1}{x}\begin{bmatrix} 1 & 0 \\ \frac{-1}{x} & \frac{b_R b_L}{x^2(b_R-b_L)} \end{bmatrix} \begin{bmatrix} h_\infty' & s_\infty' \\ h_\infty'' & s_\infty'' \end{bmatrix} \tag{5.20}$$

$$= \frac{1}{x}\begin{bmatrix} 1 & 0 \\ \frac{-1}{x} & \frac{b_R b_L}{x^2(b_R-b_L)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{a(a+1)}{M_1(1-M_1)} \end{bmatrix} \sigma_1 \hat{\Gamma}(M_1(x)), \tag{5.21}$$

where $\hat{\Gamma}$ was defined in (2.18). It is easy to show that $M_1(x)(1-M_1(x)) = \frac{-b_R b_L (x-b_L)(x-b_R)}{x^2(b_R-b_L)^2}$. Recall, from (A.15), that $\det \hat{\Gamma} = -2a-1$. So we have

$$W_x[g_{h,R}, g_{s,R}] = \frac{1}{x^2} \cdot \frac{b_R b_L}{x^2(b_R-b_L)} \cdot \frac{-a(a+1)(2a+1)}{M_1(1-M_1)} = \frac{a(a+1)(2a+1)(b_R-b_L)}{x^2(x-b_R)(x-b_L)}. \tag{5.22}$$

□

Remark 5.1.2. Since we are interested in solutions of $Lf = \omega f$ (see (5.1) for L), Theorem 5.1.1 shows that the relation between ω and λ is

$$\omega(\lambda) = \frac{(b_L + b_R)^2}{8} + b_L b_R \cdot a_-(-|\lambda|/2)(a_-(-|\lambda|/2) + 1) \tag{5.23}$$

for $\lambda \in [-1, 1]$, see Appendix B for $a(\lambda)$.

Define the function

$$f_R(x, \omega) := \frac{b_R(b_R - b_L)}{x(b_R + b_L) - 2b_R b_L} M_4(x)^{-\frac{1}{2}+i\mu} {}_2F_1\left(\begin{matrix} \frac{1}{4} + \frac{i\mu}{2}, \frac{3}{4} + \frac{i\mu}{2} \\ 1 + i\mu \end{matrix} \,\middle|\, M_4^2(x)\right), \tag{5.24}$$

where $M_4(x)$ is defined in Remark 4.1.2. If we take $b_R = -b_L = a$, where a is a constant, we obtain (4.9) of [16]. We now describe the relation of $f_R(x, \omega)$ to h'_∞, s'_∞.

Proposition 5.1.3. *For* $\lambda \in [-1, 1]$, *the functions* $f_R(x, \omega), \overline{f}_R(x, \omega)$, *defined in* (5.24), *are linearly independent solutions to ODE* (5.7) *and*

$$\frac{-b_R \alpha(\lambda)}{x\sqrt{\pi}} h'_\infty(M_1(x)) = k f_R(x, \omega), \quad \frac{-b_R \beta(\lambda)}{x\sqrt{\pi}} s'_\infty(M_1(x)) = \overline{k f}_R(x, \omega) \tag{5.25}$$

where $M_1(x)$ *is defined in Remark 4.1.2,* α, β *and* h'_∞, s'_∞ *are defined in* (4.20) *and* (2.13),(2.14), *respectively and*

$$k = \frac{\Gamma(-i\mu)}{\Gamma(\frac{1}{4} - \frac{i\mu}{2})\Gamma(\frac{3}{4} - \frac{i\mu}{2})}. \tag{5.26}$$

See (5.4) *for the relation between* ω *and* λ *and* (5.6) *for* μ.

Proof. We prove the identities (5.25) first, then it is clear that $f_R, \overline{f}_R$ solve ODE (5.7) since $x^{-1}h'_\infty, x^{-1}s'_\infty$ solve that ODE. First, we have

$$\alpha(\lambda) = \frac{\tan(a\pi)\Gamma(a)}{e^{a\pi i}4^{a+1}\Gamma(a+3/2)} = \frac{\sqrt{\pi}\Gamma(-a-\frac{1}{2})}{ae^{a\pi i}2^{a+1}\Gamma\left(-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)} = \frac{k\sqrt{\pi}}{ae^{a\pi i}2^{a+1}} \tag{5.27}$$

and, from [1] 15.3.16,

$$_2F_1\left(\begin{matrix} a+1, a+1 \\ 2a+2 \end{matrix} \,\middle|\, \frac{1}{M_1(x)}\right) = 2^{a+1}\left[\frac{x(b_R+b_L)-2b_Rb_L}{b_R(x-b_L)}\right]^{-a-1} {}_2F_1\left(\begin{matrix} \frac{a}{2}+\frac{1}{2}, \frac{a}{2}+1 \\ a+\frac{3}{2} \end{matrix} \,\middle|\, M_4^2(x)\right). \tag{5.28}$$

Now putting the previous two equations together,

$$\frac{-b_R\alpha(\lambda)}{x\sqrt{\pi}}h'_\infty\left(M_1(x)\right) = \frac{b_R\Gamma(-a-\frac{1}{2})}{x\Gamma\left(-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)}M_4(x)^{a+1}{}_2F_1\left(\begin{matrix} \frac{a}{2}+\frac{1}{2}, \frac{a}{2}+1 \\ a+\frac{3}{2} \end{matrix} \,\middle|\, M_4^2(x)\right) \tag{5.29}$$

$$= \frac{kb_R(b_R-b_L)}{x(b_R+b_L)-2b_Rb_L}M_4(x)^{-\frac{1}{2}+i\mu}{}_2F_1\left(\begin{matrix} \frac{1}{4}+\frac{i\mu}{2}, \frac{3}{4}+\frac{i\mu}{2} \\ 1+i\mu \end{matrix} \,\middle|\, M_4^2(x)\right) \tag{5.30}$$

$$= kf_R(x,\omega) \tag{5.31}$$

where we have used $i\mu = a + \frac{1}{2}$. Similarly, we have

$$\beta(\lambda) = \frac{\sqrt{\pi}e^{a\pi i}2^a\Gamma(a+\frac{1}{2})}{(a+1)\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{a}{2}+1)} = \frac{\sqrt{\pi}e^{a\pi i}2^a\overline{k}}{a+1} \tag{5.32}$$

and, from [1] 15.3.16,

$$s'_\infty\left(M_1(x)\right) = -\frac{a+1}{e^{a\pi i}}\left[\frac{x(b_R+b_L)-2b_Rb_L}{2b_R(x-b_L)}\right]^a {}_2F_1\left(\begin{matrix} -\frac{a}{2}, -\frac{a}{2}+\frac{1}{2} \\ -a+\frac{1}{2}\end{matrix}\,\middle|\, M_4^2(x)\right). \tag{5.33}$$

Now putting the previous two equations together,

$$\frac{-b_R\beta(\lambda)}{x\sqrt{\pi}} s'_\infty\left(M_1(x)\right) = \frac{b_R\overline{k}}{x} M_4(x)^{-a} {}_2F_1\left(\begin{matrix} -\frac{a}{2}, -\frac{a}{2}+\frac{1}{2} \\ -a+\frac{1}{2}\end{matrix}\,\middle|\, M_4^2(x)\right) \tag{5.34}$$

$$= \frac{\overline{k}b_R(b_R-b_L)}{x(b_R+b_L)-2b_Rb_L} M_4(x)^{-\frac{1}{2}-i\mu} {}_2F_1\left(\begin{matrix} \frac{1}{4}-\frac{i\mu}{2}, \frac{3}{4}-\frac{i\mu}{2} \\ 1-i\mu\end{matrix}\,\middle|\, M_4^2(x)\right) \tag{5.35}$$

$$= \overline{kf}_R(x,\omega). \tag{5.36}$$

□

Remark 5.1.4. Using properties of the Gamma functions, see [1] 6.1.30, it can be shown that

$$|k|^2 = \frac{\coth(\mu\pi)}{2\pi\mu}, \tag{5.37}$$

provided that $\mu \geq 0$.

Proposition 5.1.5. *The Wronskian of $f_R(x,\omega), \overline{f}_R(x,\omega)$ is*

$$W_x[f_R(x,\omega), \overline{f}_R(x,\omega)] = \frac{i\mu b_R^2(b_R-b_L)}{P(x)}, \tag{5.38}$$

where f_R is defined in (5.24) *and* $P(x) = x^2(x-b_L)(x-b_R)$.

Proof. We have already computed the Wronskian of $\frac{h'_\infty(M_1)}{x}, \frac{s'_\infty(M_1)}{x}$, see (5.10). Using the relation between h'_∞, s'_∞ and $f_R, \overline{f}_R$

$$W_x[f_R(x,\omega), \overline{f}_R(x,\omega)] = W_x\left[\frac{-b_R\alpha}{kx\sqrt{\pi}}h'_\infty(M_1), \frac{-b_R\beta}{\overline{k}x\sqrt{\pi}}s'_\infty(M_1)\right] \tag{5.39}$$

$$= \frac{b_R^2\alpha\beta}{\pi|k|^2}W_x[x^{-1}h'_\infty(M_1), x^{-1}s'_\infty(M_1)] \tag{5.40}$$

$$= \frac{\mu b_R^2\tan(a\pi)\tanh(\mu\pi)}{2a(a+1)(a+1/2)} \cdot \frac{a(a+1)(2a+1)(b_R-b_L)}{P(x)} \tag{5.41}$$

$$= \frac{i\mu b_R^2(b_R-b_L)}{P(x)}. \tag{5.42}$$

□

Define

$$\varphi_2(x,\omega) := kf_R(x,\omega) + \overline{kf}_R(x,\omega), \tag{5.43}$$

where k, f_R are defined in (5.26), (5.24), respectively.

Proposition 5.1.6. *The function $\varphi_2(x,\omega)$, defined in (5.43), solves ODE (5.7) and*

$$\varphi_2(x,\omega) = \frac{-b_RD_R(x;\lambda)}{x\sqrt{\pi}} = \frac{b_R(b_R-b_L)}{x(b_R+b_L)-2b_Rb_L}M_4(x)^{-\frac{1}{2}+i\mu}{}_2F_1\left(\begin{matrix}\frac{i\mu}{2}+\frac{1}{4}, \frac{i\mu}{2}+\frac{3}{4}\\ 1\end{matrix}\,\middle|\, 1-M_4^2(x)\right) \tag{5.44}$$

where $M_4(x)$ is defined in Remark 4.1.2 and $D_R(x;\lambda)$ was defined in (4.48). Moreover, $\varphi_2(x,\omega)$ is analytic in a neighborhood of $x=b_R$ and $\varphi_2(b_R,\omega)=1$.

Remark 5.1.7. If we take $b_R = -b_L = a$ (here a is a positive constant) in (5.44), we obtain (4.25) in [16].

Proof. It is obvious that φ_2 is a solution of ODE (5.7) since both $f_R, \overline{f}_R$ are solutions of ODE (5.7). The relationship with D_R is clear in view of (5.25), since

$$\varphi_2(x,\omega) = kf_R(x,\omega)+\overline{kf}_R(x,\omega) = \frac{-b_R}{x\sqrt{\pi}}\left(\alpha h'_\infty\left(M_1(x)\right) + \beta s'_\infty\left(M_1(x)\right)\right) = \frac{-b_R}{x\sqrt{\pi}} D_R(x;\lambda), \tag{5.45}$$

and we know $D_R(x;\lambda)$ is analytic at $x = b_R$, see Appendix C. To express φ_2 as a single hypergeometric function,

$$D_R(x;\lambda) = \alpha(\lambda)h'_\infty\left(M_1(x)\right) + \beta(\lambda)s'_\infty\left(M_1(x)\right) \tag{5.46}$$

$$= -\sqrt{\pi}\frac{\Gamma(1)\Gamma(-a-1/2)}{\Gamma\left(-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)} \cdot M_4(x)^{a+1}{}_2F_1\left(\left.\begin{matrix}\frac{a}{2}+\frac{1}{2}, \frac{a}{2}+1\\ a+\frac{3}{2}\end{matrix}\,\right|\, M_4^2(x)\right) \tag{5.47}$$

$$- \frac{\sqrt{\pi}\Gamma(1)\Gamma(a+\frac{1}{2})}{\Gamma(\frac{a}{2}+1)\Gamma(\frac{a}{2}+\frac{1}{2})} \cdot M_4(x)^{-a}{}_2F_1\left(\left.\begin{matrix}-\frac{a}{2}, -\frac{a}{2}+\frac{1}{2}\\ -a+\frac{1}{2}\end{matrix}\,\right|\, M_4^2(x)\right) \tag{5.48}$$

$$= -\sqrt{\pi}M_4(x)^{a+1}\left\{\frac{\Gamma(1)\Gamma(-a-1/2)}{\Gamma\left(-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)}{}_2F_1\left(\left.\begin{matrix}\frac{a}{2}+\frac{1}{2}, \frac{a}{2}+1\\ a+\frac{3}{2}\end{matrix}\,\right|\, M_4^2(x)\right)\right. \tag{5.49}$$

$$\left.+\frac{\Gamma(1)\Gamma(a+\frac{1}{2})}{\Gamma(\frac{a}{2}+1)\Gamma(\frac{a}{2}+\frac{1}{2})} \cdot M_4(x)^{2(-a-1/2)}{}_2F_1\left(\left.\begin{matrix}-\frac{a}{2}, -\frac{a}{2}+\frac{1}{2}\\ -a+\frac{1}{2}\end{matrix}\,\right|\, M_4^2(x)\right)\right\} \tag{5.50}$$

$$= -\sqrt{\pi}M_4(x)^{a+1}{}_2F_1\left(\left.\begin{matrix}\frac{a}{2}+\frac{1}{2}, \frac{a}{2}+1\\ 1\end{matrix}\,\right|\, 1-M_4^2(x)\right) \tag{5.51}$$

where we have used [1] 15.3.6. Making the substitution $i\mu = a + \frac{1}{2}$ gives the result.

□

Define another solution of ODE (5.7)

$$\vartheta_2(x,\omega) = l_2 f_R(x,\omega) + \bar{l}_2\overline{f}_R(x,\omega) \tag{5.52}$$

where f_R is defined in (5.24) and the constant $l_2 \in \mathbb{C}$ is to be determined.

Lemma 5.1.8. *Let $x \in [0, b_R]$, $\mu \in \mathbb{R}$, and $l_2 \in \mathbb{C}$ be such that*

$$\Im[l_2\overline{k}] = \frac{-1}{2\mu b_R^2(b_R - b_L)} \tag{5.53}$$

where k is defined in (5.26)*. Then,*

$$P(x)W_x[\vartheta_2(x,\omega), \varphi_2(x,\omega)] = 1, \tag{5.54}$$

where $P(x) = x^2(x-b_R)(x-b_L)$ and ϑ_2, φ_2 are defined in (5.52), (5.43)*, respectively.*

Proof. This statement is a consequence of Proposition 5.1.5, which states that

$$W_x[f_R(x,\omega), \overline{f}_R(x,\omega)] = \frac{i\mu b_R^2(b_R - b_L)}{P(x)}. \tag{5.55}$$

Thus,

$$\begin{aligned} W_x[\vartheta_2, \varphi_2] &= W_x[l_2 f_R + \bar{l}_2\overline{f}_R, k f_R + \overline{k f}_R] && (5.56)\\ &= W_x[l_2 f_R, \overline{k f}_R] + W_x[\bar{l}_2\overline{f}_R, k f_R] && (5.57)\\ &= 2i\Im[l_2\overline{k}]W_x[f_R(x,\omega), \overline{f}_R(x,\omega)] && (5.58)\\ &= \frac{-2\mu b_R^2(b_R - b_L)}{P(x)}\Im[l_2\overline{k}] && (5.59) \end{aligned}$$

and the result follows.

□

Remark 5.1.9. To uniquely define the constant l_2, we choose l_2 so that the order 1 term as $x \to b_R$ of φ_2 is 0. According to [1] 15.3.10,

$$\vartheta_2(x,\omega) = -2\Re[i\mu l_2\overline{k}]\ln(b_R - x) - 2\Re\left[i\mu l_2\overline{k}\left\{2\gamma + 2\Psi\left(\frac{1}{2} + i\mu\right) + \ln\left(\frac{|b_L|}{b_R(b_R - b_L)}\right)\right\}\right] + o(1) \quad \text{as } x \to b_R^-, \tag{5.60}$$

where γ is Euler's constant, Ψ is the Digamma function, and k, ϑ_2 are defined in (5.26), (5.52), respectively. Some elementary algebra shows that choosing $l_2 \in \mathbb{C}$ so that

$$\Re[l_2\overline{k}] = \frac{2\gamma + 2\Re\left[\Psi\left(\frac{1}{2} + i\mu\right)\right] + \ln\left(\frac{|b_L|}{b_R(b_R-b_L)}\right)}{-\pi\tanh(\mu\pi)} \cdot \Im[l_2\overline{k}] \tag{5.61}$$

$$= \frac{2\gamma + 2\Re\left[\Psi\left(\frac{1}{2} + i\mu\right)\right] + \ln\left(\frac{|b_L|}{b_R(b_R-b_L)}\right)}{2\mu\pi b_R^2(b_R - b_L)\tanh(\mu\pi)} \tag{5.62}$$

guarantees

$$\Re\left[i\mu l_2\overline{k}\left\{2\gamma + 2\Psi\left(\frac{1}{2} + i\mu\right) + \ln\left(\frac{|b_L|}{b_R(b_R - b_L)}\right)\right\}\right] = 0. \tag{5.63}$$

Moreover, we solve for l_2 explicitly and find

$$l_2 = \frac{k}{b_R^2(b_R - b_L)}\left[2\gamma + 2\Psi\left(\frac{1}{2} - i\mu\right) + \ln\left(\frac{|b_L|}{b_R(b_R - b_L)}\right)\right]. \tag{5.64}$$

We have now gathered the necessary ingredients to prove the main result of this subsection.

Theorem 5.1.10. *The functions* $\varphi_2(x,\omega), \vartheta_2(x,\omega)$*, defined in* (5.43),(5.52)*, respectively, satisfy properties (1)-(5) listed at the beginning of section 5.1.1.*

Proof. This statement is a collection of the results of this subsection.

1. Both φ_2, ϑ_2 are a linear combination of functions $f_R, \overline{f}_R$ which are solutions of ODE (5.7), due to Proposition 5.1.3. From Lemma 5.1.8, we can see that φ_2, ϑ_2 are linearly independent.

2. This is obvious due to definitions of ϑ_2, φ_2, which can be found in (5.52),(5.43), respectively.

3. This follows since $\varphi_2(x,\omega)$ is analytic at $x = b_R$, see Proposition 5.1.6.

4. For $x \in [0, b_R]$ and $\mu \in \mathbb{R}$, Lemma 5.1.8 tells us that $P(x)W_x[\vartheta_2(x,\omega), \varphi_2(x,\omega)] = 1$. For any fixed $x \in [0, b_R]$, $P(x)W_x[\vartheta_2(x,\omega), \varphi_2(x,\omega)]$ is real analytic for $\mu \in \mathbb{R}$ and thus can be extended analytically into the complex $\mu-$plane. But $P(x)W_x[\vartheta_2(x,\omega), \varphi_2(x,\omega)] = 1$ for $\mu \in \mathbb{R}$ so $P(x)W_x[\vartheta_2(x,\omega), \varphi_2(x,\omega)] = 1$ for $\mu \in \mathbb{C}$ as well.

5. We know from Proposition 5.1.6 that $\varphi_2(b_R,\omega) = 1$ and from Remark 5.1.9 $\vartheta_2(x,\omega)$ has log type behavior when $x \to b_R^-$, so

$$\varphi_2(x,\omega) = 1 + \mathrm{O}\left(x - b_R\right) \tag{5.65}$$

$$\vartheta_2(x,\omega) = -2\Re[i\mu l_2 \overline{k}] \ln(b_R - x) + \mathrm{o}(1) \tag{5.66}$$

as $x \to b_R^-$ and $-2\Re[i\mu l_2 \overline{k}] = \frac{-1}{b_R^2(b_R - b_L)}$ for $\mu \in \mathbb{R}$, from Lemma 5.1.8. Thus,

$$W_x\left[\vartheta_2(x,\omega), \varphi_2(x,\omega')\right] = \vartheta_2(x,\omega)\varphi_2'(x,\omega') - \vartheta_2'(x,\omega)\varphi_2(x,\omega') \tag{5.67}$$

$$= \frac{2\Re[i\mu l_2 \overline{k}]}{b_R - x} - 2\Re[i\mu l_2 \overline{k}]\varphi_2'(b_R,\omega')\ln(b_R - x) + \mathrm{O}\left(1\right) \tag{5.68}$$

so it is clear that

$$\lim_{x \to b_R^-} P(x) W_x\left[\vartheta_2(x,\omega), \varphi_2(x,\omega')\right] = 1 \tag{5.69}$$

as $x \to b_R^-$ for any $\omega' \in \mathbb{C}$ and $\omega \in ((b_L^2 + b_R^2)/8, \infty)$ (this implies $\mu \in \mathbb{R}$). But as before, $\lim_{x \to b_R^-} P(x) W_x\left[\vartheta_2(x,\omega), \varphi_2(x,\omega')\right]$ is a real analytic function of ω for $\omega \in ((b_L^2 + b_R^2)/8, \infty)$ and any $\omega' \in \mathbb{C}$. So again we can extend to the complex $\omega-$plane and see that

$$\lim_{x \to b_R^-} P(x) W_x\left[\vartheta_2(x,\omega), \varphi_2(x,\omega')\right] = 1 \tag{5.70}$$

for any $\omega, \omega' \in \mathbb{C}$.

□

Now we repeat this process on $[b_L, 0]$, where there are many similarities.

5.1.2 Left Interval

This subsection will be very similar to the last; we will construct functions φ_1, ϑ_1 on $[b_L, 0]$ that satisfy the following properties:

1. For $x \in [b_L, 0]$ and $\omega \in [(b_L^2 + b_R^2)/8, \infty)$, $\varphi_1(x,\omega), \vartheta_1(x,\omega) \in \mathbb{R}$ are linearly independent solutions of the ODE

$$\omega g(x) = [P(x)g'(x)]' + 2\left(x - \frac{b_R + b_L}{4}\right)^2 g(x), \tag{5.71}$$

2. $\varphi_1(x,\omega), \vartheta_1(x,\omega) \in \mathbb{R}$, for all $x \in [b_L, 0], \omega \in \mathbb{R}$,
3. $P(x)\varphi_1'(x,\omega) \to 0$ as $x \to b_L^+$,
4. $-P(x)W_x(\vartheta_1(x,\omega), \varphi_1(x,\omega)) = 1$ for all $\omega \in \mathbb{C}$,
5. $\lim_{x \to b_L^+} -P(x)W_x(\vartheta_1(x,\omega), \varphi_1(x,\omega')) = 1$ for all $\omega, \omega' \in \mathbb{C}$,

which are necessary in order to use the results of [10]. As before, we will build φ_1, ϑ_1 from the functions $x^{-1}h_\infty'$ and $x^{-1}s_\infty'$.

Theorem 5.1.11. *The functions*

$$g_{h,L}(x,\lambda) := \frac{1}{x}h_\infty'\left(M_3(x)\right), \ g_{s,L}(x,\lambda) := \frac{1}{x}s_\infty'\left(M_3(x)\right), \tag{5.72}$$

where $M_3(x), h_\infty', s_\infty'$ are defined in Remark 4.1.2, (2.13)*,* (2.14)*, respectively, are* *linearly independent solutions of ODE* (5.71) *and*

$$W_x[g_{h,L}, g_{s,L}] = \frac{a(a+1)(2a+1)(b_R - b_L)}{-P(x)}, \tag{5.73}$$

where $a := a_-(-|\lambda|/2)$ *(see Appendix B).*

Proof. This immediately follows from Theorem 5.1.1 by switching b_R, b_L.

□

Define function

$$f_L(x,\omega) := \frac{-b_L(b_R - b_L)}{x(b_R + b_L) - 2b_R b_L}(-M_4(x))^{-\frac{1}{2}+i\mu} {}_2F_1\left(\begin{matrix} \frac{1}{4} + \frac{i\mu}{2}, \frac{3}{4} + \frac{i\mu}{2} \\ 1 + i\mu \end{matrix} \middle| M_4^2(x)\right), \tag{5.74}$$

where $M_4(x)$ is defined in Remark 4.1.2.

Proposition 5.1.12. *For* $x \in [b_L, 0]$*, the functions* f_R, f_L*, defined in* (5.24), (5.74)*, respectively, have the relation*

$$\frac{b_L^2 f_R\left(M_2(x), \omega\right)}{x(b_R + b_L) - b_L b_R} = f_L(x,\omega), \tag{5.75}$$

where $M_2(x)$ *is defined in Remark 4.1.2.*

Proof. This statement is easy to prove because $-M_4(x) = M_4(M_2(x))$. Thus,

$$f_R(M_2(x), \omega) = \frac{b_R}{M_2(x)} M_4(M_2(x))^{\frac{1}{2}+i\mu} {}_2F_1\left(\begin{matrix} \frac{1}{4}+\frac{i\mu}{2}, \frac{3}{4}+\frac{i\mu}{2} \\ 1+i\mu \end{matrix} \middle| M_4^2(M_2(x))\right) \tag{5.76}$$

$$= \frac{b_R}{M_2(x)} (-M_4(x))^{\frac{1}{2}+i\mu} {}_2F_1\left(\begin{matrix} \frac{1}{4}+\frac{i\mu}{2}, \frac{3}{4}+\frac{i\mu}{2} \\ 1+i\mu \end{matrix} \middle| M_4^2(x)\right) \tag{5.77}$$

$$= \frac{b_R}{M_2(x)} \cdot \frac{-x(b_R - b_L)}{x(b_R+b_L) - 2b_L b_R} (-M_4(x))^{-\frac{1}{2}+i\mu} {}_2F_1\left(\begin{matrix} \frac{1}{4}+\frac{i\mu}{2}, \frac{3}{4}+\frac{i\mu}{2} \\ 1+i\mu \end{matrix} \middle| M_4^2(x)\right) \tag{5.78}$$

$$= \frac{b_R x}{b_L M_2(x)} f_L(x, \omega) \tag{5.79}$$

which is equivalent to the result.

□

Simply put, we have obtained $f_L(x, \omega)$ by interchanging b_R and b_L in $f_R(x, \omega)$, see (5.24). Thus the following Proposition immediately follows.

Proposition 5.1.13. *For* $x \in [b_L, 0]$ *and* $\lambda \in [-1, 1]$, *the functions* $f_L(x, \omega), \overline{f}_L(x, \omega)$, *defined in* (5.74), *are linearly independent solutions to ODE* (5.71) *and*

$$\frac{-b_L \alpha(\lambda)}{x\sqrt{\pi}} h'_\infty (M_3(x)) = k f_L(x, \omega), \quad \frac{-b_L \beta(\lambda)}{x\sqrt{\pi}} s'_\infty (M_3(x)) = \overline{k f}_L(x, \omega) \tag{5.80}$$

where α, β *and* k, h'_∞, s'_∞, *are defined in* (4.20) *and* (5.26), (2.13), (2.14), *respectively. See* (5.4) *for the relation between* ω *and* λ.

Proof. This follows immediately from the relation in Proposition 5.1.12, (5.25) and

Proposition 5.1.11. □

Proposition 5.1.14. *The Wronskian of* $f_L(x,\omega), \overline{f}_L(x,\omega)$ *is*

$$W_x[f_L(x,\omega), \overline{f}_L(x,\omega)] = \frac{i\mu b_L^2(b_R - b_L)}{-P(x)}, \tag{5.81}$$

where f_L *was defined in* (5.74).

Proof. This follows by switching b_R and b_L in Proposition 5.1.5. □

Define another two solutions of ODE (5.71)

$$\varphi_1(x,\omega) := kf_L(x,\omega) + \overline{kf}_L(x,\omega), \tag{5.82}$$

$$\vartheta_1(x,\omega) := l_1 f_L(x,\omega) + \bar{l}_1 \overline{f}_L(x,\omega), \tag{5.83}$$

where k, f_L are defined in (5.26), (5.74), respectively, and the constant $l_1 \in \mathbb{C}$ is to be determined.

Remark 5.1.15. Combining Proposition 5.1.12, Appendix C and Proposition 5.1.13, we can see that

$$\varphi_1(x,\omega) = \frac{b_L M_2(x)}{b_R x}\varphi_2(M_2(x),\omega) = \frac{-b_L}{x\sqrt{\pi}} D_L(x;\lambda), \tag{5.84}$$

$\varphi_1(x,\omega)$ is analytic in a neighborhood of $x = b_L$ and $\varphi_1(b_L,\omega) = 1$. See Remark 4.1.2 for $M_2(x)$, and $\varphi_1, \varphi_2, D_L$ are defined in (5.82), (5.43), (4.48), respectively.

Remark 5.1.16. If we take $b_R = -b_L$ in Remark 5.1.15, we obtain

$$\varphi_1(x,\omega) = \varphi_2(-x,\omega), \tag{5.85}$$

which was previously obtained in [16].

Lemma 5.1.17. *Let* $x \in [b_L, 0]$, $\mu \in \mathbb{R}$, *and* $l_1 \in \mathbb{C}$ *be such that*

$$\Im[l_1\overline{k}] = \frac{-1}{2\mu b_L^2(b_R - b_L)}, \tag{5.86}$$

where k *was defined in* (5.26)*. Then,*

$$-P(x)W_x[\vartheta_1, \varphi_1] = 1, \tag{5.87}$$

where φ_1, ϑ_1 *are defined in* (5.82), (5.83)*, respectively.*

Proof. From Proposition 5.1.14, we have the Wronskian of $f_L, \overline{f}_L$.

$$W_x[\vartheta_1, \varphi_1] = W_x[l_1 f_L + \overline{l}_1\overline{f}_L, kf_L + \overline{kf}_L] \tag{5.88}$$

$$= 2i\Im[l_1\overline{k}]W_x[f_L, \overline{f}_L] \tag{5.89}$$

$$= \frac{-2\mu b_L^2(b_R - b_L)}{-P(x)}\Im[l_1\overline{k}] \tag{5.90}$$

which is our result.

□

We have now gathered the necessary ingredients to prove the main result of this subsection.

Theorem 5.1.18. *The functions* $\varphi_1(x,\omega), \vartheta_1(x,\omega)$*, defined in* (5.82),(5.83)*, respectively, satisfy properties (1)-(5) listed at the beginning of section 5.1.2.*

Proof. This statement is a collection of the previous results of this section.

1. Both φ_1, ϑ_1 are a linear combination of functions $f_L, \overline{f}_L$ which are solutions of ODE (5.71), due to Proposition 5.1.13. From Lemma 5.1.17, we can see that φ_1, ϑ_1 are linearly independent.

2. This is obvious due to definitions of ϑ_1, φ_1, which can be found in (5.83),(5.82), respectively.

3. This follows since $\varphi_1(x,\omega)$ is analytic at $x = b_L$, see Proposition 5.1.15.

4. Same idea as in Theorem 5.1.10.

5. Same idea as in Theorem 5.1.10.

□

5.2 Diagonalization of $\mathcal{H}_L, \mathcal{H}_R$

According to the spectral theory developed in [10], we have gathered nearly all necessary ingredients to diagonalize $\mathcal{H}_L, \mathcal{H}_R$. The last step is to construct two functions

$m_1(\omega)$ and $m_2(\omega)$ so that

$$\vartheta_1(x,\omega) + m_1(\omega)\varphi_1(x,\omega) \in L^2([b_L,0]), \quad \vartheta_2(x,\omega) + m_2(\omega)\varphi_2(x,\omega) \in L^2([0,b_R]) \tag{5.91}$$

whenever $\Im\omega > 0$. The spectral measures ρ_1, ρ_2 are constructed via the formula

$$\rho_j(\omega_2) - \rho_j(\omega_1) = \lim_{\epsilon\to 0^+} \frac{1}{\pi} \int_{\omega_1}^{\omega_2} \Im m_j(s+i\epsilon)\ ds, \tag{5.92}$$

for $j = 1, 2$ (see [10] for more details). Recall from (5.74) that when $\Im\omega > 0(\iff \Im\mu > 0)$, $f_L \notin L^2([b_L,0])$ due to $x^{-1/2+i\mu}$ behavior at $x = 0$ (see (5.74) for the definition of f_L). This implies that $\overline{f}_L \in L^2([b_L,0])$. Inspecting $\vartheta_1 + m_1\varphi_1$, we see that

$$\vartheta_1 + m_1\varphi_1 = f_L(l_1 + m_1 k) + \overline{f}_L(\bar{l}_1 + m_1\overline{k}) \implies m_1 = -\frac{l_1}{k} \tag{5.93}$$

guarantees that $\vartheta_1 + m_1\varphi_1 \in L^2([b_L,0])$. So using Lemma 5.1.17 and Remark 5.1.4, we have that

$$m_1(\omega) = -\frac{l_1}{k} = -\frac{l_1\overline{k}}{|k|^2} \implies \Im[m_1(\omega)] = -\frac{\Im[l_1\overline{k}]}{|k|^2} = \frac{\pi\tanh(\mu\pi)}{b_L^2(b_R - b_L)}, \tag{5.94}$$

where we have used Lemma 5.1.17, Remark 5.1.4, and μ was defined in (5.6). So according to (5.92), we have

$$\rho_1'(\omega) = \frac{\Im[m_1(\omega)]}{\pi} = \frac{\tanh(\mu\pi)}{b_L^2(b_R - b_L)}. \tag{5.95}$$

Repeating this process for m_2, we find that $\vartheta_2 + m_2\varphi_2 \in L^2([0, b_R])$ for $\Im\omega > 0$ implies that

$$m_2(\omega) = -\frac{l_2}{k} = -\frac{l_2\overline{k}}{|k|^2} \implies \Im[m_2(\omega)] = -\frac{\Im[l_2\overline{k}]}{|k|^2} = \frac{\pi\tanh(\mu\pi)}{b_R^2(b_R - b_L)}, \tag{5.96}$$

where we have used Lemma 5.1.8 and Remark 5.1.4. Thus, by (5.92),

$$\rho_2'(\omega) = \frac{\Im[m_2(\omega)]}{\pi} = \frac{\tanh(\mu\pi)}{b_R^2(b_R - b_L)}. \tag{5.97}$$

According to [10], the operators $U_1 : L^2([b_L, 0]) \to L^2(J, \rho_1)$ and $U_2 : L^2([0, b_R]) \to L^2(J, \rho_2)$, where $J = \left(\frac{b_R^2 + b_L^2}{8}, \infty\right)$, defined by

$$U_1[f](\omega) := \int_{b_L}^{0} \varphi_1(x, \omega) f(x)\, dx, \quad U_1^*[f](\omega) := \int_J \varphi_1(x, \omega)\tilde{f}(\omega)\, d\rho_1(\omega) \tag{5.98}$$

$$U_2[f](\omega) := \int_{0}^{b_R} \varphi_2(x, \omega) f(x)\, dx, \quad U_2^*[\tilde{f}](\omega) := \int_J \varphi_2(x, \omega)\tilde{f}(\omega)\, d\rho_2(\omega) \tag{5.99}$$

are unitary, where φ_1, φ_2 and ρ_1', ρ_2' are defined in (5.82),(5.43) and (5.95),(5.97), respectively. According to Appendix C and Remark 5.1.15,

$$\mathcal{H}_L[\varphi_1](y, \omega) = \mathcal{H}_L\left[\frac{-b_L D_L(x; \lambda)}{x\sqrt{\pi}}\right](y, \omega) = \frac{-|\lambda| b_L}{b_R} \cdot \frac{-b_R D_R(x; \lambda)}{x\sqrt{\pi}} = \frac{-b_L}{b_R}\operatorname{sech}(\mu\pi)\varphi_2(x, \omega). \tag{5.100}$$

Following [16], eq. (3.41), we have that

$$\mathcal{H}_L U_1^*[\tilde{f}](x) = \int_J \mathcal{H}_L[\varphi_1](x,\omega)\tilde{f}(\omega)\ d\rho_1(\omega) \tag{5.101}$$

$$= \int_J -\frac{b_L}{b_R}\text{sech}(\mu\pi)\frac{\rho_1'(\omega)}{\rho_2'(\omega)}\varphi_2(x,\omega)\tilde{f}(\omega)\ d\rho_2(\omega) \tag{5.102}$$

$$= U_2^*\left[\frac{-b_L\rho_1'(\omega)}{b_R\rho_2'(\omega)}\text{sech}(\mu\pi)\tilde{f}(\omega)\right](x) \tag{5.103}$$

which is equivalent to

$$U_2\mathcal{H}_L U_1^* = -\frac{b_R}{b_L}\text{sech}(\mu\pi). \tag{5.104}$$

Thus we have proven the following:

Theorem 5.2.1. *The operators U_j, $j = 1, 2$, defined in* (5.98)*, are unitary and in the sense of operator equality on $L^2(J, \rho_2)$ one has*

$$U_2\mathcal{H}_L U_1^* = -\frac{b_R}{b_L}\text{sech}(\mu\pi), \tag{5.105}$$

where ρ_2' is defined in (5.97)*.*

Remark 5.2.2. This result was previously obtained in [16] (Theorem 3.1) but only asymptotically for large ω. The factor $\frac{a_2^3}{a_1}$ in Theorem 3.1 of [16] is a typo and should be $-\frac{a_2}{a_1}$.

Since the adjoint of $\mathcal{H}_L$ is $-\mathcal{H}_R$, we have an immediate Corollary.

Corollary 5.2.3. *In the sense of operator equality on $L^2(J, \rho_1)$ one has*

$$U_1\mathcal{H}_L^* U_2^* = -\frac{b_L}{b_R}\text{sech}(\mu\pi), \quad U_1\mathcal{H}_L^*\mathcal{H}_L U_1^* = \text{sech}^2(\mu\pi), \tag{5.106}$$

and in the sense of operator equality on $L^2(J, \rho_2)$ *one has*

$$U_2 \mathcal{H}_R^* \mathcal{H}_R U_2^* = \operatorname{sech}^2(\mu\pi), \tag{5.107}$$

where ρ_1', ρ_2' *are defined in* (5.95), (5.97), *respectively.*

Proof. The proof follows quickly from Theorem 5.2.1 because

$$(U_2 \mathcal{H}_L U_1^*)^* = U_1 \mathcal{H}_L^* U_2^* \tag{5.108}$$

and (what follows is the multiplication operator)

$$\left(-\frac{b_R}{b_L} \operatorname{sech}(\mu\pi) \right)^* = -\frac{b_R}{b_L} \operatorname{sech}(\mu\pi) \cdot \frac{\rho_2'(\omega)}{\rho_1'(\omega)} = -\frac{b_L}{b_R} \operatorname{sech}(\mu\pi). \tag{5.109}$$

□

CHAPTER 6: MATCHING RESULTS FROM CHAPTERS 4 AND 5

In chapters 4 and 5, we obtained two (seemingly) different diagonalizations of $\mathcal{H}_R^*\mathcal{H}_R$ and $\mathcal{H}_L^*\mathcal{H}_L$. We show that the diagonalizations of $\mathcal{H}_L^*\mathcal{H}_L$ and $\mathcal{H}_R^*\mathcal{H}_R$ obtained in chapters 4 and 5 are equivalent, in the sense of change of spectral variable. See (5.4) for the relation between λ and ω.

Theorem 6.0.1. *The two diagonalizations of $\mathcal{H}_L^*\mathcal{H}_L$ obtained in Theorem 4.4.1 and Corollary 5.2.3 are equivalent; that is,*

$$U_1^* \operatorname{sech}^2(\mu\pi) U_1 = U_L^* \lambda^2 U_L \tag{6.1}$$

in the sense of operator equality on $L^2([b_L, 0])$. The operators U_1, U_L are defined in (5.98),(4.85)*, respectively and* $\operatorname{sech}^2(\mu\pi), \lambda^2$ *are to be understood as multiplication operators. An identical statement about U_2 and U_R, defined in* (5.98), (4.86)*, respectively, can be made.*

Proof. We will relate the operators U_L, U_1 by using the change of variable $\lambda \to \omega$ in (5.4), which implies

$$\operatorname{sech}^2(\mu(\omega)\pi) = \lambda^2 \iff i\mu = a_-(-|\lambda|/2) + 1/2 \tag{6.2}$$

and

$$\frac{b_R(a+1/2)}{i\pi\lambda^2 b_L(b_R-b_L)}d\lambda^2 = d\rho_1(\omega). \tag{6.3}$$

Now using this change of variable,

$$\begin{aligned}
U_L^*[\tilde{f}](x) &= \int_0^1 \phi_L(x,\lambda)\tilde{f}(\lambda^2)\ d\sigma_L(\lambda^2) && (6.4)\\
&= \int_0^1 \frac{D_L(x;\lambda)}{\pi x|\lambda|D_R(\infty;\lambda)}\tilde{f}(\lambda^2)\frac{-b_L b_R(a+1/2)}{i(b_R-b_L)}D_R^2(\infty;\lambda)\ d\lambda^2 && (6.5)\\
&= \int_0^1 \frac{-b_L D_L(x;\lambda)}{\sqrt{\pi}x}\cdot \tilde{f}(\lambda^2)b_L\sqrt{\pi}|\lambda|D_R(\infty;\lambda)\cdot \frac{b_R(a+1/2)}{i\pi\lambda^2 b_L(b_R-b_L)}\ d\lambda^2 && (6.6)\\
&= \int_J \varphi_1(x,\omega)\tilde{f}(\operatorname{sech}^2(\mu\pi))c(\omega)\ d\rho_1(\omega) && (6.7)\\
&= U_1^*[c(\omega)\tilde{f}(\operatorname{sech}^2(\mu\pi)](x), && (6.8)
\end{aligned}$$

where $c(\omega) = -b_L\sqrt{\pi}|\lambda|D_R(\infty;\lambda)$ (c has a negative sign because $\lambda = 0$ implies $\omega = \infty$) and $J = [(b_L^2 + b_R^2)/8, \infty)$. Similarly,

$$\begin{aligned}
U_L[f](\lambda^2) &= \int_{b_L}^0 \phi_L(x,\lambda)f(x)\ dx && (6.9)\\
&= \int_{b_L}^0 \frac{D_L(x;\lambda)}{\pi x|\lambda|D_R(\infty;\lambda)}f(x)\ dx && (6.10)\\
&= \int_{b_L}^0 \frac{-b_L D_L(x;\lambda)}{x\sqrt{\pi}}\cdot\frac{-1}{b_L\sqrt{\pi}|\lambda|D_R(\infty;\lambda)}f(x)\ dx && (6.11)\\
&= \frac{1}{c(\omega)}\int_{b_L}^0 \varphi_1(x,\omega)f(x)\ dx && (6.12)\\
&= \frac{1}{c(\omega)}U_1[f](\omega). && (6.13)
\end{aligned}$$

So we have

$$U_L^* \lambda^2 U_L[f](x) = U_L^* \left[\lambda^2 U_L[f](\lambda^2)\right](x) \tag{6.14}$$

$$= U_1^* \left[c(\omega)\text{sech}^2(\mu\pi) U_L[f](\text{sech}^2(\mu\pi))\right](x) \tag{6.15}$$

$$= U_1^* \left[\text{sech}^2(\mu\pi) U_1[f](\omega)\right](x) \tag{6.16}$$

$$= U_1^* \text{sech}^2(\mu\pi) U_1[f](x) \tag{6.17}$$

as desired.

□

CHAPTER 7: FUTURE WORK

This chapter will focus on unfinished and unpolished future work. In section 7.1 we will discuss a so-called *bispectral property* that the operator $\hat{K}$ (see (2.3)) possesses. Then we conclude this dissertation with section 7.2 where we see the importance of the asymptotics of $\Gamma(z;\lambda)$ (obtained in Theorem 3.3.14) in the general setting, where there are $g+1$ intervals with n double points (in chapters 2 through 6 we considered only 1 interval $[b_L, b_R]$ with 1 double point at 0).

7.1 A Bispectral Problem

What is a bispectral problem? Given an operator L_x with variable x and spectral parameter λ, find a second operator J_λ with variable λ and spectral parameter x and 'nice' function $\phi(x,\lambda)$ so that

$$L_x\phi(x,\lambda) = f(\lambda)\phi(x,\lambda), \quad J_\lambda\phi(x,\lambda) = g(x)\phi(x,\lambda), \tag{7.1}$$

where f, g are typically polynomials. We refer the reader to [13] for a more refined and in-depth explanation. As an example, consider Airy's differential operator, defined by

$$L_x f = 0, \quad \text{where} \quad L_x f := \frac{d^2 f}{dx^2} - xf. \tag{7.2}$$

If we seek functions $F(x, \lambda)$ so that

$$L_x F(x, \lambda) = \lambda F(x, \lambda) \tag{7.3}$$

then it must be so that

$$\frac{d^2}{dx^2} F(x, \lambda) - (x + \lambda) F(x, \lambda) = 0. \tag{7.4}$$

It is clear that $F(x, \lambda) = f(x + \lambda)$, where f is any function satisfying $L_x f = 0$. But switching x and λ, we see that $F(x, \lambda)$ is also a solution of

$$L_\lambda F(x, \lambda) = x F(x, \lambda). \tag{7.5}$$

Thus the pair of operators L_x, L_λ and function $F(x, \lambda)$ solve a bispectral problem, as described in (7.1). Moving forward, the bispectral problem we wish to study is given operator $\hat{K}$ in (2.3), construct an operator $\hat{\tilde{K}}$ and function $\varphi(x, \lambda)$ so that

$$\hat{K}[\varphi(\cdot, \lambda)](x) = \lambda \varphi(x, \lambda), \quad \hat{\tilde{K}}[\varphi(x, \cdot)](\lambda) = x \varphi(x, \lambda). \tag{7.6}$$

We begin with a summary of $\hat{K}$ and then construct $\hat{\tilde{K}}$.

7.1.1 A Summary of the Operator $\hat{K}$

Recall that $\hat{K}$ was defined in (2.3), which can be rewritten as

$$\hat{K}[f](z) := \int_{b_L}^{b_R} K(z,x)f(x)\, dx, \quad \text{where} \quad K(z,x) := \frac{\vec{f}_1^t(z)\vec{g}_1(x)}{2\pi i(x-z)}, \tag{7.7}$$

vectors $\vec{f}_1, \vec{g}_1$ are defined as

$$\vec{f}_1(z) = \begin{bmatrix} i\chi_L(z) \\ \chi_R(z) \end{bmatrix}, \; \vec{g}_1(x) = \begin{bmatrix} -i\chi_R(x) \\ \chi_L(x) \end{bmatrix}, \tag{7.8}$$

and χ_L, χ_R are indicator functions on $[b_L, 0], [0, b_R]$, respectively. The resolvent of $\hat{K}$ is defined via the relation

$$(I + \hat{R}(\lambda))\left(I - \frac{1}{\lambda}\hat{K}\right) = I, \tag{7.9}$$

and according to Theorem 2.2.4, the kernel of $\hat{R}$ is given by

$$R(z,x;\lambda) = \frac{\vec{g}_1^t(x)\Gamma^{-1}(x;\lambda)\Gamma(z;\lambda)\vec{f}_1(z)}{2\pi i\lambda(z-x)}, \tag{7.10}$$

where $\Gamma(z;\lambda)$ is the solution to RHP 2.2.1. Observe that the jumps of $\Gamma(z;\lambda)$ can be compactly expressed in terms of $\vec{f}_1, \vec{g}_1$ as

$$\Gamma(z_+,\lambda) = \Gamma(z_-,\lambda)\left(I - \frac{1}{\lambda}\vec{f}_1(z)\vec{g}_1^t(z)\right), \quad z \in [b_L, b_R]. \tag{7.11}$$

7.1.2 Construction of Operator $\hat{\hat{K}}$

Notice that the last subsection could be read backwards; meaning if one was given a matrix $\Gamma(z;\lambda)$ with the jump structure (7.11), then the operator $\hat{K}$ could be recovered. The kernel of $\hat{K}$ (see (7.7)) and the kernel of the resolvent of $\hat{K}$ (see (7.9)) are defined explicitly in terms of $\Gamma(z;\lambda)$ and the vectors $\vec{f}_1, \vec{g}_1$ which appear in the jump of $\Gamma(z;\lambda)$. With that in mind, lets turn our attention to Theorem 4.2.4 where it was shown that

$$\Gamma(z;\lambda_+) = \Gamma(z;\lambda_-)\left[I - \frac{1}{z}\vec{f}(z,\lambda_-)\vec{g}^t(z,\lambda_-)\right], \tag{7.12}$$

for $\lambda \in (-1/2, 0) \cup (0, 1/2)$ and vectors $\vec{f}, \vec{g}$ are defined in (4.17).

Corollary 7.1.1. *For $\lambda \in (-1/2, 0) \cup (0, 1/2)$,*

$$\Gamma(z;\lambda_+) = \left[I - \frac{1}{z}\vec{f}(\infty,\lambda_-)\vec{g}^t(\infty,\lambda_-)\right]\Gamma(z;\lambda_-), \tag{7.13}$$

where $\Gamma(z;\lambda)$ is defined in (2.17) and vectors $\vec{f}, \vec{g}$ are defined in (4.17).

The proof of this corollary shares many similarities with the proof of Theorem 4.2.4 so we omit the details. Notice that the structure of the jumps in (7.11) and (7.13) is identical. We are now ready to define $\hat{\hat{K}} : L^2([-1/2, 1/2]) \to L^2([-1/2, 1/2])$ as

$$\hat{\hat{K}}[\tilde{f}](\lambda) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{K}(\mu,\lambda)\tilde{f}(\mu)\, d\mu, \quad \text{where} \quad \tilde{K}(\mu,\lambda) := \frac{\vec{f}^t(\infty;\mu_-)\vec{g}(\infty;\lambda_-)}{2\pi i(\lambda-\mu)}, \tag{7.14}$$

and vectors $\vec{f}, \vec{g}$ are defined in (4.17). It can be shown that $\hat{\tilde{K}}$ is self-adjoint because $\tilde{K}(\mu, \lambda) = \overline{\tilde{K}(\lambda, \mu)}$. As in (7.9), we define the resolvent of $\hat{\tilde{K}}$ via the relation

$$\left(I + \hat{\tilde{R}}(z)\right)\left(I - \frac{1}{z}\hat{\tilde{K}}\right) = I. \tag{7.15}$$

Comparing with Theorem 2.2.4, we have an analogous result.

Theorem 7.1.2. *With the resolvent operator* $\hat{\tilde{R}}$ *defined by* (7.15)*, let the kernel of* $\hat{\tilde{R}}$ *be denoted by* $\tilde{R}$*. Then,*

$$\tilde{R}(\mu, \lambda; z) = \frac{\vec{g}^t(\infty; \lambda_-)\Gamma(z; \lambda_-)\Gamma^{-1}(z; \mu_-)\vec{f}(\infty; \mu_-)}{2\pi i z(\lambda - \mu)}, \tag{7.16}$$

where $\Gamma(z; \lambda)$ *is defined in* (2.17) *and vectors* $\vec{f}, \vec{g}$ *are defined in* (4.17).

The proof mirrors that of Theorem 2.2.4 so it is omitted here.

7.1.3 Solution of Bispectral Problem

In the previous two subsections we have constructed the pair of operators $\hat{K}$ (see (7.7)) and $\hat{\tilde{K}}$ (see (7.14)). It is left to find a function $\varphi(x, \lambda)$ so that

$$\hat{K}[\varphi(\cdot, \lambda)](x) = \lambda\varphi(x, \lambda), \quad \hat{\tilde{K}}[\varphi(x, \cdot)](\lambda) = x\varphi(x, \lambda). \tag{7.17}$$

The author wildly speculates that the function $\varphi(x, \lambda)$ is expressed in terms of the kernels of the unitary operators U_1, U_2, defined in (5.98), which diagonalize the operators $\mathcal{H}_R, \mathcal{H}_L$ (see Theorem 5.2.1 and Corollary 5.2.3).

7.2 General Setting with n Double Points

The goal of this section is to (loosely) describe how one can use the knowledge of the solution of RHP 2.2.1 (and asymptotics) in the general scenario where there are multiple intervals with touching endpoints. We show that the parametrix near a 'double' point can be built using hypergeometric functions. In this section only, we will refer to $\Gamma_4(z;\lambda)$ as the solution of RHP 2.2.1, where 4 is indicative of the number of endpoints. Similarly, we call g_4, Ψ_4 the g_4-function defined in (3.106) and Ψ_4 the solution to RHP 3.3.4 defined in (3.101).

7.2.1 Setting and Notation

Let $g \in \mathbb{N}$ and choose real numbers $a_1, a_2, \cdots, a_{2g+2}$ so that $-\infty < a_1 < a_2 < \cdots < a_{2g+2} < \infty$. These $2g+2$ points form $g+1$ standardly oriented intervals $I_j = [a_{2j-1}, a_{2j}]$ for $1 \leq j \leq g+1$. Let $I = \cup_{j=1}^{g+1} I_j$. Define matrices

$$V(\lambda) = \begin{bmatrix} 1 & -\frac{i}{\lambda} \\ 0 & 1 \end{bmatrix} \text{ and } V^*(\lambda) = \begin{bmatrix} 1 & 0 \\ \frac{i}{\lambda} & 1 \end{bmatrix}. \tag{7.18}$$

We will frequently refer to $V(\lambda), V^*(\lambda)$ as just V, V^* for convenience. For each interval I_j, $1 \leq j \leq g+1$, assign a matrix V or V^*, with the condition that at least one interval has been assigned V and V^*. Let the set of all intervals associated

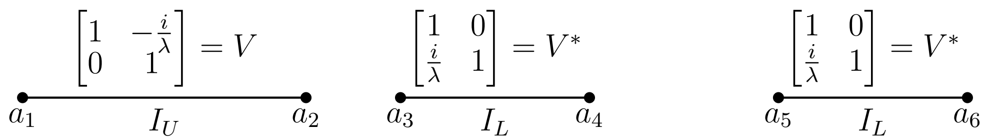

Figure 7.1: One possible I_U and I_L when $g = 2$.

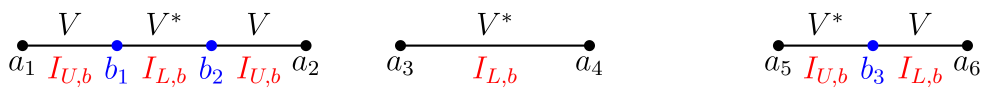

Figure 7.2: Introduction of 3 double points.

with V be labeled I_U and the set of all intervals associated with V^* be labeled I_L. In the Figure 7.1, we display a possible setup with $g = 2$. Now introduce 'double' points $b_1, b_2, \cdots, b_n$ with $b_1 < b_2 < \cdots < b_n$ and place each b_k in the interior of $I_U \cup I_L$. When a double point is placed into an interval I_j, the matrix associated with I_j (either V or V^*) will switch to the right of the double point. For example, if $I_1 = [a_1, a_2]$ has been assigned V and we introduce one double point b_1 to I_1, then the interval $[a_1, b_1]$ is still assigned V and the interval $[b_1, a_2]$ is now assigned V^*. After placing all double points, we will denote $I_{U,b}$ the set of all intervals which have been assigned V and $I_{L,b}$ the set of all intervals which have been assigned V^*. To illustrate, we show one possible way to add 3 double points to the scenario in Figure 7.2.

We begin by stating the general RHP.

Riemann-Hilbert Problem 7.2.1. *Find a* 2×2 *matrix-function* $\Gamma = \Gamma(z;\lambda)$, $\lambda \in \mathbb{C} \setminus \{0\}$, *which is analytic in* $\overline{\mathbb{C}} \setminus I$ *and satisfies*

$$\Gamma(z_+;\lambda) = \Gamma(z_-;\lambda) \begin{bmatrix} 1 & -\frac{i}{\lambda} \\ 0 & 1 \end{bmatrix}, \text{ for } z \in I_{U,b}, \tag{7.19}$$

$$\Gamma(z_+;\lambda) = \Gamma(z_-;\lambda) \begin{bmatrix} 1 & 0 \\ \frac{i}{\lambda} & 1 \end{bmatrix}, \text{ for } z \in I_{L,b}, \tag{7.20}$$

$$\Gamma(z;\lambda) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(\log(z-a_j)) \end{bmatrix} \text{ as } z \to a_j \text{ for each } a_j \in I_U, \tag{7.21}$$

$$\Gamma(z;\lambda) = \begin{bmatrix} \mathrm{O}(\log(z-a_j)) & \mathrm{O}(1) \end{bmatrix} \text{ as } z \to a_j \text{ for each } a_j \in I_L, \tag{7.22}$$

$$\Gamma(z;\lambda) \in L^2([a_1, a_{2g+2}]), \tag{7.23}$$

$$\Gamma(z;\lambda) = I + \mathrm{O}\left(z^{-1}\right) \text{ as } z \to \infty. \tag{7.24}$$

Remark 7.2.2. As it was shown in Remark 2.2.3, the solution $\Gamma(z;\lambda)$ of the RHP 7.2.1 possesses the symmetry $\Gamma(z;\lambda) = \overline{\Gamma(\bar{z};\bar{\lambda})}$.

Remark 7.2.3. Note that if $\Gamma(z;\lambda)$ solve the RHP 7.2.1 (with the piece-wise constant jump matrix $V(z)$), then $\sigma_2\Gamma(z;\lambda)\sigma_2$ solves the same type RHP with the only distinction that the jump matrix now is $\sigma_2 V \sigma_2 = V^*(z)$. That is, interchanging intervals $I_{U,b}, I_{L,b}$ in RHP 7.2.1 is equivalent to the conjugating the solution with σ_2.

7.2.2.1 Reduction to the Model Problem and g-function

Define the g-function $g(z)$ as the solution of the following scalar RHP:

- For $z \in I_{L,b}$, $g_+(z) + g_-(z) = -1$,
- For $z \in I_{U,b}$, $g_+(z) + g_-(z) = 1$,
- For $z \in (a_{2j}, a_{2j+1})$, $g_+(z) - g_-(z) = i\Omega_j$, $j = 1, \cdots, g$ where constants Ω_j are to be determined,
- g is analytic in $\overline{\mathbb{C}} \setminus I$,
- $g \in L^2([a_1, a_{2g+2}])$.

The solution of this RHP exists and is given by

$$g(z) = \frac{R(z)}{2\pi i}\left(\int_I \frac{\chi(z)d\zeta}{(\zeta - z)R_+(\zeta)} + \sum_{j=1}^{g}\int_{a_{2j}}^{a_{2j+1}} \frac{i\Omega_j d\zeta}{(\zeta - z)R(\zeta)}\right), \tag{7.25}$$

where

$$\chi(z) = \begin{cases} 1, & \text{if } z \in I_{U,b} \\ -1, & \text{if } z \in I_{L,b} \end{cases}, \qquad R(\zeta) = \prod_{j=1}^{2g+2}(\zeta - a_j)^{\frac{1}{2}}. \tag{7.26}$$

with the branch of R satisfying $R(z) \sim z^{g+1}$ as $z \to \infty$ and the constants Ω_j are (uniquely) chosen so that $g(z)$ is analytic at $z = \infty$. Note that $g(z)$ has $\mathrm{O}\left(\ln(z - b_k)\right)$ behavior near the double points b_k. Also note that $\Re g(z)$ is a Schwarz symmetrical harmonic function in $\overline{\mathbb{C}} \setminus I$. Because of the jump conditions and the Schwarz symmetry,

$$\Re g_+(z) = \Re g_-(z) = \pm\frac{1}{2} \tag{7.27}$$

on $I_{U,b}, I_{L,b}$ respectively. Thus, $|\Re g(z)| < \frac{1}{2}$ on $\overline{\mathbb{C}} \setminus I$. The transformation

$$Y(z;\varkappa) = e^{-\varkappa g(\infty)\sigma_3}\Gamma(z;e^{-\varkappa})e^{\varkappa g(z)\sigma_3}, \tag{7.28}$$

where $\varkappa = -\ln\lambda$, reduces the RHP 7.2.1 to the following RHP.

Riemann-Hilbert Problem 7.2.4. *Find a 2×2 matrix-function $Y(z;\varkappa)$ with the following properties:*

(a) *$Y(z;\varkappa)$ is analytic in $\mathbb{C}\setminus[a_1, a_{2g+2}]$;*

(b) *$Y(z;\varkappa)$ satisfies the jump conditions*

$$\begin{aligned} Y_+ &= Y_- \begin{bmatrix} e^{\varkappa(g_+-g_-)} & 0 \\ ie^{\varkappa(g_++g_-+1)} & e^{-\varkappa(g_+-g_-)} \end{bmatrix}, \quad z\in I_{L,b}, \\ Y_+ &= Y_- \begin{bmatrix} e^{\varkappa(g_+-g_-)} & -ie^{-\varkappa(g_++g_--1)} \\ 0 & e^{-\varkappa(g_+-g_-)} \end{bmatrix}, \quad z\in I_{U,b}, \\ Y_+ &= Y_-e^{i\varkappa\Omega_j\sigma_3}, \quad z\in(a_{2j}, a_{2j+1}),\ j=1,\dots,g; \end{aligned} \tag{7.29}$$

(c) *non-tangential boundary values of $Y(z,\varkappa)$ from the upper/lower half-planes belong to $L^2_{loc}(I)$, and;*

(d) *$Y = 1 + O(z^{-1}) \quad as \quad z\to\infty$.*

The jumps for Y on $I_{U,b}$ and $I_{L,b}$ can be written as

$$Y_+(z;\varkappa) = Y_-(z;\varkappa)\begin{bmatrix}1 & \frac{1}{i}e^{-\varkappa(2g_-+1)}\\ 0 & 1\end{bmatrix}(i\sigma_1)\begin{bmatrix}1 & \frac{1}{i}e^{-\varkappa(2g_++1)}\\ 0 & 1\end{bmatrix} \text{ on } I_{L,b}, \tag{7.30}$$

$$Y_+(z;\varkappa) = Y_-(z;\varkappa)\begin{bmatrix}1 & 0\\ ie^{\varkappa(2g_--1)} & 1\end{bmatrix}(-i\sigma_1)\begin{bmatrix}1 & 0\\ ie^{\varkappa(2g_+-1)} & 1\end{bmatrix} \text{ on } I_{U,b}, \tag{7.31}$$

This decomposition can be verified by direct matrix multiplication and by using the jump properties of $g(z)$. We now follow the standard procedure of opening lenses around each subinterval of $I_{U,b}, I_{L,b}$. First we introduce the new unknown matrix

$$Z(z;\varkappa) = \begin{cases} Y(z;\varkappa) & \text{outside the lenses,}\\ Y(z;\varkappa)\begin{bmatrix}1 & 0\\ \mp ie^{\varkappa(2g-1)} & 1\end{bmatrix} & z\in\mathcal{L}^{(\pm)}_{U,b},\\ Y(z;\varkappa)\begin{bmatrix}1 & \pm ie^{-\varkappa(2g+1)}\\ 0 & 1\end{bmatrix} & z\in\mathcal{L}^{(\pm)}_{L,b}, \end{cases} \tag{7.32}$$

where $\mathcal{L}^{(\pm)}_{U,b,L,b}$ denote regions inside the lenses around intervals $I_{U,b}, I_{L,b}$ and in the upper or lower half planes respectively, see Figure 3.5 for the 1 interval scenario.

Riemann-Hilbert Problem 7.2.5. *Find a matrix* $Z(z;\varkappa)$, *analytic on* $\overline{\mathbb{C}}\setminus\left(\mathcal{L}^{(\pm)}_{U,b}\cup\mathcal{L}^{(\pm)}_{L,b}\right)$, *satisfying the jump conditions*

$$Z_+(z;\varkappa) = Z_-(z;\varkappa)\begin{cases} \begin{bmatrix} 1 & 0 \\ ie^{\varkappa(2g-1)} & 1 \end{bmatrix} & z \in \partial\mathcal{L}_{U,b}^{(\pm)} \setminus \mathbb{R}, \\ \begin{bmatrix} 1 & \frac{1}{i}e^{-\varkappa(2g+1)} \\ 0 & 1 \end{bmatrix} & z \in \partial\mathcal{L}_{L,b}^{(\pm)} \setminus \mathbb{R}, \\ -i\sigma_1 & z \in I_{U,b}, \\ i\sigma_1 & z \in I_{L,b}, \\ e^{i\varkappa\Omega_j\sigma_3} & z \in (a_{2j}, a_{2j+1}),\ j = 1, \ldots, g \end{cases} \tag{7.33}$$

normalized by

$$Z(z;\varkappa) = 1 + \mathrm{O}\left(z^{-1}\right),\ \text{as } z \to \infty, \tag{7.34}$$

with same endpoint behavior as $Y(z;\varkappa)$*.*

Then the approximation of $Z(z,\varkappa)$ outside small discs around the endpoints and double points is given by the outer parametrix (solution of the model RHP) $\Psi(z;\varkappa)$. The approximation of $Z(z,\varkappa)$ near the endpoints and double points is given by local parametrices. Following [3], the model problem for $\Psi(z;\varkappa)$ is:

Riemann-Hilbert Problem 7.2.6 (Model problem)**.** *Find a matrix* $\Psi = \Psi(z;\varkappa)$*,* *analytic on* $\mathbb{C} \setminus [a_1, a_{2g+2}]$ *and satisfying the following conditions:*

$$\Psi_+ = \Psi_-(-i\sigma_1), \ \ on \ I_{U,b} \tag{7.35}$$

$$\Psi_+ = \Psi_-(i\sigma_1), \ \ on \ I_{L,b} \tag{7.36}$$

$$\Psi_+ = \Psi_- e^{i\varkappa\Omega_j\sigma_3}, \ \ on \ (a_{2j}, a_{2j+1}) \tag{7.37}$$

$$\Psi(z) = 1 + \mathrm{O}\left(z^{-1}\right) \ \ as \ z \to \infty \tag{7.38}$$

$$\Psi(z) = \mathrm{O}\left(|z - b_k|^{-1/2}\right) \ \ as \ z \to b_k \tag{7.39}$$

$$\Psi(z) = \mathrm{O}\left(|z - a_j|^{-1/4}\right) \ \ as \ z \to a_j \tag{7.40}$$

and $\Psi_\pm(z) \in L^2_{loc}(I)$ *except for neighborhoods of the double points* b_k*.*

Note that solution of the RHP 7.2.6 is not unique because $\Psi(z) \notin L^2_{loc}$ near any double point b_k.

7.2.2.2 Solution of the Model Problem

Let $z \in I_j$, $1 \leq j \leq g+1$. Define another g function $\tilde{g}(z;\varkappa)$ as the solution of the following RHP:

$$\tilde{g}(z_+;\varkappa) + \tilde{g}(z_-;\varkappa) = \begin{cases} 0, & \text{if } z \in I_U \cap I_{U,b} \text{ or } z \in I_L \cap I_{L,b}, \\ \mathrm{sgn}(\Im\varkappa) i\pi, & \text{if } z \in I_L \cap I_{U,b}, \\ -\mathrm{sgn}(\Im\varkappa) i\pi, & \text{if } z \in I_U \cap I_{L,b}; \end{cases} \tag{7.41}$$

$$\tilde{g}(z_+;\varkappa) - \tilde{g}(z_-;\varkappa) = W_j, \quad z \in (a_{2j}, a_{2j+1}), \ \text{ where } W_j \text{ is TBD;} \tag{7.42}$$

$\tilde{g}(z;\varkappa)$ is analytic at $\overline{\mathbb{C}} \setminus [a_1, a_{2g+2}]$ and is an L^2_{loc} function on the jump contour. $R(\zeta) = \prod_{j=1}^{2g+2}(\zeta - a_j)^{1/2}$, where we choose the branch of R by $R(z) \sim z^{g+1}$ as $z \to \infty$. Then

$$\tilde{g}(z;\varkappa) = \frac{R(z)}{2\pi i}\left(\int_{I_U \cap I_{L,b}} \frac{-\mathrm{sgn}(\Im\varkappa) i\pi d\zeta}{(\zeta - z)R_+(\zeta)} + \int_{I_L \cap I_{U,b}} \frac{\mathrm{sgn}(\Im\varkappa) i\pi d\zeta}{(\zeta - z)R_+(\zeta)} \right.$$
$$\left. + \sum_{j=1}^{g} \int_{a_{2j}}^{a_{2j+1}} \frac{W_j d\zeta}{(\zeta - z)R(\zeta)}\right), \tag{7.43}$$

where the constants $W_j \in \mathbb{R}$ are (uniquely) chosen (in the standard way), so that $\tilde{g}(z;\varkappa)$ is analytic at $z = \infty$. Let $z \to b_k$. Then, according to (7.43),

$$\tilde{g}(z;\varkappa) = \pm\frac{1}{2}\mathrm{sgn}(\Im\varkappa)\mathrm{sgn}(\Im z)\ln(z - b_k) + O(1) \tag{7.44}$$

where the sign "-" if we have $I_{L,b}$ to the left of b_k and $I_{U,b}$ to the right and the sign is plus in the opposite case. It is to be understood that if $\Im\varkappa = 0$, then $\mathrm{sgn}(\Im\varkappa) = \pm 1$ where $\pm$ is taken when $\varkappa$ approaches the real axis from above/below. Denote

$$\tilde{\Psi}(z;\varkappa) = e^{-\tilde{g}(\infty;\varkappa)\sigma_3}\Psi(z;\varkappa)e^{\tilde{g}(z;\varkappa)\sigma_3}. \tag{7.45}$$

Then $\tilde{\Psi}(z;\varkappa)$ satisfies the following RHP.

Riemann-Hilbert Problem 7.2.7. *Find a matrix* $\tilde{\Psi} = \tilde{\Psi}(z;\varkappa)$*, analytic on* $\mathbb{C} \setminus$

$[a_1, a_{2g+2}]$ *and satisfying the following conditions:*

$$\begin{aligned}
&\tilde{\Psi}_+ = \tilde{\Psi}_-(-i\sigma_1)\ , \quad \textit{for } z \in I_U; \\
&\tilde{\Psi}_+ = \tilde{\Psi}_-(i\sigma_1)\ , \quad \textit{for } z \in I_L; \\
&\tilde{\Psi}_+ = \tilde{\Psi}_- e^{(i\varkappa\Omega_j + W_j)\sigma_3} \ \textit{on } (a_{2j}, a_{2j+1}); \\
&\tilde{\Psi}(z) = 1 + O(z^{-1})\ , \quad z \to \infty; \\
&\tilde{\Psi}_\pm(z) \in L^2_{loc}(I).
\end{aligned} \tag{7.46}$$

Lemma 7.2.8. *There exist a solution* $\Psi(z;\varkappa)$ *to RHP 7.2.6 such that the matrix function* $\tilde{\Psi}(z;\varkappa)$ *given by* (7.45) *solves the RHP 7.2.7.*

Proof. Analyticity of $\tilde{\Psi}(z;\varkappa)$ and its asymptotics at $z = \infty$ are clear. We only need to check the jump conditions. Let $z \in I_L \cap I_{L,b}$. Then we have

$$\tilde{\Psi}_+(z;\varkappa) = \Psi_+(z;\varkappa) e^{\tilde{g}_+(z;\varkappa)\sigma_3} \tag{7.47}$$

$$= \Psi_-(z;\varkappa) i\sigma_1 e^{\tilde{g}_+(z;\varkappa)\sigma_3} \tag{7.48}$$

$$= \Psi_-(z;\varkappa) e^{-\tilde{g}_+(z;\varkappa)\sigma_3} i\sigma_1 \tag{7.49}$$

$$= \tilde{\Psi}_-(z;\varkappa) i\sigma_1 \tag{7.50}$$

since $-\tilde{g}_+(z;\varkappa) = \tilde{g}_-(z;\varkappa)$. Consider now the case $z \in I_U \cap I_{L,b}$. Then $-\tilde{g}_+(z;\varkappa) = \tilde{g}_-(z;\varkappa) \pm \operatorname{sgn}(\Im\varkappa) i\pi$, so that $e^{-\tilde{g}_+(z;\varkappa)\sigma_3} = -e^{\tilde{g}_-(z;\varkappa)\sigma_3}$ and, similarly to (7.47), we obtain

$$\tilde{\Psi}_+(z;\varkappa) = \tilde{\Psi}_-(z;\varkappa)(-i\sigma_1). \tag{7.51}$$

Finally, on any gap (a_{2j}, a_{2j+1}) we have

$$\tilde{\Psi}_+(z;\varkappa) = \Psi_+(z;\varkappa)e^{\tilde{g}_+(z;\varkappa)\sigma_3} = \Psi_-(z;\varkappa)e^{i\varkappa\Omega_j\sigma_3}e^{(\tilde{g}_-(z;\varkappa)+W_j)\sigma_3} = \tilde{\Psi}_-(z;\varkappa)e^{(i\varkappa\Omega_j+W_j)\sigma_3}. \tag{7.52}$$

According to the last requirement of (7.46), the RHP 7.2.7 has a unique solution, which can be constructed using Riemann Theta functions as shown in [3], Section 5. Thus, we completed the proof. □

We fix the solution to the model RHP 7.2.6 as

$$\Psi(z;\varkappa) = e^{\tilde{g}(\infty;\varkappa)\sigma_3}\tilde{\Psi}(z;\varkappa)e^{-\tilde{g}(z;\varkappa)\sigma_3}. \tag{7.53}$$

Remark 7.2.9. Note that $\det \Psi \equiv 1$ and $\tilde{\Psi}(z;\varkappa)$ is analytic (and invertible) near $z = b_k$ on any shore of the cut I. Therefore, the RHP 7.2.7 has a unique solution $\tilde{\Psi}(z,\varkappa)$, that, in general, can be constructed in terms of the Riemann Theta functions.

7.2.2.3 Parametrix at a Double Point

Construction of the local parametrices at the endpoints a_j, $j = 1, \dots, 2j+2$ is essentially the same as in [3]. Label this parametrix $\mathcal{P}_{a_j}(z;\varkappa)$. Here we consider a parametrix at $b_k \in (a_{2j-1}, a_{2j})$. After opening of the lenses, the RHP 7.2.4 was naturally transform to another RHP, satisfied by $Z(z;\varkappa)$. Let us define the approximate solution to this RHP, called $\hat{Z}(z;\varkappa)$, as follows:

- $\hat{Z}(z;\varkappa) = \Psi(z;\varkappa)$, where $\Psi(z;\varkappa)$ is given by (7.53), everywhere outside small discs $\mathcal{D}_{a_j}$, $\mathcal{D}_{b_k}$ around the branchpoints a_j, $j = 1, \ldots, 2g+2$ and double points b_k, $k = 1, \ldots, n$;

- inside each $\mathcal{D}_{a_j}$ the approximate solution $\hat{Z}(z;\varkappa)$ is given by the standard Bessel type parametrix, constructed, for example, in [3];

- inside each $\mathcal{D}_{b_k}$ the approximate solution $\hat{Z}(z;\varkappa) = \mathcal{P}_{b_k}(z;\varkappa)$, where the parametrix $\mathcal{P}_{b_k}(z;\varkappa)$ will be constructed below.

The requirements for the sectorial analytic parametrix $\mathcal{P}_{b_k}(z;\varkappa)$ are:

1. $\mathcal{P}_{b_k}(z;\varkappa)$ has to have exactly the same jump matrices in $\mathcal{D}_{b_k}$ as $Z(z;\varkappa)$;

2. $\mathcal{P}_{b_k}(z;\varkappa)$ has to be in L^2_{loc} on the jump contours;

3. the jump matrix between $\Psi(z;\varkappa)$ and $\mathcal{P}_{b_k}(z;\varkappa)$ on $\partial\mathcal{D}_{b_k}$ should approach 1 as $\varkappa \to \infty$.

The error matrix $\mathcal{E}(z;\varkappa)$ is defined as $\mathcal{E} = Z\hat{Z}^{-1}$. If we denote by M, $\hat{M}$ the jump matrices of Z, $\hat{Z}$ respectively, then we have

$$\mathcal{E}_+ = Z_+\hat{Z}_+^{-1} = Z_-M\hat{M}^{-1}\hat{Z}_-^{-1} = Z_-\hat{Z}_-^{-1}\hat{Z}_-M\hat{M}^{-1}\hat{Z}_-^{-1} = \mathcal{E}_-M_{\mathcal{E}}, \tag{7.54}$$

where the jump matrix $M_{\mathcal{E}}$ for $\mathcal{E}$ is

$$M_{\mathcal{E}} = \hat{Z}_-M\hat{M}^{-1}\hat{Z}_-^{-1}. \tag{7.55}$$

Since the jump matrices M, $\hat{M}$ coincide inside the discs $\mathcal{D}_{a_j}, \mathcal{D}_{b_k}$, as well as on the interval (a_1, a_{2g+2}) outside these discs, the jump contours for $\mathcal{E}$ consists of the boundaries $\partial\mathcal{D}_{a_j}, \partial\mathcal{D}_{b_k}$, as well as of the lenses outside the discs, see Figure 3.6 for the 1 interval scenario. By the standard arguments, $M_{\mathcal{E}} = 1 + O(\varkappa^{-1})$ on the lenses and on $\partial\mathcal{D}_{a_j}$. Since Z has no jump on $\partial\mathcal{D}_{b_k}$, we conclude that

$$M_{\mathcal{E}} = \hat{Z}_- \hat{Z}_+^{-1} = \Psi \mathcal{P}_{b_k}^{-1} \tag{7.56}$$

provided that the contour $\partial\mathcal{D}_{b_k}$ is positively (counter clockwise) oriented. We now construct the parametrix $\mathcal{P}_{b_k}$ so that $M_{\mathcal{E}}^{-1} = \mathcal{P}_{b_k}\Psi^{-1}$ approaches 1 as $\varkappa \to \infty$. Let $\Gamma_4(z;\varkappa)$ denote the solution RHP 2.2.1 with $b_L = -a$, $b_R = a$, (here a is a positive constant) $b_1 = 0$ and jump matrices V on $(-a, 0)$ and V^* on $(0, a)$. We will call the RHP for $\Gamma_4(z;\varkappa)$ as the 4 point RHP. *Let the double point* b_k *have* $I_{U,b}$ *subinterval on the left and* $I_{L,b}$ *subinterval on the right.* Denote by g_4 the g-function (3.106), constructed for the 4 point RHP. Then

$$\mathcal{P}_{b_k}(z;\varkappa) = \begin{cases} \Psi(z;\varkappa)\Psi_4^{-1}(\zeta_k;\varkappa)\Gamma_4(\zeta_k;\varkappa)e^{\varkappa g_4(\zeta_k)\sigma_3}\begin{bmatrix} 1 & \pm i e^{-\varkappa(2g_4(\zeta_k)+1)} \\ 0 & 1 \end{bmatrix}, & (1)_\pm, \\ \Psi(z;\varkappa)\Psi_4^{-1}(\zeta_k;\varkappa)\Gamma_4(\zeta_k;\varkappa)e^{\varkappa g_4(\zeta_k)\sigma_3}, & (2)_\pm, \\ \Psi(z;\varkappa)\Psi_4^{-1}(\zeta_k;\varkappa)\Gamma_4(\zeta_k;\varkappa)e^{\varkappa g_4(\zeta_k)\sigma_3}\begin{bmatrix} 1 & 0 \\ \mp i e^{\varkappa(2g_4(\zeta_k)-1)} & 1 \end{bmatrix}, & (3)_\pm, \end{cases} \tag{7.57}$$

where Ψ_4 is given by (3.3.5) and see Figure 7.3 for regions (1),(2),(3). The function

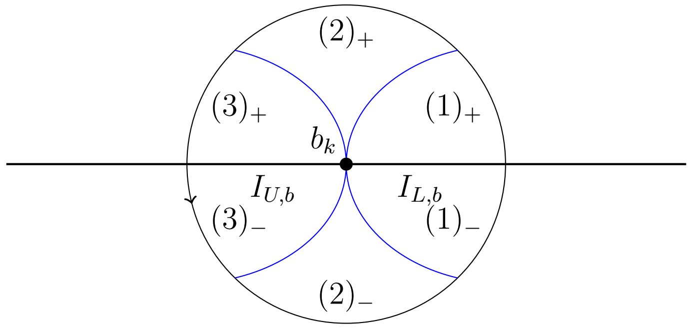

Figure 7.3: The region $\mathcal{D}_{b_k}$ with lenses(blue).

$\zeta_k(z)$ is defined by the condition

$$g(z) = g_4(\zeta_k(z)), \tag{7.58}$$

where $g(z)$ is given by (7.25). The existence of such $\zeta_k(z)$ will be shown in Lemma 7.2.11 below.

Theorem 7.2.10. *The parametrix $\mathcal{P}_{b_k}$, defined in* (7.57)*, satisfies the above mentioned conditions 1 - 3.*

Proof. Indeed, according to (7.58), the jumps matrices of (7.57) coincide with that of $Z(z;\varkappa)$ inside $\mathcal{D}_{b_k}$ so we have proven the first requirement. According to Theorem 3.2.6, when z is on $\partial\mathcal{D}_{b_k}$ in sectors $(2)_\pm$,

$$\Psi_4^{-1}(\zeta_k;\varkappa)\Gamma_4(\zeta_k;\varkappa)e^{\varkappa g_4(\zeta_k)\sigma_3} = 1 + O(\varkappa^{-1}) \qquad \text{as} \quad \varkappa \to \infty \tag{7.59}$$

Then

$$M_{\mathcal{E}}^{-1}(z;\varkappa) = \mathcal{P}_k(z;\varkappa)\Psi^{-1}(z;\varkappa) = \Psi(z;\varkappa)[1+O(\varkappa^{-1})]\Psi^{-1}(z;\varkappa) = 1+O(\varkappa^{-1}) \tag{7.60}$$

since $\Psi^{\pm 1}(z;\varkappa)$ are bounded on $\partial\mathcal{D}_{b_k}$ when $\varkappa$ is restricted to the horizontal strip $-\pi/2 < \Im[\varkappa] < \pi/2$. The idea when z is in any of the remaining sectors is similar. Thus, we have proven the third requirement for the parametrix.

Finally, to prove the remaining second requirement we notice that

$$\Gamma_4(\zeta_k;\varkappa)e^{\varkappa g_4(\zeta_k)\sigma_3} = e^{\varkappa g_4(\infty)}Y_4(z;\varkappa) \text{ is an } L^2_{loc}$$

matrix valued function provided that $\varkappa \notin \mathbb{R}$. So, it is sufficient show that $\Psi(z;\varkappa)\Psi_4^{-1}(\zeta_k;\varkappa)$ is bounded on $\mathcal{D}_{b_k}$. Note that, according to (7.44), $\tilde{g}(z;\varkappa) - \tilde{g}_4(\zeta_k;\varkappa)$ does not have logarithmic singularity at $z = b_k$, so that $e^{\tilde{g}(z;\varkappa)-\tilde{g}_4(\zeta_k;\varkappa)}$ is bounded in a neighborhood of b_k. Taking into the account the fact that $\tilde{\Psi}^{\pm 1}$ from (7.53) is bounded in $\mathcal{D}_{b_k}$, we obtain

$$\Psi(z;\varkappa)\Psi_4^{-1}(\zeta_k;\varkappa) = e^{\tilde{g}(\infty;\varkappa)\sigma_3}\tilde{\Psi}(z,\varkappa)e^{-(\tilde{g}(z;\varkappa)-\tilde{g}_4(\zeta_k;\varkappa))\sigma_3}\left(\frac{\zeta_k+1}{\zeta_k-1}\right)^{-\frac{\sigma_1}{4}} e^{-\tilde{g}_4(\infty;\varkappa)\sigma_3} \tag{7.61}$$

is also bounded in $\mathcal{D}_{b_k}$. □

The parametrix when the double point b_k has $I_{L,b}$ subinterval on the left and $I_{U,b}$ subinterval on the right can be constructed in a similar manner and will be omitted here.

Lemma 7.2.11. *Let functions* $\hat{\phi}(z), \hat{\psi}(z)$ *be analytic in a disc* $\mathcal{D}$ *centered at the origin and let* $\phi(z) = \frac{\ln z}{i\pi} + \bar{\phi}(z)$, $\psi(z) = \frac{\ln z}{i\pi} + \bar{\psi}(z)$. *Then, there exist a function* $\zeta(z) = az(1 + y(z))$ *analytic in, perhaps, a smaller disk* $\tilde{\mathcal{D}} \subset \mathcal{D}$ *centered at* $z = 0$, *where* $a \neq 0$ *and* $y(0) = 0$, *such that*

$$\phi(\zeta(z)) = \psi(z). \tag{7.62}$$

Proof. Substituting $\zeta(z)$ in (7.62) and taking $a = e^{i\pi(\hat{\psi}(0) - \hat{\phi}(0))}$, we obtain equation

$$F(z, y) = \frac{1}{i\pi} \ln(1 + y) + \hat{\phi}(az(1 + y)) - \hat{\psi}(z) - \hat{\psi}(0) + \hat{\phi}(0) = 0, \tag{7.63}$$

which is true for $(z, y) = (0, 0)$. Since

$$\frac{\partial F}{\partial y} = \frac{1}{i\pi(1 + y)} + az\hat{\phi}'(az(1 + y)) \neq 0 \tag{7.64}$$

at $(z, y) = (0, 0)$, the conclusion follows from the Implicit Function Theorem. □

APPENDIX A: CONSTRUCTION OF $\Gamma(z;\lambda)$

Recall that the Hypergeometric ODE (see [7] 15.10.1) is

$$\eta(1-\eta)\frac{\mathrm{d}^2 w}{\mathrm{d}\eta^2} + (c-(a+b+1)\eta)\frac{\mathrm{d}w}{\mathrm{d}\eta} - abw = 0, \tag{A.1}$$

which has exactly three regular singular points at $\eta = 0, 1, \infty$. The idea is to choose parameters a, b, c so that the monodromy matrices of the fundamental matrix solution solution of the ODE will match (up to similarity transformation) the jump matrices of RHP 2.2.1. We orient the real axis of the $\eta-$plane as described in Figure A.1.

A.1 Solutions of ODE (A.1) near Regular Singular Points and Connection Formula

According to [7] 15.10.11 - 15.10.16, three pairs of linearly independent solutions of ODE (A.1) when $\eta = 0, 1, \infty$, respectively, are

$$h_0(\eta) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, \eta\right),\ s_0(\eta) = \eta^{1-c}{}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix} \,\middle|\, \eta\right), \tag{A.2}$$

$$h_1(\eta) = {}_2F_1\left(\begin{matrix} a, b \\ a+b+1-c \end{matrix} \,\middle|\, 1-\eta\right),\ s_1(\eta) = (1-\eta)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a, c-b \\ c-a-b+1 \end{matrix} \,\middle|\, 1-\eta\right), \tag{A.3}$$

$$h_\infty(\eta) = e^{a\pi i}\eta^{-a}{}_2F_1\left(\begin{matrix} a, a-c+1 \\ a-b+1 \end{matrix} \,\middle|\, \frac{1}{\eta}\right),\ s_\infty(\eta) = e^{b\pi i}\eta^{-b}{}_2F_1\left(\begin{matrix} b, b-c+1 \\ b-a+1 \end{matrix} \,\middle|\, \frac{1}{\eta}\right). \tag{A.4}$$

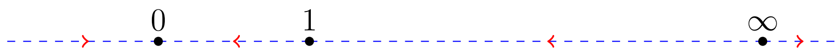

Figure A.1: Orientation of the real axis of the η-plane.

From [7] 15.10.7, we have that

$$W_\eta\left[h_\infty(\eta), s_\infty(\eta)\right] = e^{(a+b)\pi i}(a-b)\eta^{-c}(1-\eta)^{c-a-b-1}. \tag{A.5}$$

Kummer's 20 connection formula are listed in [7] 15.10.17 - 15.10.36. We will list only what is necessary in this construction. The connection between solutions at $\eta = 0$ and $\eta = \infty$ is (see [7] 15.10.19, 15.10.20, 15.10.25, 15.10.26)

$$\begin{bmatrix} h_0(\eta) & s_0(\eta) \end{bmatrix} = \begin{bmatrix} h_\infty(\eta) & s_\infty(\eta) \end{bmatrix} C_{\infty 0}, \tag{A.6}$$

$$\begin{bmatrix} h_\infty(\eta) & s_\infty(\eta) \end{bmatrix} = \begin{bmatrix} h_0(\eta) & s_0(\eta) \end{bmatrix} C_{0\infty}, \tag{A.7}$$

where

$$C_{\infty 0} = \begin{bmatrix} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} & e^{(1-c)\pi i}\frac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(1-a)\Gamma(b-c+1)} \\ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} & e^{(1-c)\pi i}\frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(1-b)\Gamma(a-c+1)} \end{bmatrix}, \tag{A.8}$$

$$C_{\infty 0}^{-1} = C_{0\infty} = \begin{bmatrix} \frac{\Gamma(1-c)\Gamma(a-b+1)}{\Gamma(a-c+1)\Gamma(1-b)} & \frac{\Gamma(1-c)\Gamma(b-a+1)}{\Gamma(b-c+1)\Gamma(1-a)} \\ e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(a-b+1)}{\Gamma(a)\Gamma(c-b)} & e^{(c-1)\pi i}\frac{\Gamma(c-1)\Gamma(b-a+1)}{\Gamma(b)\Gamma(c-a)} \end{bmatrix}. \tag{A.9}$$

The connection between solutions at $\eta = 1$ and $\eta = \infty$ is (see [7] 15.10.23, 15.10.24, 15.10.27, 15.10.28)

$$\begin{bmatrix} h_1(\eta) & s_1(\eta) \end{bmatrix} = \begin{bmatrix} h_\infty(\eta) & s_\infty(\eta) \end{bmatrix} C_{\infty 1}, \tag{A.10}$$

$$\begin{bmatrix} h_\infty(\eta) & s_\infty(\eta) \end{bmatrix} = \begin{bmatrix} h_1(\eta) & s_1(\eta) \end{bmatrix} C_{1\infty}, \tag{A.11}$$

where

$$C_{\infty 1} = \begin{bmatrix} e^{-a\pi i}\frac{\Gamma(a+b-c+1)\Gamma(b-a)}{\Gamma(b)\Gamma(b-c+1)} & e^{(b-c)\pi i}\frac{\Gamma(c-a-b+1)\Gamma(b-a)}{\Gamma(1-a)\Gamma(c-a)} \\ e^{-b\pi i}\frac{\Gamma(a+b-c+1)\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)} & e^{(a-c)\pi i}\frac{\Gamma(c-a-b+1)\Gamma(a-b)}{\Gamma(1-b)\Gamma(c-b)} \end{bmatrix}, \tag{A.12}$$

$$C_{\infty 1}^{-1} = C_{1\infty} = \begin{bmatrix} e^{a\pi i}\frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)} & e^{b\pi i}\frac{\Gamma(b-a+1)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(c-a)} \\ e^{(c-b)\pi i}\frac{\Gamma(a-b+1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(a-c+1)} & e^{(c-a)\pi i}\frac{\Gamma(b-a+1)\Gamma(a+b-c)}{\Gamma(b)\Gamma(b-c+1)} \end{bmatrix}. \tag{A.13}$$

A.2 Selection of Parameters a, b, c

Define

$$\hat{\Gamma}(\eta) := \eta^{\frac{c}{2}}(1-\eta)^{\frac{a+b-c+1}{2}} \begin{bmatrix} h_\infty(\eta) & s_\infty(\eta) \\ h'_\infty(\eta) & s'_\infty(\eta) \end{bmatrix}. \tag{A.14}$$

Notice that for any $\eta \in \mathbb{C}$

$$\det\left(\hat{\Gamma}(\eta)\right) = e^{(a+b)\pi i}(a-b) \tag{A.15}$$

according to (A.5). Our solution $\Gamma(z;\lambda)$ to RHP 2.2.1 has singular points at $z = b_L, 0, b_R$. Notice that the Möbius transform

$$\eta = M_1(z) := \frac{b_R(z-b_L)}{z(b_R-b_L)} \tag{A.16}$$

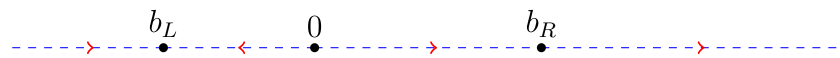

Figure A.2: Orientation of the real axis of the z-plane.

maps $b_L \to 0$, $b_R \to 1$, and $0 \to \infty$ where the orientation of the $z-$axis is described in Figure A.2. Thus we are interested in the matrix

$$\hat{\Gamma}\left(M_1(z)\right) = \left(\frac{b_R(z-b_L)}{|b_L|(z-b_R)}\right)^{\frac{c}{2}} \left(\frac{|b_L|(z-b_R)}{z(b_R-b_L)}\right)^{\frac{a+b+1}{2}} \begin{bmatrix} h_\infty\left(M_1(z)\right) & s_\infty\left(M_1(z)\right) \\ h'_\infty\left(M_1(z)\right) & s'_\infty\left(M_1(z)\right) \end{bmatrix}. \tag{A.17}$$

We need to determine parameters a, b, c such that $\hat{\Gamma}(M_1(z))$ is L^2_{loc} at $z = b_L, 0, b_R$, so we are interested in the bi-resonant case, which is

$$c \in \mathbb{Z}, \quad c - b - a \in \mathbb{Z}. \tag{A.18}$$

When $z = b_L$, to guarantee that $\hat{\Gamma}$ is L^2_{loc} we must have that (use the connection formula of section A.1 to easily inspect the local behavior)

$$\frac{c}{2} > -\frac{1}{2}, \quad -\frac{c}{2} > -\frac{1}{2}. \tag{A.19}$$

Since $c \in \mathbb{Z}$, it must be so that $c = 0$. Now for $z = b_R$, we must have that

$$\frac{a+b-c+1}{2} = \frac{r+1}{2} > -\frac{1}{2}, \tag{A.20}$$

$$\frac{1-a-b}{2} = \frac{-r-1}{2} > -\frac{1}{2} \tag{A.21}$$

where $a + b = r \in \mathbb{Z}$. Since $r \in \mathbb{Z}$, the only possibility is $r = -1$. So we have that $b = -1 - a$ and $c = 0$. Lastly, as $z \to 0$,

$$h_\infty\left(M_1(z)\right) = \mathrm{O}\left(z^a\right), \tag{A.22}$$

$$h'_\infty\left(M_1(z)\right) = \mathrm{O}\left(z^{a+1}\right), \tag{A.23}$$

$$s_\infty\left(M_1(z)\right) = -e^{-a\pi i}\left(\frac{-b_R b_L}{z(b_R - b_L)}\right)^{a+1} + \mathrm{O}\left(z^{-a}\right), \tag{A.24}$$

$$s'_\infty\left(M_1(z)\right) = -(a+1)e^{-a\pi i}\left(\frac{-b_R b_L}{z(b_R - b_L)}\right)^{a} + \mathrm{O}\left(z^{-a+1}\right), \tag{A.25}$$

so we see that it is not possible for $\hat{\Gamma}(M_1(z))$ to have L^2 behavior at $z = 0$. On the other hand, observe that

$$s_\infty\left(\frac{b_R(z-b_L)}{z(b_R-b_L)}\right) + \frac{b_L b_R}{z(b_R-b_L)(a+1)} s'_\infty\left(\frac{b_R(z-b_L)}{z(b_R-b_L)}\right) = \mathrm{O}\left(z^{-a}\right), \quad z \to 0. \tag{A.26}$$

Thus the matrix

$$\begin{bmatrix} 1 & \frac{b_L b_R}{z(b_R-b_L)(a+1)} \\ 0 & 1 \end{bmatrix} \hat{\Gamma}(M_1(z)) \tag{A.27}$$

is L^2_{loc} as $z \to 0$ provided that $|\Re(a)| < 1/2$. In the next section, we solve for a explicitly in terms of λ and the condition $|\Re(a)| < 1/2$ will be met provided that

$\lambda \notin [-1/2, 1/2]$, see Appendix B.

A.3 Monodromy

The monodromy matrices of $\hat{\Gamma}(z)$ about the singular points $z = 0, 1, \infty$ are

$$M_0 = C_{\infty 0} e^{i\pi c\sigma_3} C_{0\infty}, \quad M_1 = C_{\infty 1} e^{i\pi(a+b-c+1)\sigma_3} C_{1\infty}, \quad M_\infty = e^{i\pi(b-a-1)\sigma_3}, \quad \text{(A.28)}$$

where $C_{\infty 0}, C_{0\infty}, C_{\infty 1}, C_{1\infty}$ are defined in (A.8), (A.9), (A.12), (A.13), respectively. With some effort it can be shown that

$$M_0 = \begin{bmatrix} \cos \pi c \left(1 - 2i \frac{\sin \pi a \sin \pi b}{\sin \pi (b-a)}\right) & \frac{2\pi i \Gamma(b-a)\Gamma(b-a+1)}{\Gamma(b)\Gamma(c-a)\Gamma(b-c+1)\Gamma(1-a)} \\ \frac{2\pi i \Gamma(a-b)\Gamma(a-b+1)}{\Gamma(a)\Gamma(c-b)\Gamma(a-c+1)\Gamma(1-b)} & \cos \pi c \left(1 + 2i \frac{\sin \pi a \sin \pi b}{\sin \pi (b-a)}\right) \end{bmatrix} + \sin \pi c \frac{\sin \pi (a+b)}{\sin \pi (b-a)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{(A.29)}$$

$$M_1 = e^{\frac{i\pi}{2}(b-a)\sigma_3} M_0 \big|_{c \to a+b-c+1} e^{-\frac{i\pi}{2}(b-a)\sigma_3}. \quad \text{(A.30)}$$

From the previous section, we take $c = 0$ and $b = -1 - a$. It is important to note that the connection matrices $C_{\infty 0}, C_{0\infty}, C_{\infty 1}, C_{1\infty}$ are singular when $c = 0$ and/or $b = -1 - a$, but we can see that M_0, M_1 are not. Taking $c = 0$ and $b = -a - 1$, we obtain

$$M_0 = \begin{bmatrix} 1 - i\tan(a\pi) & \frac{\tan^2(a\pi)\Gamma(a)\Gamma(a+2)}{i4^{2a+1}\Gamma(a+\frac{1}{2})\Gamma(a+\frac{3}{2})} \\ \frac{i4^{2a+1}\Gamma(a+\frac{1}{2})\Gamma(a+\frac{3}{2})}{\Gamma(a)\Gamma(a+2)} & 1 + i\tan(a\pi) \end{bmatrix}, \tag{A.31}$$

$$M_1 = \sigma_3 e^{-i\pi a\sigma_3} M_0 e^{i\pi a\sigma_3}\sigma_3, \tag{A.32}$$

$$M_\infty = e^{-2\pi i a\sigma_3}. \tag{A.33}$$

Let

$$Q(\lambda) = \begin{bmatrix} -\tan(a\pi) & 0 \\ 0 & 4^{2a+1}e^{a\pi i}\frac{\Gamma(a+3/2)\Gamma(a+1/2)}{\Gamma(a)\Gamma(a+2)} \end{bmatrix} \begin{bmatrix} 1 & e^{a\pi i} \\ -e^{a\pi i} & 1 \end{bmatrix} \tag{A.34}$$

so then we have

$$Q^{-1}M_0Q = \begin{bmatrix} 1 & 0 \\ -\frac{e^{2\pi i a}-1}{e^{a\pi i}} & 1 \end{bmatrix}, \quad Q^{-1}M_1Q = \begin{bmatrix} 1 & -\frac{e^{2\pi i a}-1}{e^{a\pi i}} \\ 0 & 1 \end{bmatrix}, \tag{A.35}$$

$$Q^{-1}M_\infty Q = \begin{bmatrix} \frac{e^{4\pi i a}-e^{2\pi i a}+1}{e^{2\pi i a}} & \frac{1-e^{2\pi i a}}{e^{a\pi i}} \\ \frac{1-e^{2\pi i a}}{e^{a\pi i}} & 1 \end{bmatrix} \tag{A.36}$$

The match requires

$$\frac{e^{2\pi i a}-1}{e^{a\pi i}} = \frac{i}{\lambda} \tag{A.37}$$

which implies

$$a(\lambda) = \frac{1}{i\pi}\ln\left(\frac{i+\sqrt{4\lambda^2-1}}{2\lambda}\right). \tag{A.38}$$

In Appendix B we have listed all the important properties of $a(\lambda)$.

A.4 RHP 2.2.1 Solution

We will now construct $\Gamma(z;\lambda)$ so that it is the solution of RHP 2.2.1. Notice that the matrix

$$\begin{bmatrix} 1 & \frac{b_L b_R}{z(b_R-b_L)(a+1)} \\ 0 & 1 \end{bmatrix} \hat{\Gamma}(M_1(z))Q(\lambda)\sigma_2, \tag{A.39}$$

where $\hat{\Gamma}, M_1, a$ are defined in (A.14), (A.16), (A.38), respectively, satisfies the following properties:

- L^2 behavior at $z = 0$, provided $\lambda \notin [-1/2, 1/2]$, due to (A.27) and properties of $a(\lambda)$ (see Appendix B),
- jump matrix $\begin{bmatrix} 1 & -\frac{i}{\lambda} \\ 0 & 1 \end{bmatrix}$ for $z \in (b_L, 0)$ with positive orientation, see (A.35) and Figure A.2 for orientation,
- jump matrix $\begin{bmatrix} 1 & 0 \\ \frac{i}{\lambda} & 1 \end{bmatrix}$ for $z \in (0, b_R)$ with positive orientation, see (A.35),
- column-wise behavior $\begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(\ln(z - b_L)) \end{bmatrix}$ as $z \to b_L$, because the first column has no jump on $(b_L, 0)$ and is analytic for $z \notin [b_L, b_R]$ so the first column is $\mathrm{O}(1)$ as $z \to b_L$. The Sokhotski-Plemelj formula can be used to inspect the behavior of the second column.
- column-wise behavior $\begin{bmatrix} \mathrm{O}(\ln(z - b_R)) & \mathrm{O}(1) \end{bmatrix}$ as $z \to b_R$, same idea as above,
- behavior $\hat{\Gamma}(M_1(\infty))Q(\lambda)\sigma_2(I + \mathrm{O}(z^{-1}))$ as $z \to \infty$,
- analytic for $z \in \overline{\mathbb{C}} \setminus [b_L, b_R]$, due to properties of hypergeometric functions.

Thus, we conclude that the matrix

$$\Gamma(z;\lambda) := \sigma_2 Q^{-1}(\lambda)\hat{\Gamma}^{-1}(M_1(\infty))\begin{bmatrix}1 & \frac{b_L b_R}{z(b_R-b_L)(a+1)}\\ 0 & 1\end{bmatrix}\hat{\Gamma}(M_1(z))Q(\lambda)\sigma_2 \tag{A.40}$$

is a solution of RHP 2.2.1 provided that $\lambda \notin [-1/2, 1/2]$.

APPENDIX B: DEFINITION AND PROPERTIES OF $a(\lambda)$

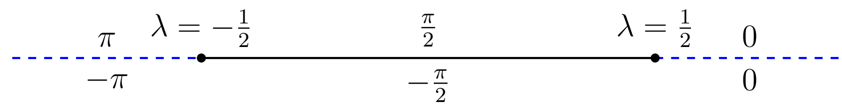

Figure B.1: Branch cut and argument of $\sqrt{4\lambda^2 - 1}$ for $\lambda \in \mathbb{R}$.

Define function

$$a(\lambda) = \frac{1}{i\pi} \ln \left(\frac{i + \sqrt{4\lambda^2 - 1}}{2\lambda} \right) \tag{B.1}$$

where the branch cut of $\sqrt{4\lambda^2 - 1}$ is from $[-1/2, 1/2]$ and the principle value of the logarithm is taken $\left(-\pi < \arg\left(\frac{i+\sqrt{4\lambda^2-1}}{2\lambda}\right) < \pi\right)$. In Figure B.1, the branch cut is the black line, the blue dashed line is the remaining real axis, and the argument of $\sqrt{4\lambda^2 - 1}$ is displayed for each shore of the real axis.

The following proposition lists all relevant properties of $a(\lambda)$, none of which are difficult to prove.

Proposition B.0.1. *The function $a(\lambda)$ has the following properties:*

1. *$a(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus [-1/2, 1/2]$.*
2. *$a(\lambda)$ is Schwarz symmetric.*
3. *$a_+(\lambda) + a_-(\lambda) = 1$ for $\lambda \in \left(0, \frac{1}{2}\right)$.*
4. *$a_+(\lambda) + a_-(\lambda) = -1$ for $\lambda \in \left(-\frac{1}{2}, 0\right)$.*
5. *$\Re[a_\pm(\lambda)] = \frac{1}{2}$ for $\lambda \in (0, 1/2)$.*

6. $\Re[a_\pm(\lambda)] = -\frac{1}{2}$ *for* $\lambda \in (-1/2, 0)$.

7. $-\frac{1}{2} < \Re\left[a(\lambda)\right] < \frac{1}{2}$ *for* $\lambda \in \mathbb{C} \setminus \left[-\frac{1}{2}, \frac{1}{2}\right]$.

8. *For* $\lambda \in (-1/2, 1/2)$, $\Im[a_+(\lambda)] \le 0$, $\Im[a_-(\lambda)] \ge 0$, *and* $\Im[a_+(\lambda)] = -\Im[a_-(\lambda)]$.

9. *If* $\lambda \to 0$, *then*

$$a(\lambda) = \begin{cases} -\frac{1}{i\pi}\ln(\lambda) + \frac{1}{2} + \mathrm{O}\left(\lambda^2\right) = \frac{\varkappa}{i\pi} + \frac{1}{2} + \mathrm{O}\left(e^{-2\varkappa}\right), & \Im\lambda \ge 0, \\ \frac{1}{i\pi}\ln(\lambda) + \frac{1}{2} + \mathrm{O}\left(\lambda^2\right) = -\frac{\varkappa}{i\pi} + \frac{1}{2} + \mathrm{O}\left(e^{-2\varkappa}\right), & \Im\lambda \le 0, \end{cases} \tag{B.2}$$

   where $\varkappa = -\ln\lambda$.

10. *As* $\lambda \to 0$,

$$e^{i\pi a(\lambda)} = \begin{cases} ie^{\varkappa}(1 + \mathrm{O}\left(e^{-2\varkappa}\right)), & \Im\lambda \ge 0, \\ ie^{-\varkappa}(1 + \mathrm{O}\left(e^{-2\varkappa}\right)), & \Im\lambda \le 0, \end{cases} \tag{B.3}$$

   where $\varkappa = -\ln\lambda$.

APPENDIX C: PROPERTIES OF $d_L(z;\lambda), d_R(z;\lambda)$

In this Appendix we compute and list the important properties of the functions $d_L(z;\lambda)$ and $d_R(z;\lambda)$. Recall that (from eq. (4.18))

$$d_R(z;\lambda) := \alpha(\lambda)h'_\infty\left(\frac{b_R(z-b_L)}{z(b_R-b_L)}\right) + \beta(\lambda)s'_\infty\left(\frac{b_R(z-b_L)}{z(b_R-b_L)}\right), \tag{C.1}$$

$$d_L(z;\lambda) := -e^{a\pi i}\alpha(\lambda)h'_\infty\left(\frac{b_R(z-b_L)}{z(b_R-b_L)}\right) + e^{-a\pi i}\beta(\lambda)s'_\infty\left(\frac{b_R(z-b_L)}{z(b_R-b_L)}\right), \tag{C.2}$$

where

$$\alpha(\lambda) := \frac{e^{-a\pi i}\tan(a\pi)\Gamma(a)}{4^{a+1}\Gamma(a+3/2)}, \quad \beta(\lambda) := \frac{4^a e^{a\pi i}\Gamma(a+1/2)}{\Gamma(a+2)} \tag{C.3}$$

and h'_∞, s'_∞ are defined in (2.13), (2.14).

Proposition C.0.1. *For $\lambda \in (-1/2, 0)$,*

1. $d_R\left(M_2(x);\lambda\right) = d_L(x;\lambda)$, *where* $M_2(x) = \frac{b_R b_L x}{x(b_R+b_L)-b_R b_L}$,

2. $d_L(z;\lambda)$ *and* $d_R(z;\lambda)$ *are single-valued in* λ,

3. $d_L(z;\lambda)$ *is analytic for* $z \in \overline{\mathbb{C}} \setminus [0, b_R]$, $\overline{d_L(z;\lambda)} = d_L(\overline{z};\overline{\lambda})$ *for* $z \in \overline{\mathbb{C}}$, *and*

$$d_L(z_+;\lambda) - d_L(z_-;\lambda) = \frac{i}{\lambda}d_R(z;\lambda), \quad \textit{for } z \in (0, b_R), \tag{C.4}$$

4. $d_R(z;\lambda)$ *is analytic for* $z \in \overline{\mathbb{C}} \setminus [b_L, 0]$, $\overline{d_R(z;\lambda)} = d_R(\overline{z};\overline{\lambda})$ *for* $z \in \overline{\mathbb{C}}$, *and*

$$d_R(z_+;\lambda) - d_R(z_-;\lambda) = -\frac{i}{\lambda}d_L(z;\lambda), \quad \textit{for } z \in (b_L, 0), \tag{C.5}$$

5. *We have the SVD system*

$$\mathcal{H}_R\left[\frac{d_R(y;\lambda)}{y}\right](x) = 2\lambda\frac{d_L(x;\lambda)}{x}, \quad \mathcal{H}_L\left[\frac{d_L(x;\lambda)}{x}\right](y) = 2\lambda\frac{d_R(y;\lambda)}{y} \tag{C.6}$$

where $x \in (b_L, 0)$ *and* $y \in (0, b_R)$.

6. $\displaystyle\int_0^{b_R} \frac{d_R(x;\lambda)}{x}\,dx = 2\pi\lambda d_L(\infty;\lambda), \int_{b_L}^{0} \frac{d_L(x;\lambda)}{x}\,dx = -2\pi\lambda d_R(\infty;\lambda)$

To obtain the corresponding identities for $D_R(z;\lambda) := d_R(z, -|\lambda|/2)$ *and* $D_L(z;\lambda) := d_L(z, -|\lambda|/2)$ *for* $\lambda \in (-1, 1)$*, replace* λ *with* $-|\lambda|/2$.

Proof. 1. Recall from Remark 4.1.2 that

$$M_1(x) := \frac{b_R(x - b_L)}{x(b_R - b_L)}, \tag{C.7}$$

$$M_2(x) := \frac{b_R b_L x}{x(b_R + b_L) - b_R b_L}, \tag{C.8}$$

$$M_3(x) := M_1(M_2(x)) = \frac{-b_L(x - b_R)}{x(b_R - b_L)}. \tag{C.9}$$

It is easy to show that

$$1 - M_3(x) = M_1(x), \quad 1 - \frac{1}{M_3(x)} = \frac{-M_1(x)}{M_3(x)}. \tag{C.10}$$

From [1] 15.5.7, 15.5.13, (taking $a := a_-(\lambda)$ and $-1 = e^{-i\pi}$)

$$ {}_2F_1\left(\begin{matrix} a+1, a+1 \\ 2a+2 \end{matrix} \middle| \frac{1}{M_3(x)} \right) = \left(1 - \frac{1}{M_3(x)}\right)^{-a-1} {}_2F_1\left(\begin{matrix} a+1, a+1 \\ 2a+2 \end{matrix} \middle| \frac{1}{1 - M_3(x)} \right) \tag{C.11} $$

$$ = \left(\frac{-M_1(x)}{M_3(x)}\right)^{-a-1} {}_2F_1\left(\begin{matrix} a+1, a+1 \\ 2a+2 \end{matrix} \middle| \frac{1}{M_1(x)} \right) \tag{C.12} $$

$$ = -e^{a\pi i} \left(\frac{M_1(x)}{M_3(x)}\right)^{-a-1} {}_2F_1\left(\begin{matrix} a+1, a+1 \\ 2a+2 \end{matrix} \middle| \frac{1}{M_1(x)} \right) \tag{C.13} $$

Thus we have

$$ h'_\infty(M_3(x)) = -ae^{a\pi i} M_3(x)^{-a-1} {}_2F_1\left(\begin{matrix} a+1, a+1 \\ 2a+2 \end{matrix} \middle| \frac{1}{M_3(x)} \right) = -e^{a\pi i} h'_\infty(M_1(x)) \tag{C.14} $$

and taking $a \to -a-1$ in the identity above yields

$$ s'_\infty(M_3(x)) = e^{-a\pi i} s'_\infty(M_1(x)), \tag{C.15} $$

because $h_\infty(\eta)|_{a\to -a-1} = s_\infty(\eta)$. Since

$$ d_R(x;\lambda) = \alpha h'_\infty(M_1(x)) + \beta s'_\infty(M_1(x)) \tag{C.16} $$

then we have that

$$d_R(M_2(x),\lambda) = \alpha h'_\infty(M_1(M_2(x))) + \beta s'_\infty(M_1(M_2(x))) \tag{C.17}$$

$$= \alpha h'_\infty(M_3(x)) + \beta s'_\infty(M_3(x)) \tag{C.18}$$

$$= -e^{a\pi i}\alpha h'_\infty(M_1(x)) + e^{-a\pi i}\beta s'_\infty(M_1(x)) \tag{C.19}$$

$$= d_L(x;\lambda). \tag{C.20}$$

2. Notice that $h_\infty(z)\big|_{a\to -a-1} = s_\infty(z)$ so we have $h'_\infty(z,\lambda_+) = s'_\infty(z,\lambda_-)$. To compute the jumps of the coefficients α, β, we use [7] 5.5.3.

$$\alpha_+(\lambda) = \frac{\tan(a_+\pi)\Gamma(a_+)}{e^{a_+\pi i}4^{a_++1}\Gamma(a_+ + 3/2)} \tag{C.21}$$

$$= \frac{-e^{a_-\pi i}\tan(\pi(-a_- - 1))\Gamma(-a_- - 1)}{4^{-a_-}\Gamma(1/2 - a_-)} \tag{C.22}$$

$$= \frac{4^{a_-}e^{a_-\pi i}\tan(a_-\pi)\Gamma(-a_- - 1)}{\Gamma(1/2 - a_-)} \tag{C.23}$$

$$= \frac{4^{a_-}e^{a_-\pi i}\Gamma(a_- + 1/2)}{\Gamma(a_- + 2)} \tag{C.24}$$

$$= \beta_-(\lambda) \tag{C.25}$$

and, similarly, $\beta_+(\lambda) = \alpha_-(\lambda)$. This can now be used to prove the statement.

3. Recall, from the proof of Theorem 4.2.4, that $d_L(z;\lambda), d_R(z;\lambda)$ was defined in terms of the $(2,1), (2,2)$ element, respectively, of the matrix

$$M(z,\lambda) = \hat{\Gamma}(M_1(z),\lambda)Q(\lambda)\sigma_2. \tag{C.26}$$

Using the definition of $\Gamma(z;\lambda)$, see (2.17), some simple algebra shows that

$$M(z,\lambda) = \begin{bmatrix} 1 & \frac{-b_L b_R}{z(b_R - b_L)(a+1)} \\ 0 & 1 \end{bmatrix} \hat{\Gamma}^{-1}(M_1(\infty)) Q \sigma_2 \Gamma(z;\lambda). \tag{C.27}$$

Since $\Gamma(z;\lambda)$ is a solution of RHP 2.2.1, we know $M(z,\lambda)$ is analytic for $z \in \overline{\mathbb{C}} \setminus [b_L, b_R]$ and

$$M_+ = M_- \begin{bmatrix} 1 & -\frac{i}{\lambda} \\ 0 & 1 \end{bmatrix} \; z \in (b_L, 0), \qquad M_+ = M_- \begin{bmatrix} 1 & 0 \\ \frac{i}{\lambda} & 1 \end{bmatrix} \; z \in (0, b_R), \tag{C.28}$$

which immediately gives the jumps and analyticity of both d_L, d_R. For the symmetry, notice that

$$\begin{aligned} d_R(z;\lambda) = &\frac{-a\tan(a\pi)\Gamma(a)}{4^{a+1}\Gamma(a+\frac{3}{2})} M_1(z)^{-a-1} {}_2F_1\left(\begin{matrix} a+1, a+1 \\ 2a+2 \end{matrix} \,\middle|\, \frac{1}{M_1(z)} \right) \\ &- \frac{(a+1)4^a \Gamma(a+\frac{1}{2})}{\Gamma(a+2)} M_1(z)^a {}_2F_1\left(\begin{matrix} -a, -a \\ -2a \end{matrix} \,\middle|\, \frac{1}{M_1(z)} \right), \end{aligned} \tag{C.29}$$

and $a = a(\lambda), M_1(z)$ are Schwarz symmetric. Thus $\overline{d_R(z;\lambda)} = d_R(\overline{z};\overline{\lambda})$ and the symmetry of $d_L(z;\lambda)$ follows from the relation $d_R(M_2(x);\lambda) = d_L(x;\lambda)$.

4. This was proven above.

5. Let γ_R be the circle with center and radius of $b_R/2$ with negative orientation.

$$\mathcal{H}_R\left[\frac{d_R(y;\lambda)}{y}\right](x) = \frac{2\lambda}{2\pi i}\int_0^{b_R}\frac{\frac{i}{\lambda}d_R(y;\lambda)}{y(y-x)}\,dy \tag{C.30}$$

$$= \frac{2\lambda}{2\pi i}\int_0^{b_R}\frac{\Delta_y d_L(y;\lambda)}{y(y-x)}\,dy \tag{C.31}$$

$$= \frac{2\lambda}{2\pi i}\int_{\gamma_R}\frac{d_L(y;\lambda)}{y(y-x)}\,dy \tag{C.32}$$

$$= 2\lambda d_L(x;\lambda) \tag{C.33}$$

where we have deformed γ_R through $z=\infty$ and into the circle of center and radius $b_L/2$ with positive orientation and then apply residue theorem. The other computation is nearly identical.

6. The idea is similar to that of the last proof;

$$\int_0^{b_R}\frac{d_R(x;\lambda)}{x}\,dx = \frac{2\pi\lambda}{2\pi i}\int_0^{b_R}\frac{\frac{i}{\lambda}d_R(x;\lambda)}{x}\,dx = \frac{2\pi\lambda}{2\pi i}\int_{\gamma_R}\frac{d_L(x;\lambda)}{x}\,dx = 2\pi\lambda d_L(\infty;\lambda), \tag{C.34}$$

and the remaining identity is proven analogously.

□

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