MULTI-LEVEL OPTIMIZATION AND APPLICATIONS WITH NON-TRADITIONAL GAME THEORY

by

GUANXIANG YUN
M.S. University of Florida, 2014
B.S. Peking University, 2010

A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Industrial Engineering and Management Systems
in the College of Engineering and Computer Science
at the University of Central Florida
Orlando, Florida

Fall Term
2019

Major Professor: Qipeng Zheng

c 2019 Guanxiang Yun

ABSTRACT

We study multi-level optimization problem on energy system, transportation system and information network. We use the concept of boundedly rational user equilibrium (BRUE) to predict the behaviour of users in systems. By using multi-level optimization method with BRUE, we can help to operate the system work in a more efficient way. Based on the introducing of model with BRUE constraints, it will lead to the uncertainty to the optimization model. We generate the robust optimization as the multi-level optimization model to consider for the pessimistic condition with uncertainty. This dissertation mainly includes four projects. Three of them use the pricing strategy as the first level optimization decision variable. In general, our models' first level's decision variables are the measures that we can control, but the second level's decision variables are users behaviours that can only be restricted within BRUE with uncertainty.

Keywords: Boundedly Rational, Robust Optimization, Non-linear Programming, Linear Programming

ACKNOWLEDGMENTS

First, I would like to present my deepest acknowledgment to my advisor, Dr. Qipeng P. Zheng and co-advisor Dr. Vladimir Boginski for their professional advising, education and kindness. I am also grateful to all of my group colleagues.

I would like also to express my thanks to my committee members: Dr. Waldemar Karwowski and Dr. Jiongmin Yong. for their great efforts and supports in my qualify exam and candidacy exam.

I would also like to thanks to my collaborator Dr. Eduardo Pasiliao and Dr. Lihui Bai.

Finally, I would like to thanks to my family, My father Gang Yun, my mother Yurong Qiao for their encourages and supports.

TABLE OF CONTENTS

LIST OF FIGURES ix

LIST OF TABLES xi

CHAPTER 1: INTRODUCTION 1

CHAPTER 2: ROBUST PRICING AND BOUNDEDLY RATIONAL USEREQUILIBRIUM FOR MARKET STUDIES OF RESIDENTIALELECTRICITY CONSUMERS IN SMART GRID 6

Introduction 7

Mathematical Models 11

Two Energy Consumption Models 11

Two Boundedly Rational User Equilibrium Models 13

Pricing Strategies in Boundedly Rational User Equilibrium for Electricity Consumer Market 15

Mathematical Property and Sensitivity Analysis Based on the Models 17

Hidden Convexity 17

Algorithms for BRUE and Pricing Models 22

Cutting Plane with Penalty Method 22

Lagrangian Dual Cutting Plane Method 24

Computational Experiments and Results 25

Results under Different Data Sets 26

A Simple Example With Four Time Period 26

Results for Instances with 24 Time Periods 29

The Results in 24 Hours for Multiple Users 32

Sensitivity Analysis 34

Impacts of p and r to the System 34

The relative Difference of the Total Cost With or Without b 36

The Influence of the Change for the Customer Demand 38

Compare of the Game Theory Model and BRUE with Pricing Strategy 43

CHAPTER 3: ROBUST OPTIMIZATION WITH SURPLUS PRICE TRANSPORTATION UNDER BOUNDEDLY RATIONALITY USER EQUILIBRIUM 45

Introduction 46

BRUE For the Static Network Models 50

Counter part for the Difference Between Path Flow And Link-based Flow Distribution 63

Algorithm 65

Result 71

Simple Example 71

Check the Theorems 74

Nine Nodes Example 75

Sioux Fall System 76

CHAPTER 4: INFORMATION NETWORK CASCADING ANDNETWORK RE-CONSTRUCTION WITH BOUNDEDLY RATIONALUSER BEHAVIORS 82

Introduction 83

Math Model Formulation 87

Linear Threshold Model for Information Cascading 87

BRUE Model 89

Pessimistic Condition 89

Optimistic Condition 93

BRUE Model with Budget Restriction 94

Algorithm 95

Computation Result 97

Data Set 97

Result of Connection Utility for Information provider 99

Post plan of information provider 99

In uence of BRUE Coef cientr 100

Compare BRUE and Game Theory 100

Result by Using Large Neighbour Search Method 103

Result Without Budget and Cost Penalty 103

Result With Budget and Cost Penalty 104

CHAPTER 5: RESIDENTS ENERGY TRANSACTION WITHBLOCKCHAIN TECHNOLOGY 106

Introduction 106

Mathematical Model 107

Theorem 110

Algorithm 113

Computational Results 114

CHAPTER 6: CONCLUSION 119

LIST OF REFERENCES 123

LIST OF FIGURES

2.1 Upper Bound and Lower Bound of Model (PR-P) by k Increasing 28

2.2 Upper Bound and Lower Bound by Iteration in 24 Hours Model 31

2.3 Residual For BRUE Constraints with Tow Users 32

2.4 Residual For BRUE Constraints with 10 Users 34

2.5 Sensitivity Analysis 36

2.6 The Relative Improvement of the Best and Worst Condition by Using Pricing Strategy 37

2.7 The In uence of Optimal Utility W by Change of Demand 43

2.8 The In uence of Total Cost by Change of Demand 43

2.9 The Difference of BRUE and Game Theory Model 44

3.1 Parameters For the Four Nodes Example 64

3.2 Four Nodes Flow Distribution 64

3.3 The nine nodes network 76

3.4 The Sioux Fall Network 77

3.5 The Sioux Fall Network 2 79

3.6 The Sioux Fall Network 3 80

4.1 Cascading for the information and the nal connection network 88

4.2 Relation of Information Utility to Times User Receive this information 92

4.3 Utility for Different Post Plan 100

4.4 In uence of BRUE coef cient 101

4.5 Compare of BRUE and Game Theory Model 1 102

4.6 Compare of BRUE and Game Theory Model 2 102

4.7 Compare of BRUE and Game Theory Model 3 103

4.8 Large Neighbour hood Search without budget 104

4.10 Large Neighbourhood Search with budget with Cost Penalty1 105

4.9 Result of Large Neighbourhood Search without budget with Cost Penalty . . 105

5.1 The Demand and Generation Amount for each user in different time periods . 115

5.2 Pro t of Central Company in Minimization and Maximization Condition for Different Pricing Strategy 116

5.3 Pro t of Central Company in Minimization and Maximization Condition for DifferentPricing Strategy 117

5.4 In uence to Price with Different Generation and Demand Amount 118

LIST OF TABLES

2.1 Parameters of Simple Example 27

2.2 Results of Simple Example 27

2.3 Parameters of 24 Hours Example 29

2.4 The Results of 24 Hours Example 30

2.5 The Results of 24 Hours for Multiple Users Example 33

2.6 The De nition of Variables for the Simple Example Used for Sensitivity Analysis 38

3.1 The Results of Four Nodes Example 74

3.2 The Results of Four Nodes Example 75

3.3 The Results of Nine Nodes Example 76

3.4 The Results for Sioux Fall network with 4 OD pairs 77

3.5 The Results for Sioux Fall Network With 4 OD Pairs By Add Constraints . . 78

3.6 The Results for Sioux Fall Network With 4 OD Pairs By Add Constraints With Different Accuracy 80

3.7 The Results for Sioux Fall network with multiple OD pairs 81

4.1 Data Structure 98

CHAPTER 1: INTRODUCTION

Mathematical optimization is a widely used method in multiple areas. The general formulation of optimization problem is shown as follows,

(GO):

$$\min_{x} \quad f(x) \tag{1.1}$$

$$\text{s.t.} \quad g(x) \quad 0; \tag{1.2}$$

Where x is the decision variable in $\mathbb{R}^n$. $f(x)$ is the relative objective function respect to x. $g(x)$ is the constraints with m dimension.

In this proposal, we also use the robust optimization model, which is a special case of the general optimization problem. It has two levels for the optimization problem. The general formulation of robust optimization is shown as follows,

(RO-GO):

$$\max_{b \in B} \min_{x} \quad f(b;x) \tag{1.3}$$

$$\text{s.t.} \quad g(b;x) \quad 0; \tag{1.4}$$

$$x \in X; \tag{1.5}$$

For this two-level optimize model b is the first level decision variable. B is the feasible region for b. X is the feasible region for x.

The application fields of mathematical optimization include but not limit to the energy system, transportation system and information network system. Four projects in this proposal are also in these three areas.

The method robust optimization can be traced back to 1950s to the decision theory which use for worst case analysis under uncertainty. Over the years, especially the recent two decades after the work of Ben-Tal [11, 12], it is widely applied in the areas includes operations research, statistics, control theory, finance, logistics and computer science.

Stochastic optimization is also another method to deal with the problem with uncertainty. But it can only be used when the probability of each scenario is known. But under many cases in the real world we can not know these probability. Then the study of robust optimization becomes extremely important.

In this dissertation, the concept of boundedly rational user equilibrium (BRUE) is introduced to estimate the users' behaviour. BRUE model is proposed by Simon in year 1957 [77, 80, 81, 79], which means for one individual, when the difference of the utilities of different options that the individual can choose are below a level, this individual will regard the utilities of such different options as the same. He or she may choose any options within that level as his or her final decisions. The following is a mathematical constraint for BURE. We suppose one user in the system have multiple choices, for choice i it has utility $U(i)$. With out loss of generality, we can set the choice i has the optimal utility value. Then BRUE tells us that for any choice i has the following property will be deemed as the possible future choice for this user.

$$U(i) \quad r \quad U(i\ ): \tag{1.6}$$

Where r is called the bounded rationality coef cient. And we must have α 1 because of the optimality of the choice i .

Here in our model of the transportation system, the utility includes the travel time and the surplus price. The concept of boundedly rational can be used in many elds, such as the energy system [96], psychology [44], military [69], transportation [58, 24, 56]and so on [70, 58, 36, 27].

The structure of this dissertation is that the rst chapter is the introduction part of the whole dissertation. The second to ve chapter is the work for four projects. And the six chapter is the conclusion.

The rst project studies a new time-of-day pricing framework to reduce the Peak-To-Average ratio in residential electricity usage while considering consumers' boundedly rational behaviors. Instead of always choosing the optimal electricity consumption pro les as described in traditional game theory models, consumers tends to simply pick solutions that are acceptable in terms of cost or preference in reality. To address this, this paper proposes a Boundedly Rational User Equilibrium (BRUE) to model residential electricity consumption in smart grid with advanced metering infrastructure. Upon the BRUE models, this paper studies two pricing strategies, i.e., optimistic and robust, to minimize the total system cost, via bilevel optimization models. In order to address the computational dif culty caused by the nonconexity of the lower level problem, this paper studies three cutting plane methods, i.e., direct cutting-plane method, penalty-based cutting-plane method and Lagrangian-dual-based cutting plane method. Due to the property of hidden convexity, the Lagrangian dual method outperforms the other two methods. Numerical experiments show that by introducing the time-of-day pricing, it can decrease the total cost of the system. The results also suggest that the more users with exible preferred time windows for electricity usage, the lower total cost the system can achieve by pricing.

The second project gives a model for the static transportation path based problem. We suppose

that all the users in the system will obey the bounded rational principle. In real world instances, people will feel just ne even if they do not reach the best utility they can achieve but only attain a certain level. We propose totally four conditions for our static models with two of them having the pricing strategy. By using the pricing strategy, the total time cost of the system can be reduced. And we also have the robust optimization model by using the pricing strategy, we solve it by using the column and constraint generation method. For transportation path based problem, it is also a large scale problem because the number of pathes have the exponentially relation with the number of acres in the system.

The third project talks about the social media platforms, which have become very popular for people to share information and make new friends. By expanding their own connections to new users in the social network, commercial users can greatly increase their in uences leading to much higher pro ts. In order to optimize a information provider's network connections, we establish a mathematical model to simulate behaviours of other users within the information provider's network. The behaviours include the information repost as well as following/unfollowing other users. We apply the linear threshold propagation model to determine the action of repost. In addition, the action of following or unfollowing other users is restricted by boundedly rational user equilibrium (BRUE). The topology of the network can change depending on the information provider's plan of posting information. The connections for the information provider, therefore, may change as well. We establish a three-level optimization model for the information provider. The rst level is to maximize the information provider's connections. The second level is to simulate users' behaviours under BRUE. The third level is to maximize the other users' utility that need to be used in the second level. We solve this problem by using exact algorithms for a small-scale synthetic network. For a large-scale problem, we tackle it by using the heuristic large neighbourhood search algorithms. In this paper, we discuss possible reasons why the BRUE model may be a more accurate simulation of users' actions compared to game theory. We compare results from the BRUE

model to game theory, and nd that the BRUE model performs signi cantly better than game theory.

The fourth project is research about blockchain technology used in energy transaction. BlockChain technology guarantees the safety of transactions between two users who do not know each other without any central institute. We apply this characteristic of blockchain to power system. It can help prosumers within the power networks transact electricity and money. The users who have redundant amount of power can sell it to other users who need power with the price lower than the power company. It is double win for both these two users. This paper establish a mathematical game theory model for user's decision to buy or sell power in the system. We can see what is the in uence to the price of the central company by introducing the blockchain to the power transaction system. In addition, we generate the simulation with hyperledge to see its indluence to the price. We use the KKT condition algorithm to solve the multiple level game theory model. We nd the price of the power can decrease dramatically by applying the transaction among prosumers unless the amounts of generation power from prosumers are much more less than their demands.

CHAPTER 2: ROBUST PRICING AND BOUNDEDLY RATIONAL USEREQUILIBRIUM FOR MARKET STUDIES OF RESIDENTIALELECTRICITY CONSUMERS IN SMART GRID

NOMENCLATURE

A. Sets, Indices, Parameters and Variables for the Equilibrium Model

A	Set of appliances indexed by a
I	Set of users indexed by i
i	Denote all other users in I except user i
T	Set of time periods indexed by t
$T^0_{i;a}$	The unacceptable time periods for user i to use appliance a
$T^1_{i;a}$	The preferred time periods for user i to use appliance a

B. Parameters

$D_{i;a}$	The total daily demand of user i on appliance a
$E_{i;a}$	The maximum electricity that can be consumed by user i on appliance a in one time period
p_i	The momentary value of the time-of-use convenience for user i
c_0; c	Electricity price coef cients in the cost function

C. Variables

$x^t_{i;a}$ Electricity consumption of user i on appliance a at time t

x_i The electricity usage pro le of user i, i.e., a vector of $x^t_{i;a}$ for all appliances and time periods

$l_t; l^t_i$ Total electricity loads/consumptions at time t for all users and user i, respectively

$p_{i;a}$ Electricity consumption of user i on appliance a during preferred times

b_t Surplus price for time period t

m_i Lagrangian dual multiplier for user i

p_i Penalty coef cient for user i

s_i Auxiliary variable for user i when using the penalty cutting plane method

D. Functions

$f(\ )$ Unit electricity cost as of a function of total load

$u_i(\ )$ Utility function for user i

Introduction

With increasing world population and rapid development and use of new electrical appliances, residential power consumption has signi cantly increased in the past decades. According to U.S. Energy Information Administration (EIA), the residential consumption accounts for about 37% of the total electricity end use in the past decade (2008-2017), rising from around around 33%

in the 1980s and 25% in the 1950s. Because the introduction of electrical vehicles (EV) [20] is shifting from gas consumption toward electricity consumption, it is estimated that mass adoption of EVs will double the residential electricity consumption. In the foreseeable future the percentage of residential use in total electricity end use will continue to increase. Therefore, it is imperative to achieve high ef ciency in residential electricity consumption for a sustainable energy system. Peak-to-Average (PTA) ratio of electricity demands indirectly re ects the system redundancy and additional cost for system stability due to the fact electrical power being a instantly perishable commodity. Hence it is considered as an important index for power systems' ef ciency. Demand Side Management (DSM) by leveraging time-of-day electricity prices, has long been proposed since 1980s to reduce PTA ratio in the commercial or industrial use. The introduction of Advanced Metering Infrastructure (AMI) in smart grid has enable DSM in the residential sector by sending price signals to residential consumers in real time. Various pricing schemes [59] such as time-of-use, critical peak pricing, day ahead pricing [55, 42, 47] and real-time pricing [72, 8] have been adopted in practice (e.g., ComEd in Chicago area) and all have achieve some level of successes in peak load reduction.

In the literature, many have built mathematical models to study the optimal electricity consumption pro les and the optimal design of pricing schemes for DSM [17]. Since DSM is centered around altering consumer behaviors via nancial incentives, many have devoted to develop game theoretical models to describe electricity consumption pro les of residential users. For example, [3] proposed a system optimum model and a Nash Equilibrium model considering both electricity costs and convenience of electricity consumption pro les. Similar works based on Nash Equilibrium also appeared in [71, 29]. [23] studied the use of energy management controller for electric vehicles, and conclude that the game-theory-based controller on the New European Driving Cycle (NEDC) works better than the existing baseline controller. Further, [100] proposed an integer linear programming models based on game theory for optimal scheduling the use of power-shiftable

appliances. In addition to modeling user behaviors and optimal scheduling, many have also studied the pricing problems. For example, [37] showed that a proper pricing strategy is important to minimize the total cost and proposes optimal pricing policies under certain conditions. [72] demonstrated that pricing strategy has big in uence in the electricity system and proposed strategies that can effectively shift users' consumption from peak to off-peak time. In addition, similar work also appeared in [73, 18]. At the same time, many have combined the use of game theory and pricing model to determine the optimal pricing strategies in smart grids. For example, [92] integrated distributed generations to reduce the energy losses. They achieve this in two ways: making a game theory-based loss reduction allocation and making a load feedback control with price elasticity. [97], on the other hand, combined game theory and pricing strategies using virtual machines (VMs) placement. They propose new algorithms to solve the problem of dynamic placement of VMs for energy consumptions' optimization.

This paper considers a mathematical model framework to re ect the reality that not every user or even no user seeks to perfectly minimize their electricity bill (e.g., [44]) or maximize their personal utility (including electricity cost, comfort and convenience). This is because humans have limited cognitive ability to solve optimization problem in practice and that usage of some home appliances can be exible so long as certain threshold or range is maintained. In other words, most users are satis ed or have no incentive to change their consumption pro les so long as the total utility (including electricity cost, comfort and convenience) of their consumption pro les reaches a certain threshold. In economic literature, this phenomenon is referred to as “bounded rationality.” In contrast to Nash equilibrium [64] where individual players optimize their own problem until no unilateral change of strategies occurs and the system reaches an equilibrium, [78] de nes “boundedly rational user equilibrium” (BRUE) as the system reaches an equilibrium when no unilateral change is needed when individual players accept the utility to be at least at a certain percentage of the maximum value of their optimal utility. The notion of BRUE has been widely

used in modeling users behaviors. For example, [70] use the BRUE concept in the bank system, while [58] use the boundedly rationality in their transportation planning models. In addition, [56] used the BRUE concept in the static traf c assignment problem, and they solved the resulting mathematical program with equilibrium constraints (MPEC) by using the penalty method. [36] also use the BRUE in a dynamic traf c assignment problem. Examples of BRUE related works in other elds include [44] in psychology, [26] in industrial organization, and [96] in energy systems. To our best knowledge, this paper is the rst to use the BRUE framework to study residential electricity consumption and related pricing strategies in DSM in a smart grid. Under the BRUE framework, this paper studies four core problems, extending from the system optimum model, which aims to minimize the total generation cost while satisfying all shift-able demands. The four core problems are actually categorized into two groups. The rst group includes two models built to explore the best and the worst possible performances of the BRUE conditions in terms of total generation cost. The second group contains two models (with pessimistic v.s. optimistic viewpoints) that aim to provide pricing strategies via bi-level optimization.

All four core problems are very dif cult to solve due to nonlinearity and nonconvexity. Of particular, the proposed bi-level pricing models are even more challenging. However, we show that the special structure of the lower-level electricity consumption game allows the bi-level model to satisfy the “hidden convexity” property rst introduced by [13] in 1996. By exploiting the hidden convexity of the BRUE models, it is guaranteed that the proposed Lagrangian dual cutting plane method will produce optimal solution with zero duality gap. They especially focused on the nonconvex, quadratically constrained problem. In a more recent study, [10] found that under some conditions, the nonconvex quadratic problem is equivalent to a convex problem. Since then, [53] and [95] analyze the conditions of hidden convex in more general cases beyond the quadratic constrained problems. [25] uses the relaxation techniques to transform the nonconvex to an approximately hidden convex problem. [15] discusses the hidden convex property under the

condition with positive eigenfunctions. [63] applied the hidden convexity to communication problems. To our best knowledge, this paper is the rst to exploit hidden convexity to solve hard BRUE related pricing models in a smart grid.

The contributions of this paper are summarized as follows. First, we propose mathematical models under the principle of bounded rationality user equilibrium in residential electricity consumption games compared to the most existing works in the literature that are under the Nash equilibrium principle. The BRUE models are more realistic in that they adequately acknowledge that electricity consumers do not necessarily optimize their energy consumption in real life. Second, we consider four cases best-performance and worst-performance system optimal models with BRUE constraints, and pessimistic and optimistic pricing models with BRUE constraints. Furthermore, we show that by introducing a carefully chosen time-varying surcharge, electricity users will change their energy consumption behavior ultimately leading to higher system ef ciency (i.e., lower peak-to-average ratio). Third, we show that even though BRUE pricing models are non-convex, the Lagrangian method still satis es the property of strong duality due to its hidden convexity. Finally, we conduct extensive sensitivity analysis to provide managerial insights for stakeholders of the DSM program in a smart grid.

Mathematical Models

Two Energy Consumption Models

We introduce two basic energy consumption models, i.e., the system optimal and user equilibrium models in energy consumption. The de nition of the sets, indices, parameters and variables are listed at the beginning of this chapter. As in [3], we considers a local residential electrical power system with n users and a set of appliances A for each user. In the system optimal model (SO),

the energy company want to minimize their total electricity cost based on the certain customer demand. In this model $T = \{1, 2, \ldots, 24\}$ has 24 time periods in a daily cycle.

$$\text{(SO):} \quad \min \sum_{t \in T} f\left(\sum_{i \in I} \sum_{a \in A} x_{i,a}^t\right) \qquad \sum_{i \in I, a \in A} x_{i,a}^t \tag{2.1a}$$

$$\text{s.t.} \quad \sum_{t \in T} x_{i,a}^t = D_{i,a}, \qquad \forall i \in I, a \in A, \tag{2.1b}$$

$$x_{i,a}^t \le E_{i,a}, \qquad \forall i \in I, a \in A, t \in T, \tag{2.1c}$$

$$x_{i,a}^t = 0, \qquad \forall i \in I, a \in A, t \in T_{i,a}^0, \tag{2.1d}$$

$$x_{i,a}^t \ge 0, \qquad \forall i \in I, a \in A, t \in T, \tag{2.1e}$$

where $f(l_t)$ is the utility cost function, which is a monotone increasing function of the total electricity consumption l_t at time t. In this paper, we let $f(l_t) = p \cdot l_t + q$. p and q are constant value here. In the SO model, we suppose the central power company can control all users consume behavior to let the system work in the best way. The constraints here means the power supply meet the customer demands.

On the other hand, unlike the central controller's system optimal model, the user equilibrium model assumes each user optimizes her/his own objective which is a combination of electricity cost and self convenience based on electricity consumption profile, i.e., when and how much the consumer uses his/her appliances. Hence each user i maximizes the following payoff or utility:

$$U_i = \left[\sum_{t \in T} f(l_t) \cdot l_i^t\right] + u_i(x_i) \tag{2.2}$$

where l_i^t is the total electricity consumption by user i at time t, and x_i is the electricity usage profile of user i, a vector of $x_{i,a}^t$ for all appliances and time periods. In this payoff U_i, the first term is the total power cost, which is actually the disutility for user i. The second term defines the convenience utility by user i, it means when a user use the appliance within his/her desired time period, it has

positive convenience utility. We define $u_i(x_i) = \sum_{a \in A} \sum_{t \in T} p^t_{i;a} n^t_{i;a}(x^t_{i;a})$, where $p^t_{i;a}$ and $n^t_{i;a}$ are the parameter of convenience. Hence, in the equilibrium model, each user needs to solve the following UO_i model:

$$(UO_i): \quad \max \quad U_i = \sum_{t \in T} \left(\sum_{a \in A} x^t_{i;a} + \sum_{j \in Inf\{i\}; a \in A} x^t_{j;a} \right) \left(\sum_{a \in A} x^t_{i;a} \right) + u_i(x_i) \qquad (2.3a)$$

$$\text{s.t.} \quad \sum_{t \in T} x^t_{i;a} = D_{i;a}; \qquad \forall a \in A; \qquad (2.3b)$$

$$x^t_{i;a} \quad E_{i;a}; \qquad \forall a \in A;\ t \in T; \qquad (2.3c)$$

$$x^t_{i;a} = 0; \qquad \forall a \in A;\ t \in T^0_{i;a}; \qquad (2.3d)$$

$$x^t_{i;a} \quad 0; \qquad \forall a \in A;\ t \in T: \qquad (2.3e)$$

The UO_i model means each user want to maximize their unilities. Thus, the objective (2.3a) for user i is to minimize the total disutility, but each users decision also depends on other user's decision. It is a game theory model.

Finally, in an n-user system where each user solves their own UO_i model, we define the user equilibrium for the energy consumption game as follows. In game theory model, each user does not have the incentive to change his/her decision, it means the equilibrium will have the following inequality.

$$(UE) \quad U_i(x_i; x_{-i}) \quad U_i(x_i; x_{-i}); \forall x_i \in X_i;\ \forall i = 1; \quad ; n: \qquad (2.4)$$

Two Boundedly Rational User Equilibrium Models

While the UE model (2.4) represents the decentralized behavior for all energy users instead of a centralized control scheme by the utility $\ \ rm$ as in the (SO) model, still one drawback of the UE is that in practice no user has either the desire or cognitive ability to optimize a utility function. Based

on Simon's notion of bounded rationality, people are assumed to be happy if they can reach some level of utility without maximizing his/her utility. Hence, the boundedly rational user equilibrium (BRUE) is de ned by a set of constraints compared to simultaneous optimization problems for equilibrium with perfect rationality. Assume user i has a satisfaction level $x_i \in [0;1]$, i.e., he/she is happy if his/her utility is within the range $[x_i R_i; R_i]$. Here, R_i is the upper bound of user i's utility and can be computed by assuming zero consumptions from others, $R_i = U_i(x_i; 0)$. Hence, any $x \in F$ satis es the BRUE condition if the following holds:

$$\sum_{t \in T} \left(\sum_{i \in I; a \in A} x^t_{i;a} - \sum_{a \in A} x^t_{i;a} + u_i(x_i) \right) \geq x_i R_i; \quad \forall i \in I: \tag{2.5}$$

The BRUE constraint (2.5) enforces a lower bound on the utility of user i. Alternatively, the above BRUE condition can be rewritten in terms of disutility as the following:

$$\sum_{t \in T} \left(\sum_{i \in I; a \in A} x^t_{i;a} - \sum_{a \in A} x^t_{i;a} - u_i(x_i) \right) \leq r_i W_i; \quad \forall i \in I: \tag{2.6}$$

where $r_i \geq 1$ is a scalar and W_i is the minimum disutility for user i assuming no other users exist.

Because essentially bounded rationality (BR) is represented by a lower (or upper) bound constraint on individual's utility (or disutility), it gives rise to subsequent optimization models with such BR constraint. Below we introduce two modeles representing the best and worst BRUE conditions, respectively.

To formulate the best performance BRUE conditions, we aim to minimize the total system generation cost while respecting the BR constraint.

$$\text{(B-BRUE):} \quad \min \sum_{t \in T} \left(\sum_{i \in I} \sum_{a \in A} x^t_{i;a} - \sum_{i \in I; a \in A} x^t_{i;a} \right)$$

$$\text{s.t.} \quad \sum_{t \in T} f\left(\sum_{i \in I; a \in A} x_{i;a}^t\right) - \sum_{a \in A} x_{i;a}^t \le u_i(x_i) - r_i W_i; \quad \forall i \in I \qquad (2.7a)$$

$$x \in F; \qquad (2.7b)$$

The above (B-BRUE) is a non-convex quadratic problem where both the objective function and the constraints (2.7a) are quadratic.

On the other hand, the worst performance BRUE conditions are found by maximizing the total system genration cost with the BR constraints.

$$\text{(W-BRUE):} \quad \max \sum_{t \in T} f\left(\sum_{i \in I} \sum_{a \in A} x_{i;a}^t\right) - \sum_{i \in I; a \in A} x_{i;a}^t$$

$$\text{s.t.} \quad (2.7a), (2.7b)$$

Pricing Strategies in Boundedly Rational User Equilibrium for Electricity Consumer Market

Building on the BRUE energy consumption models the previous section, we now consider the pricing strategies to be employed by the utility rm under the two BRUE energy consumption behavior scenarios. Because under BRUE, consumer behaviors are within a given range, the performance of the system also falls into a range. Hence it is necessary to discuss pessimistic and optimistic pricing strategies acknowledging varying system performances under BRUE.

The optimistic pricing model below determines an optimal pricing scheme (or surcharge on top of the generation cost) b^t, so that the resulting total system cost is minimum given any users' behaviors under BRUE falling within their own rational bounds.

$$\text{(O-P):} \quad \min_{b^t \in B; \forall t \in T} \quad \min \sum_{t \in T} f(l_t) - l_t \qquad (2.9a)$$

$$\text{s.t.} \quad \sum_{t \in T} \left[b^t + f\left(l_i^t + l_i^t \right) \right] l_i^t \quad u_i(x_i) \quad r_i W_i; \qquad \forall i \in I \tag{2.9b}$$

$$l_i^t = \sum_{a \in A} x_{i;a}^t; \qquad \forall t \in T; i \in I \tag{2.9c}$$

$$x \in F; \tag{2.9d}$$

where b^t is the surcharge price at time t for all users, and b is the vector composed by all b^t and B is the set of b^t.

The (O-P) problem is a bi-level optimization problem. The upper level minimizes the total system cost for the utility rm to select an optimal price strategy b, and the lower level minimizes the same objective for users to select an optimal consumption profile x_i. As will be discussed in Section 2, the solution method for solving the (O-P) is similar to that for solving (B-BRUE) as the two levels of minimization in (O-P) can be merged into a single level objective.

Similarly the pessimistic/robust pricing strategy determines a proper pricing strategies so that the resulting maximum system cost due to varying consumer behaviors under BRUE can be minimized. In other words, we assess the highest possible system cost given users' rational bounds of their satisfactory utilities, and then minimize this worst case system cost. The pricing problem can be formulated as a two-stage robust optimization problem as follows:

$$\text{(PR-P):} \quad \min_{b^t \in B; \forall t \in T} \quad \max \sum_{t \in T} \left[f\left(\sum_{i \in I} \sum_{a \in A} x_{i;a}^t \right) \quad \sum_{i \in I} \sum_{a \in A} x_{i;a}^t \right] \tag{2.10a}$$

$$\text{s.t.} \quad \sum_{t \in T} \left[b^t + f\left(\sum_{i \in I} \sum_{a \in A} x_{i;a}^t \right) \quad \sum_{a \in A} x_{i;a}^t \right] \quad u_i(x_i) \quad r_i W_i; \forall i \in I \tag{2.10b}$$

$$x \in F \tag{2.10c}$$

Mathematical Property and Sensitivity Analysis Based on the Models

The (PR-P) model is a non-linear and non-convex problem due to the nonlinearity of the objective function and the BRUE constraint. This section investigates conditions under which the so-called "hidden convexity" [13] holds for the (PR-P) problem and the next section presents its solution algorithm by exploiting the hidden convexity property.

Hidden Convexity

"Hidden convexity" [13] refers to a non-convex optimization problem for which the Lagrangian dual has zero duality gap. Hence, computationally it can be treated as a convex optimization problem. Below we present three main results. First, the (W-BRUE) problem has the hidden convexity property when there is $n = 1$ user in the system. Second, in general though there is no guarantee that the hidden convexity holds for a system with $n > 1$. Third, when $n > 1$, under certain conditions with respect to the BRUE constraint in the (W-BRUE) problem, strong duality can still satisfy with and without the pricing strategy.

Lemma 1. Recall the following (W-BRUE) problem.

$$\text{(W-BRUE):} \quad \min \sum_{t \in T} f\left(\sum_{i \in I} \sum_{a \in A} x_{i;a}^{t} \left(\sum_{i \in I; a \in A} x_{i;a}^{t} \right) \right)$$

$$\text{s.t.} \quad \sum_{t \in T} f\left(\sum_{i \in I; a \in A} x_{i;a}^{t} \left(\sum_{a \in A} x_{i;a}^{t} \right) \right) \quad u_i(x_i) \quad r_i W_i; \quad \forall i \in I \quad (2.7a)$$

$$x \in F;$$

Let $g(x) = \left[\sum_{t \in T} f\left(\sum_{i \in I; a \in A} x_{i;a}^{t} \left(\sum_{a \in A} x_{i;a}^{t} \right) \right) \quad u_i(x_i) \quad r_i W_i \right]$ and l_i be the associated Lagrangian multiplier for the ith constraint in (2.7a). Suppose x and l are the optimal solutions to the primal and Langrangian dual problems, respectively, then (W-BRUE) has strong duality if the

following holds: $g_i(x) \quad 0; u_i \quad g_i(x) = 0; \quad 8i \ 2 \ I:$

Proof. Let Z and Z_D be, respectively, the optimal objective values for the (W-BRUE) and its Lagrangian dual after relaxing constraint 72 with Lagrangian multiplier l. Let

$$h(x) = \sum_{t \ 2 \ T} f \left(\sum_{i \ 2 \ I} \sum_{a \ 2 \ A} x^t_{i;a} \quad \sum_{i \ 2 \ I; a \ 2 \ A} x^t_{i;a} \right)$$

be the objective function. From weak duality, $Z_D \quad Z$. Further, if $g_i(x) \quad 0; 8i \ 2 \ I$, then x is a feasible solution to (W-BRUE) and thus $Z \quad h(x)$. Hence, $Z_D = h(x) \quad \sum_{i \ 2 \ I} u_i \quad g_i(x) = h(x)$ Z. Therefore, $Z = Z_D$, i.e., the Lagrangian dual method has the strong duality. □

Lemma 2. [39]

Let A and B be two real symmetric matrices. If there exists $a, b \ 2 \ R$ such that $a A + b B > 0$, then there exists a nonsingular matrix $C \ 2 \ R^{n \ n}$ such that both $C^T AC$ and $C^T BC$ are diagonal.

Lemma 3. [13]

Consider the following nonlinear program:

$$\begin{aligned} \text{(BT):} \quad & \min \quad 1=2x^T Ax + c^T x \\ & \text{s.t.} \quad 1=2x^T Bx \quad d; \\ & \qquad\quad 1=2x^T Gx + h^T x + k \quad 0; \end{aligned}$$

where $A, B; G \ 2 \ R^{n \ n}$ are symmetric, and $c; d; k; h \ 2 \ R^n$. The following holds:

(1) If one of the matrices $A, B; G$ is a zero matrix and the other two are simultaneous diagonalizable, then separability is obtained.

(2) Let $n_1 \quad 0$ and $n_2 \quad 0$ be the KKT multipliers of the two constraints of (BT), respectively, corresponding to an optimal solution x. Then, either $n_1 = 0$ or $n_2 = 0$ holds.

We rewrite the W-BRUE model as follows:

$$
\begin{aligned}
\text{(W-BRUE-Q):}\quad \min\quad & p\ x^T A x + q_1^T x; \\
\text{s.t.}\quad & x^T B^k x + d_1^T x \quad d^k \quad 0; \quad k = 1; 2; \quad ; n \\
& H^T x + m \quad 0; \\
& x \ 2\ R^{nT}; q_1 = [q; q; \quad ; q] \\
& A;\ B^k\ 2\ R^{nT\ \ nT}; d^k\ 2\ R; m\ 2\ R
\end{aligned}
$$

Where p and q are relative value in function $f(l_t) = p\ l_t + q.\ d_1^T x = q \left[\sum_{t2T} \sum_{a2A} x_{i;a}^t \quad u_i(x_i) \right]$, it has a linear relation with x. $d^k = r_k W_k$. We can have linear transformation for the variable x to let d_1=0. n is the number of users, T, is the number of time periods and A and B^k are symmetric whose $(i; j)^{th}$ elements are de ned below.

$$
a_{ij} = \begin{cases} 1; & 8(i; j)\ 2\ A_1 \\ 0; & \text{else} \end{cases} \quad ; \quad b_{ij}^l = \begin{cases} 1; & 8(i; j)\ 2\ B_1^l \\ 1=2; & 8(i; j)\ 2\ B_2^l\ [\ B_3^l \\ 0; & \text{else} \end{cases}
$$

$$
A_1 = \left\{ (i; j) : n(t \quad 1) + 1 \quad i \quad nt; n(t \quad 1) + 1 \quad j \quad nt; 8t = \{1; 2; \quad ; T\} \right\}.
$$

$$
B_1^l = \left\{ (i; j) : i = j = n(t \quad 1) + l; 8t = \{1; 2; \quad ; T\} \right\}.
$$

$$
B_2^l = \left\{ (i; j) : i \ 6 = j; i = n(t \quad 1) + l; n(t \quad 1) + 1 \quad j \quad n(t \quad 1) + T; 8t = \{1; 2; \quad ; T\} \right\}.
$$

$$
B_3^l = \left\{ (i; j) : i \ 6 = j; j = n(t \quad 1) + l; n(t \quad 1) + 1 \quad i \quad n(t \quad 1) + T; 8t = \{1; 2; \quad ; T\} \right\}.
$$

To illustrate, when $n = 2$ the matrices A and B^1 are as follows.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}; \quad B^1 = \begin{pmatrix} 1 & 1/2 & 0 & 0 & \cdots & 0 & 0 \\ 1/2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1/2 & \cdots & 0 & 0 \\ 0 & 0 & 1/2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1/2 \\ 0 & 0 & 0 & 0 & \cdots & 1/2 & 0 \end{pmatrix}$$

If there is more than one person in the system, the BRUE constraints for the model (W-BRUE) are nonconvex constraint. Because with the de nition of convex feasible region in such form, we need the matrix B^k to be an positive semi de nite (PSD) matrix. But actually in our case if we choose $x = [1; \quad 2; 0; 0; \quad ; 0]$, then now $x^T B^1 x = \quad 1 < 0$, and now $x \in 0$, so B^1 is not a PSD matrix, then the feasible region for such constraint is not a convex set.

Theorem 1. Consider the (W-BRUE) problem and its Lagrangian dual after relaxing constraint (2.7a).

(1) If $n = 1$, then strong duality holds for the (W-BRUE) and the Lagrangian dual.

(2) If $n > 1$ and at least $n \quad 1$ constraints in (2.7a) are non-binding at the optimal solution, then strong duality holds for the (W-BRUE) and the Lagrangian dual.

Proof. First consider the special case where $n = 1$. In this case, A and B are both an identity matrix. According to Lemma 2, there exists a nonsingular matrix C such that both $C^T AC$ and $C^T BC$ are diagonal. We call C as the simultaneous diagonalization matrix for A and B. Subsequently, applying Lemma 3's result (1), one concludes that the W-BRUE problem with $n = 1$ is convex.

On the other hand, if $n > 1$ and at least $n \quad 1$ (2.7a) constraints are non-binding at the optimal solution, then there is only one constraint that is active in the system. Hence, the problem also reduces to a convex problem. □

Note that Theorem 1 suggests that when the value of n in constraint (2.7a) is chosen such that conditions (1) or (2) in the theorem holds, then strong duality holds between W-BRUE and its Lagrangian dual. Below is a counterexample when neither (1) nor (2) is satis ed, then there may be duality gap between W-BRUE and its dual.

Finally we present some strong duality results for the pessimistic pricing problem (PR-P) in (2.10a)-(2.10c). Let Z and b be the optimal objective value and the associated optimal pricing, respectively. Further, let Z_{b^0} denote the optimal objective value for the inner maximum problem when b is xed at b^0. When relaxing constraint (2.10b) for the Lagrangian dual problem (PR-PD) (see details in section 2), we use Z_D to denote the optimal objective value for the dual problem (PR-PD), and b_D to denote the corresponding optimal value of b for (PR-PD). Finally, let $Z_{D;\, b^0}$ be the optimal objective value for problem (PR-PD) when b is xed at b^0.

Theorem 2. The following holds for the pessimistic/robust pricing model (PR-P).

(1): $Z = Z_D$ if and only if $Z_b = Z_{D;\, b}$, when it has the unique solution.

(2): Strong duality holds for (PR-P) and its Lagrangian dual when relaxing constraint (2.10b) if and only if it has the strong duality to $b = b$.

Proof. For part (1), rstly, if $Z = Z_D$ is satis ed, then we show that $Z_b = Z_{D;\, b}$. This is because $Z = Z_b$; $Z_D = Z_{D;b_D}$ From the de nition of b, one has $Z_b \quad Z_{b_D}$ Further, by weak duality $Z_{b_D} \quad Z_{D;b_D}$. Hence $Z = Z_b \quad Z_{b_D} \quad Z_{D;b_D} = Z_D = Z$. Therefore, all inequalities hold as equality, i.e., $Z_{b_D} = Z_b = Z_{D;\, b_D}$. Because it has the solution, one obtains $b_D = b$, thus $Z_b = Z_{D;\, b}$.

Secondly, we show that if $Z_b = Z_{D;\, b}$ then $Z = Z_D$. This is because $Z = Z_b \quad Z_{b_D} \quad Z_{D;\, b_D} =$

$Z_D \quad Z_{D;\,b} = Z_b$. Similar to the above, all inequalities must hold as equalities, therefore $Z = Z_D$.

For part (2), this is a direct result of part (1). □

Algorithms for BRUE and Pricing Models

We have four BRUE related models. (B-BRUE) is the Best Performance of the BRUE Conditions, whereas (W-BRUE) is the Worst Performance of the BRUE Conditions. When pricing is considered, (O-P) solves for the optimistic Pricing Strategies whereas (PR-P) determins the pessimistic/Robust Pricing Strategies. For these four models, we apply three calculation methods for different models. First method is using solver BARON [87] directly. BARON can be used tor solve (B-BRUE), (W-BRUE) and (O-P). Second method is that we develop a penalty cutting plane method for each model and then apply BARON to solve the sub-problems. Third method is applying the lagrangian dual cutting plane method for each model and then solve sub-problems with BARON.

Cutting Plane with Penalty Method

In model (B-BRUE), (W-BRUE) and (O-P), we can use BARON to solve the problem directly. However, since (PR-P) is a robust optimization, BARON cannot be used directly to solve it. In this section, we rst propose a penalty cutting plan method for solving the (RP-P) and then propose a Lagrangian dual cutting plane method, and nally compare the ef ciency of the two methods.

In devising the penalty cutting plane, we penalize all BRUE constraints in (PR-P) into the objective

function, and therefore obtain the following problem (PR-PP).

$$\text{PR-PP:} \quad \min_{b^t \in B; \forall t \in T} \quad \max_{x \in F} \quad P(x; b); \tag{2.12a}$$

where $P(x; b)$ is new penalty objective function. More speci cally,

$$P(x; b) = \sum_{t \in T} f\left(\sum_{i \in I} \sum_{a \in A} x^t_{i;a}\right) - \left(\sum_{i \in I} \sum_{a \in A} x^t_{i;a}\right) \cdot \left(\sum_{i \in I} p_i \left(r_i W_i + \sum_{t \in T} b_t + f\left(\sum_{i \in I} \sum_{a \in A} x^t_{i;a}\right) - \sum_{a \in A} x^t_{i;a} u_i(x_i) + s_i\right)\right);$$

where p_i is the penalty value for the corresponding constraint (2.10b), $u_i = [u_i; \forall i \in I]^T$ is as before the utility coef cient, and $s_i \geq 0$ is the slack variable for the constraint (2.10b).

Hence the model of the cutting plane method can be formulated as follows.

$$\text{(PR-MPP):} \quad \min_{b \in B;\ p} \quad p \tag{2.13a}$$

$$\text{s.t.} \quad p \geq a_k b + b_k b^2 + c_k; \quad \forall k = 0; 1; 2; \ldots; l \tag{2.13b}$$

a_k, b_k, c_k are the corresponding coef cient for b, b^2 and constant value in $P(x; b)$ when $x = x_k$, and p is the cutting plane value. Below is the outline for the resulting penalty cutting plane method.

Remark. Under optimal pricing strategies, the optimal system cost falls into the range of $[Z_{SO-P}, Z_{PR-P}]$ due to user's behavior under BRUE.

Algorithm 1 ALG PR PP

1: Set $k=0, b_0 = 0$, $UB = M$, $LB = -M$.
2: Solve $\max_x \{P(x; b_k) \mid x_i \in X_i; \forall i\}$ by BARON. We can get x_k and the upper bound of PR-PP, denote as UB_k.
3: If $UB_k < UB$, $UB = UB_k$.
4: Get a_k, b_k, c_k by using x_k.
5: Add the cutting plane (2.13b) by using the relative a_k, b_k, c_k.
6: Solve PR-MPP by BARON and get b_{k+1}. And p_{k+1} is the lower bound of PR-PP, denote as LB.
7: If $UB > LB$, $k = k + 1$, go to Step 2. Otherwise, stop.

Lagrangian Dual Cutting Plane Method

We recognize that the second stage in (PR-P) is a maximization problem for a given b. This is indeed the same problem as (W-BRUE). Therefore, hidden convexity holds if we relax constraint (2.10b), and this has motivated us to study the Lagrangian dual cutting plan method as an alternative to the penalty cutting plan method.

After obtaining the Lagrangian dual for the inner maximization problem of the (PR-P) by relaxing constraint (2.10b), the new equivalent problem (PR-PD) is as follows.

$$\text{(PR-PD):} \quad \min_{b^t \in B;\ m \geq 0} \ \max_{x \in F} \ L(x; b; m);$$

where $L(x; b; m)$ is the new objective function. More speci cally,

$$L(x; b; m) = \sum_{t \in T} f\left(\sum_{i \in I} \sum_{a \in A} x^t_{i;a} - \sum_{i \in I} \sum_{a \in A} x^t_{i;a}\right) - \sum_{i \in I} m_i \left(r_i W_i + \sum_{t \in T} b^t + f\left(\sum_{i \in I} \sum_{a \in A} x^t_{i;a} - \sum_{a \in A} x^t_{i;a}\right) - u_i(x_i)\right);$$

where m_i is the lagrange dual variable corresponding to constraint (2.10b) and $m = [m_i; \forall i]^T$.

Hence the Lagrangian dual cutting plane model can be formulated as follows.

$$\text{(PR-MP):} \quad \min_{b \in B,\ m \geq 0,\ p} \quad p$$

$$p \leq a_k \cdot m + b_k \cdot b + c_k \cdot m \cdot b + d_k, \quad \forall k = 0, 1, 2, \ldots, l,$$

where as previously a_k, b_k, c_k and d_k are corresponding coefficient for m, b, $m \cdot b$ and the constant in $L(x; b; m)$ when $x = x_k$. Also similar to previously p is the cutting plane value. Below is the outline of the Lagrangian dual cutting plane method.

Algorithm 2 ALG_PR_PD

1: Set $k=0$, $b_0 = 0$, $m_0 = 0$, UB = M, LB = -M.
2: Solve $\max_x \{L(x; b_k; m_k) \mid x_i \in X_i; \forall i\}$ by BARON. We can get x_k and the upper bound of PR-PD, denote as UB_k.
3: If $UB_k < UB$, $UB = UB_k$.
4: Get a_k, b_k, c_k and d_k by using x_k.
5: Add the cutting plane (2.14) by using the relative a_k, b_k, c_k and d_k.
6: Solve PR-MP by BARON and get b_{k+1}, m_{k+1}. And p_{k+1} is the lower bound of PR-PD, denote as LB.
7: If UB > LB, $k = k + 1$, go to Step 2. Otherwise, stop.

Computational Experiments and Results

In this section, we compute the result for all of the four former cases. We solved it in Matlab using the solver BARON. All of the calculations were run on a Intel(R) Core(TM)2 Duo CPU serve with 4GB of memory in 64-bit Operating System.

We define the preferred usage utility function as below,

$$u_i(x_i) = p_i \sum_{a \in A} \frac{p_{i;a}}{D_{i;a}} \tag{2.14}$$

where p_i is the corresponding utility coef cient for user i and $p_{i;a} = \sum_{t \in T^1_{i;a}} x^t_{i;a}$ is the total amount for the user i to use appliance a in the preferred time period. We de ne the unit cost function as $f_t(l_t) = c_0 + c_1 l_t$, here f_t means the total load at the time period t.

Results under Different Data Sets

We will use differ data set to calculate for our four different models, and will analysis the calculation time and accuracy by using three algorithms mentions before.

A Simple Example With Four Time Period

We consider a quick example with two users, two appliances and four time periods. We use $[a; b]$ to demonstrate $T = T^0_{i;a}$, which means the time period that available for the user i with appliance a. Use $[a_p; b_p]$ to demonstrate $T^1_{i;a}$, which means the preferred time periods for the user i with appliance a. In this case, we have $1 \le a \le a_p \le b_p \le b \le 4$. The relative parameters are shown in Table 2.1. And we set $c_0 = 10, c_1 = 3$, which is used in the unit price function $f_t(l_t) = c_0 + c_1 l_t$ in example 1. Set $r_1 = 1.214, r_2 = 1.054, p_1 = p_2 = 100$, penalty value $p_1 = p_2 = 100$, set B is $1 \le \sum_t b^t \le 1$, $0.5 \le b^t \le 0.5$.

Table 2.1: Parameters of Simple Example

User	Appliance	$D_{i;a}$	$E_{i;a}$	a	b	a_p	b_p
1	1	7	3	1	3	1	1
1	2	3	1	2	4	4	4
2	1	6	4	2	4	3	3
2	2	4	3	1	3	2	2

We solve this problem with solver BARON directly and penalty method, the results are shown in Table 2.2 and Figure 2.1(a), 2.1(b), 2.1(c). w_1 is the minimum optimal cost for the user 1, if we suppose user 2 do not need to obey the BRUE. It has the similar de nition w_2 for

Table 2.2: Results of Simple Example

$w_1 = 164.81$ $w_2 = 123.30$	(B-BRUE)	(W-BRUE)	(O-P)	(PR-P)
Cost	513.2	542.9	510.2	LB: 533.5 UB: 536.3
Solve Method	BARON	BARON	BARON	Penalty
Iterations	N/A	N/A	N/A	1000
Time Cost	0.79s	0.7s	1.12s	8 days

(a) Upper Bound (b) Lower Bound (c) Upper Bound and Lower Bound

Figure 2.1: Upper Bound and Lower Bound of Model (PR-P) by k Increasing

Table 2.2 shows that the result for four cases is different and the cost of (B-BRUE) = 513.2 is greater than the cost of (O-P) = 510.2, the cost of (W-BRUE) = 542.9 is greater than the cost of (PR-P) =[533.5; 536.3]. The values within the brackets are relatively the lower bound and upper bound of model (PR-P). This is reasonable because the optimal solution of (B-BRUE) must be a solution of (O-P) = 510.2, it just means $b_t^s = 0; 8t \in T$ in (O-P). And the same reason with (W-BRUE) and (PR-P).

The calculation time cost for the first three cases are just seconds, but the calculation time for the (PR-P) is about 8 days, and it also has the gap of 2.8. This is slow first because we use the cutting plane method and the iterations are 1000. Second reason is that we also use the penalty method, then in this case we introduce new slack variables $s_t^s \in I$ and make a square for the BRUE constraints, now the objective function become more complex.

Figure 2.1(a) shows that upper bound is decreasing when k increasing, it is because we use the judgement (if $UB(k) < UB; UB(k) = UB$). Figure 2.1(b) shows that lower bound is increasing when k increasing, it is because for a new iteration, we add a new constraint to the former iteration, and it is a minimum problem, so the result for the Lower Bound is increasing. Figure 2.1(c) shows that when k increasing, the upper bound and Lower Bound go to a convergency value.

Results for Instances with 24 Time Periods

From the result of four time periods case study. We know that the calculation time for the model (PR-P) which is a max-min problem with the penalty and cutting plane method is 8 days. And the gap is still 2.8 between the upper bound and lower bound from the cutting plane method. Then we use the lagrangian cutting plane method to solve our problem. In this example, the data are based on daily energy consumptions of three appliances: dishwasher, vehicle and air conditioner [1]. The unit cost here $f_t(l_t) = c_0 + c_1 l_t$; $c_0 = 7.43$ cents, $c_1 = 1.55$ cents per KWh. $p_1 = p_2 = 100$, $r_1 = r_2 = 1.7$. The upper bound of the lagrangian dual is $L_0 = 100$. set B is $1 \quad \sum_t b^t \quad 1, \quad 0.1 \quad b^t \quad 0.1$. The other parameters are based on the Table 2.3.

Table 2.3: Parameters of 24 Hours Example

User	Appliance	$D_{i,a}$	$E_{i,a}$	a	b	a_p	b_p
1	1	6.0753	1.1703	1	24	3	10
1	2	13.0305	3.2684	1	24	4	9
1	3	18.5805	2.3439	1	24	5	15
2	1	6.0753	1.1703	1	24	3	13
2	2	13.0305	3.2684	1	24	4	12
2	3	18.5805	2.3439	1	24	5	18

First under the current parameters, we can calculate w_1, w_2. Then we calculate the four cases by different methods. The results are shown in Table 2.4.

Table 2.4: The Results of 24 Hours Example

$w_1 = 150.2$ $w_2 = 120.0$		Solve Method		
		Baron	Penalty	Lagrange Dual
(B-BRUE)	Lower Bound	N/A	-	936.677
	Upper Bound	936.675	-	936.704
	Time Spend	1 day	-	2.5 hours
	Iteration	N/A	-	11
(W-BRUE)	Lower Bound	N/A	-	987.678
	Upper Bound	987.696	-	987.697
	Time Spend	1 day	-	2.5 hours
	Iteration	N/A	-	10
(O-P)	Lower Bound	N/A	-	933.985
	Upper Bound	933.991	-	934.047
	Time Spend	1 day	-	2.5 hours
	Iteration	N/A	-	11
(PR-P)	Lower Bound	N/A	853.471	972.371
	Upper Bound	N/A	978.151	973.030
	Time Spend	N/A	7 days	3 hours
	Iteration	N/A	100	12

`N/A': There has no meaning for such condition.

`-': Not calculate the result under that case.

Figure 2.2: Upper Bound and Lower Bound by Iteration in 24 Hours Model

From the Table 2.4, we can know that the calculation speed of penalty method is much slower than the lagrangian method, using the penalty method, we get the result with a gap 1651.978 853.471 = 124.68 after 100 iterations and the time cost is about 7 days for model (PR-P). But using the lagrangian method, we get the result with a gap of 1037.3 972.371 = 0.659 only in 12 iterations and the time cost is about 3 hours for model (PR-P). We found that the lagrangian method can get more accuracy result compare to the penalty method and also spend less time to calculate. Figure 2.2 gives out the upper bound and lower bound by increasing the iteration using algorithm 2ALG PR PD.

We also found that the time to solve model (PR-P) and model (W-BRUE) is almost the same with lagrangian method, and the numbers of iterations to be converged is also the same(for (W-BRUE) is 10, for model (PR-P) is 12). Without using the lagrangian method, the model (PR-P) is a min-max problem and (W-BRUE) is just a max problem. But using the lagrangian method, calculation speed of them become similar. It is because we introduce a lagrangian multi in the system, and in model (PR-P), we can mix the master problem variable unit price and m together, so we can

solve the master problem just like the (W-BRUE). Then we can know that the lagrangian method is an ef cient method for the min-max or max-min problem if there is no lagrangian gap exists.

Figure 2.3 give us the result for the residual values for each BRUE constraint for different iteration k. From the results we can know that the tendency of the residual is becoming 0 when the iteration is increasing. And from the theorem 1 and 2 we can know that now our lagrangian dual method has the strong duality.

Figure 2.3: Residual For BRUE Constraints with Tow Users

The Results in 24 Hours for Multiple Users

In the real world, we can divided the people in different group of people with different rationality coef cient r . In table 2.5, we give the results for 4 users and 10 users system with the lagrange dual method. For the 4 users system, we set two users with r = 1:7 and other two users with r = 1:8. For the 10 users system. we set the ten users in ve groups with the different and relatively r = 1:7, 1:8, 1:9, 2:0, 2:1.

Table 2.5: The Results of 24 Hours for Multiple Users Example

		(B-BRUE)	(W-BRUE)	(O-P)	(PR-P)
4 users	Lower Bound	2613.7	2756.6	2604.6	2714.5
	Upper Bound	2623.4	2766.2	2614.4	2724.2
	Error(%)	0.37	0.35	0.38	0.36
	Time Spend	8 hours	8 hours	8 hours	9 hours
	Iteration	23	23	24	29
10 users	Lower Bound	12099	12618	12053	12428
	Upper Bound	12297	12812	12248	12626
	Error(%)	1.6	1.5	1.6	1.6
	Time Spend	1 day	1 day	1 day	30 hours
	Iteration	73	76	74	94

We also calculation for the residual value for each BRUE constraints when the iteration k increasing in 10 users system. If we set

$$\text{Violation}_k = \max\{(g_i(x_k); 0) \mid i = 1; 2; \quad ; 10\}; \quad 8k$$

Then we have the gure 2.4, Which shows that the convergency solution almost to be zero, that means now it has the strong duality.

Figure 2.4: Residual For BRUE Constraints with 10 Users

Sensitivity Analysis

As we know, if the coef cient parameter's of human beings was changed, the nal result will also change. We want to know the relations for the in uence by changing the coef cient parameter's of human such as the boundedly rationality coef cient r, preference coef cient p and the customer's demand D. Here we give some theorems about the sensitivity analysis. And we will test them in the computer results part.

Impacts of p and r to the System

As we know, p is the utility coef cient for the user and r is the Boundedly Rational coef cient for the user. These two parameters are determined by the user, it changes with different groups of users. So we want to discuss how the different values of p and r in uence the system and total cost. The results for the in uence of p are shown in Figure 2.5(a). The results for the in uence of r are shown in Figure 2.5(b). The followings are some remarks for the in uence of p and r to the

energy system.

Remark. When the utility coefficient p is increasing, the minimum optimal objective value is also increasing.

This is because when p is increasing, actually it means the users have more incentive to use the energy in their prefer using time period because the utilities for them to use the energy in such time period are higher. Then this will lead to the peak of the energy consumption is higher, so the minimum optimal objective value is also higher.

Remark. The influence by introducing the pricing strategy is better when the utility coefficient p is greater in the minimization problem of BRUE model.

As we discussed before, when p is smaller, the users prefer to use the energy in average for each time period. So the pricing strategy does not give much influence to the user's behaviour. But when p is larger, the user prefer to use the energy in their prefer time period, at this time, we can using pricing strategy to encourage some users to move their time to use the energy to their unlike time period. So the effect for the pricing strategy is better when p is larger.

Remark. When the BRUE coefficient r is increasing, the minimum optimal objective value is decreasing together with the maximum optimal objective value is increasing.

When r is increasing, it will lead that the feasible region for the BRUE constraint is increasing. And the objective function and other constraints are keep the same. So for minimization problem, the optimal value become smaller. And for maximization problem, the optimal value become greater.

Remark. The influence by introducing the pricing strategy is better when the BRUE coefficient r is smaller in the minimization problem of BRUE model.

When r is greater enough, it means actually we do not have the BRUE constraint, now we will

have no in uence by using the pricing strategy. But when become smaller, the uncertainty set for the users behaviour that restricted by the BRUE constraint also becomes smaller. This will lead to the peak hours energy consumption become higher. Now the pricing strategy can make atten for the energy consumption curve.

(a) In uence by p (b) In uence by r

Figure 2.5: Sensitivity Analysis

Figure 2.5(a) shows Remark 2 and Remark 2, when is increasing, the cost is also increasing. And it also shows that for the maximization problem, it does not have this property. And the gap of use or not use the pricing strategy is increasing when is increasing.

Figure 2.5(b) shows Remark 2 and Remark 2. When is increasing the optimal cost of the minimum problem is decreasing, and for the maximum problem the optimal cost is increasing. And the gap of use or not use the pricing strategy is decreasing when is increasing.

The relative Difference of the Total Cost With or Without

From the above we know that, if we use the to the unit price, the total cost can decrease no matter in the best condition or the worst condition. And we also want to know that how much

improvement by introducing b, we want to calculate the relative improvement in both the best and worst condition. We use the formulation that. The relative improvement of best condition improvement

$$IB = \frac{(\text{Total cost of best without } b - \text{Total cost of best with } b)}{\text{Total cost of best without } b} \; 100\%$$

The relative improvement of best condition improvement

$$IW = \frac{(\text{Total cost of worst without } b - \text{Total cost of worst with } b)}{\text{Total cost of worst without } b} \; 100\%$$

Figure 2.6(a), 2.6(b) shows the relative improvements. We can know from the result that when p increasing, the relative improvement to introduce b is also increasing. This is because when p increasing, the w is decreasing, then $(n - 1)w$ is decreasing, it lead to the impact of b is increasing. When $p = 100$, the we can improve the system in about 0.6% at best condition and about 5.7% at the worst condition.

(a) The Relative Improvement of the Best Condition (b) The Relative Improvement of the Worst Condition

Figure 2.6: The Relative Improvement of the Best and Worst Condition by Using Pricing Strategy

The In uence of the Change for the Customer Demand

In this section, we make changes to the customer demand for different users. For some theoretical analysis, we list them in following. It is the theory for the special case with two users and two appliances. We compare their in uence to the total cost in minimization case and maximization case under different customer demand. In numerical result, we make changes to the demand by the following rules, $D_2^0 = D_2 \quad s$ or $D_4^0 = D_4 \quad s$, here D_2 is $D_{1;2}$ and D_4 is $D_{2;2}$ in table 2.1. The results are shown in the following gures. Figure 2.7 and 2.8 shows the results for the w_1/w_2 and the relative minimization/maximization cost when we change the customer demand. We can know that in our example, the change of D_2 has more in uence to the system compare than the change of D_4 no matter in the minimum or the maximum cases. And the relation of w_1/w_2 and the relative minimization/maximization cost have linear relation respect to the change of demand with sensitivity analysis.

In the case with two users and two appliances model, the de nition of the variables and the preference time period are shown in Table 2.6. We will name this model as (Sim) and we have our model as

Table 2.6: The De nition of Variables for the Simple Example Used for Sensitivity Analysis

Variables	T_1	T_2	Preference
Person 1	x_{11}	x_{12}	T_1
Person 2	x_{21}	x_{22}	T_2

$$
\begin{aligned}
\text{(SW1):}\quad & w_1 = \min \quad [c_1 \ (x_{11} + x_{21}) + c_0] \ x_{11} + [c_1 \ (x_{12} + x_{22}) + c_0] \ x_{12} \quad p_1 \ x_{11} \\
& \text{s.t.} \quad x_{11} + x_{12} = D_1; \\
& \qquad x_{21} + x_{22} = D_2; \\
& \qquad x_{11}; x_{12}; x_{21}; x_{22} \quad 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{(SW2):}\quad & w_2 = \min \quad [c_1 \ (x_{11} + x_{21}) + c_0] \ x_{21} + [c_1 \ (x_{12} + x_{22}) + c_0] \ x_{22} \quad p_1 \ x_{22} \\
& \text{s.t.} \quad x_{11} + x_{12} = D_1; \\
& \qquad x_{21} + x_{22} = D_2; \\
& \qquad x_{11}; x_{12}; x_{21}; x_{22} \quad 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{(Sim):}\quad & \min = \max \quad (c_1 \ (x_{11} + x_{21}) + c_0) \ (x_{11} + x_{21}) + (c_1 \ (x_{12} + x_{22}) + c_0) \ (x_{12} + x_{22}) \\
& \text{s.t.} \quad x_{11} + x_{12} = D_1; \\
& \qquad x_{21} + x_{22} = D_2; \\
& \qquad (c_1 \ (x_{11} + x_{21}) + c_0) \ x_{11} + (c_1 \ (x_{12} + x_{22}) + c_0) \ x_{12} \quad p_1 x_{11} \quad r \ w_1; \\
& \qquad (c_1 \ (x_{11} + x_{21}) + c_0) \ x_{21} + (c_1 \ (x_{12} + x_{22}) + c_0) \ x_{22} \quad p_2 x_{22} \quad r \ w_2; \\
& \qquad x_{11}; x_{12}; x_{21}; x_{22} \quad 0;
\end{aligned}
$$

Where c_1, c_0 is the coef cient for the unit energy price, and p_1, p_2 is the relative preference coef cient for the users, D_1, D_2 is the relative customer demand.

Theorem 3. It has more in uence to the optimal value of the problem (Sim) when we change D_1 compare to we change D_2, when any one of the following two conditions is achieved.

Condition 1, $2D_1 \quad D_2 + p_1 = c_1$, $2D_2 \quad D_1 + p_2 = c_1$, $2D_2 \quad (p_2 \quad c_0) = c_1$.

Condition 2, $2D_1 \quad D_2 + p_1 = c_1$, $2D_1 \quad (p_1 \quad c_0) = c_1$, $2D_2 \quad D_1 + p_2 = c_1$, $2D_2 \quad (p_2 \quad c_0) = c_1$.

Proof. First we make the de nition that $(x^1_{11}; x^1_{12}; x^1_{21}; x^1_{22})$ is the optimal solution to calculate w_1, and $(x^2_{11}; x^2_{12}; x^2_{21}; x^2_{22})$ is the optimal solution to calculate w_2, and

$$f_1 = (c_1 \ (x_{11} + x_{21}) + c_0) \ x_{11} + (c_1 \ (x_{12} + x_{22}) + c_0) \ x_{12} \quad p_1 \ x_{11};$$

$$f_2 = (c_1 \ (x_{11} + x_{21}) + c_0) \ x_{21} + (c_1 \ (x_{12} + x_{22}) + c_0) \ x_{22} \quad p_2 \ x_{22};$$

We have that $x^1_{11} \quad x^1_{12}$, because if not we can choose $(\tilde{x}^1_{11}; \tilde{x}^1_{12}; \tilde{x}^1_{21}; \tilde{x}^1_{22}) = (x^1_{12}; x^1_{11}; x^1_{22}; x^1_{21})$ which is also feasible to the problem $SW1$, but the value of the objective function is obviously less than the optimal value with $(x^1_{11}; x^1_{12}; x^1_{21}; x^1_{22})$, that is contradict to it is the optimal solution to problem $SW1$. Now obviously we have $(x^1_{12}; x^1_{22}) = (0; D_2)$, this is because rst the feasible set for the person 1 and person 2 is separated. Second, the coef cient for x_{12} is greater or equal to the coef cient for x_{22}. Now if we let $x_{12} = D_1 \quad x_{11}$, then

$$f_1(x_{11}) = c_1 \ (x_{11} \ x_{11} + (D_1 \quad x_{11} + D_2) \ (D_1 \quad x_{11})) + c_0 \ D_1 \quad p_1 \ x_{11}$$

Then

$$d(f_1(x_{11})) = d(x_{11}) = 4c_1 x_{11} \quad (2c_1 D_1 + c_1 D_2 + p_1)$$

$$d^2(f_1(x_{11})) = d(x_{11})^2 = 4c_1 > 0:$$

So if we let $d(f_1(x_{11})) = d(x_{11}) = 0$ and the solution $(x^1_{11}; x^1_{12}) = (D_1 = 2 + (D_2 + p_1 = c_1) = 4; D_1 = 2 \ (D_2 + p_1 = c_1) = 4)$ for this is also feasible for the problem $SW1$, then this solution must be the optimal solution for $SW1$, if the solution not optimal for problem $SW1$, then $(x^1_{11}; x^1_{12}) = (D_1; 0)$. The

feasibility need that $2D_1 \quad D_2 + p_1 = c_1$, so we can get the result that

$$w_1 = \begin{cases} c_1 D_1^2 + c_0 D_1 \quad p_1 D_1; & 2D_1 \quad D_2 + p_1 = c_1 \\ c_1 \left((D_1 = 2 + (D_2 + p_1 = c_1) = 4)^2 + \right. & \\ \left. (D_1 = 2 + (D_2 + p_1 = c_1) = 4 + D_2) \right) \left((D_1 = 2 + (D_2 + p_1 = c_1) = 4) \right) & \\ + c_0 D_1 \quad p_1 (D_1 = 2 + (D_2 + p_1 = c_1) = 4); & 2D_1 \quad D_2 + p_1 = c_1 \end{cases}$$

And similarly for the problem SW2, we can get the result

$$w_2 = \begin{cases} c_1 D_2^2 + c_0 D_2 \quad p_2 D_2; & 2D_2 \quad D_1 + p_2 = c_1 \\ c_1 \left((D_2 = 2 + (D_1 + p_2 = c_1) = 4)^2 + \right. & \\ \left. (D_2 = 2 + (D_1 + p_2 = c_1) = 4 + D_1) \right) \left((D_2 = 2 + (D_1 + p_2 = c_1) = 4) \right) & \\ + c_0 D_2 \quad p_2 (D_2 = 2 + (D_1 + p_2 = c_1) = 4); & 2D_2 \quad D_1 + p_2 = c_1 \end{cases}$$

For condition 1,

$$\begin{aligned} d(w_1)=d(D_1) &= c_1(4D_1 + 2D_2 \quad 2p_1 = c_1) = 4 + c_0; \\ d(w_1)=d(D_2) &= c_1(2D_1 \quad D_2 \quad p_1 = c_1) = 4; \\ d(w_2)=d(D_1) &= 0; \\ d(w_2)=d(D_2) &= 2c_1 D_2 + c_0 \quad p_2; \end{aligned}$$

And

$$\begin{aligned} d(w_1)=d(D_1) \quad d(w_1)=d(D_2) &= c_1(2D_1 + 3D_2 \quad p_1 = c_1) = 4 + c_0 \quad 0; \\ d(w_2)=d(D_1) \quad d(w_2)=d(D_2) &= (2c_1 D_2 + c_0 \quad p_2) \quad 0; \end{aligned}$$

For condition 2,

$$d(w_1)=d(D_1) \quad = \quad 2c_1D_1 + c_0 \quad p_1;$$

$$d(w_1)=d(D_2) \quad = \quad 0;$$

$$d(w_2)=d(D_1) \quad = \quad 0;$$

$$d(w_2)=d(D_2) \quad = \quad 2c_1D_2 + c_0 \quad p_2;$$

And

$$d(w_1)=d(D_1) \quad d(w_1)=d(D_2) \quad = \quad 2c_1D_1 + c_0 \quad p_1 \quad 0;$$

$$d(w_2)=d(D_1) \quad d(w_2)=d(D_2) \quad = \quad (2c_1D_2 + c_0 \quad p_2) \quad 0;$$

So either the condition 1 or condition 2 happens, we will have that the change for w_1, or w_2 by changing D_1 is larger than by changing D_2. So the in uence is greater to the optimal value of the problem (Sim) when we change D_1 compare to we change D_2.

□

Figure 2.7: The In uence of Optimal Utility W by Change of Demand

Figure 2.8: The In uence of Total Cost by Change of Demand

Compare of the Game Theory Model and BRUE with Pricing Strategy

In this section, we will compare the result of different pricing strategy that relatively determined by the game theory model and the BRUE model. We will rst calculate the optimal price under

the game theory model. Then we use this price to our BRUE model and get the result under such pricing strategy. And we can get what is the error that we will get if we use the game theory model to determine the pricing strategy.

(a) Compare of Game Theory (GT) and BRUE with Different r and p

(b) Difference(%) of Game Theory and BRUE with Different r and p

Figure 2.9: The Difference of BRUE and Game Theory Model

Figure 2.9(a) gives the difference of the cost when we use the price that get from the game theory model to our BRUE model for different r and different p. We can know that all of the difference is positive.

CHAPTER 3: ROBUST OPTIMIZATION WITH SURPLUS PRICE TRANSPORTATION UNDER BOUNDEDLY RATIONALITY USER EQUILIBRIUM

NOMENCLATURE

A. Sets, Indices, Parameters and Variables for the Static Model

$I; J$ Set of the node in the network, indexed by i, j

$(I; J)$ Set of arcs, indexed by $(i; j)$

$O; D$ Set of original and destination, indexed by o, d

$(O; D)$ Set of arcs, indexed by (od)

W_{OD} Possible pairs of routes for (od), indexed by w_{od}

B. Parameters

D_{od} Demand for (od)

$a_{ij;w_{od}}$ Binary value equals to 1 when the path w_{od} go through arc $(i; j)$. Else equals to 0.

$b_{i;j}$, $k_{i;j}$ Relative coef cient of arc $(i; j)$ for the Time function.

C. Variables

$x_{r;(od);w_{od}}$ Flows go into the system with rationality on (od) on the path w_{od}

f_{ij} Flows on arc $(i; j)$

Binary b_{ij} Whether or not to change the direction of the road of (i,j)

U_{od} Minimum utility for (od)

D. Functions

$T(\ )$ Time cost function

Introduction

In recent world, the transportation methods are very important to our lives. We need transport the food, clothes, daily necessities and so on everyday all over the world. Also we have a lot of different kinds of methods to transport them, such as through the air, rail, road, water or tube. In this paper, we will emphasis on the road transportation. In 2010, there were over 1 billion automobiles in the world [93]. Road transport undertakes the most part of the transportation. We know that under some conditions, the road will become crowed and the travelling speed will reduce. Such as when the evacuation happens, the ows on the road will in uence the evacuation speed dramatically. There were some research in this eld before [76, 75, 94, 84]. But only few researchers [56] consider the boundedly rational to the problem and models. We will solve our problem under the boundedly rational user equilibrium (BRUE) by introducing the pricing strategy to our models. Different with Lou's work [56] with arc based ows, our model is path based ows, which is the actually conditions for the BRUE model.

BRUE model is proposed by Simon in year 1957 [77, 80, 81, 79], which means for one individual, when the difference of the utilities of different options that the individual can choose are below

a level, this individual will regard the utilities of such different options as the same. He or she may choose any options within that level as his or her nal decisions. Here in our model of the transportation system, the utility includes the travel time and the surplus price. The concept of boundedly rational can be used in many elds, such as the energy system [96], psychology [44], military [69], transportation [58, 24, 56]and so on [70, 58, 36, 27]. When a individual in the transportation system, the individual will have a bunch of choices for the path go from the original place to the destination. Different pathes will have different utilities. And the individual may choose any path that ful ll the BRUE constraints. It means that for the pathes that do not ful ll the BRUE constraints, the ows will be zero. Actually the difference between BRUE and game theory is just the tolerance level. In the well known game theory model, the tolerance level can only be zero. But in the BRUE model, the tolerance can be greater or equal to zero. So game theory is just a special case of BRUE. Some former research was also done under the game theory [3] to the energy system. And also some work was done under game theory to the transportation system [32, 9, 61]. The transportation problem is a classical illustration for the game theory model.

As discussed above, the ows maybe only aggregate in some pathes. This will lead the time cost for the total system large. We introduced the pricing strategy to disperse the ows. The pricing strategy is a well known method [49, 51] to make the system work better. Especially in the transportation system [54, 60], by introducing the price to the arcs in the network, we can control the behaviour of the individuals under the BRUE conditions by controlling the price that we can determine for each arc. But after introducing the price to the system, our model will be a two level optimization model. The outer is to make minimization for the price. And the inner level is to make maximization under the worst case and minimization under the best case for the individual behaviour. When we want to consider the worst case, actually our model become a robust optimization model, the price become the robust price. We gave the algorithms [99] how to solve the

robust problem.

In our models, we proposed a path ow based system. It means that our main decision variables are the ows for each path. As we know, the number of pathes have the exponential relations with the number of arcs in the system. So the number of pathes are actually a large value even if the network itself is not so large. It will be hard for us to solve the problem directly, then we use the column generation and branch and price method to solve this problem. The column generation [6, 22, 91] is a well known method to solve the big problem. And in our problem, originally the BRUE constraints are complementary nonlinear constraints, we make transform for them and make them to the mixed integer linear problem(MILP). Because of the integer introduced to our model, we also use the branch and price method [74, 5] to solve our problem.

And in our models, we also use a method to nd the kth shortest path in a known network. we found some former researchers who also get some results for this. The rst article for this problem was done by Hoffman and Pavley [38] in 1959. Yen [98] did this in year 1971, the author gave us a method that with the computational linear time relations to the number of k. Afm et al.'s work [88] gave us a method to nd out all of the pathes by order between two nodes in the network, and apparently it can also nd out the kth shortest path in the system. By using Eppstein's method [28] we can also nd out the kth shortest path in time $O(m + n\log n + k)$, where m is the number edges in the network and n is the number of nodes in the network. Aljazzar and Leue's work [1] did this by using the heuristic method to solve this problem. In their methods, we do not need to store the whole graph in the main memory, just part of the graph need to be generated. So their advantage is that they do not need too much memory to calculate and also less time for calculation. But their disadvantage is their method is heuristic method. Also we can nd some other researcher did the work in this direction [2, 35, 16, 40, 41]. Because of the work [62], in general, people can only make choice from 7-2 to 7+2. Thus, we only need to consider the value of k from 5 to 9.

We proposed a model for the static transportation path ow based problem when the evacuation happens. We supposed that all the users in the system will obey the BRUE constraint. We totally consider about four cases 1, Best case without the price. 2, Worst case without the price. 3, Best case with the price. 4, Worst case with the price. For the st two cases, we want to nd if we do not introduce the price to our system, how best and how worst the total system can be under BRUE. For the last two cases, we tried to nd out the situations when we introduce the pricing strategy to the system. In our model, rst we changed the originally nonlinear constraints to the MILP constraints in order to use the cplex to solve. But now the objective function is still a quadratic form. The cplex still can not be used directly, then we make the linear transform as the rst level's algorithm. After this our model become the robust MILP optimization problem. Then we use the algorithm 4 for the robust part. This algorithm is our second level algorithm. After this, we change our problem to a MILP problem with huge number of variables and huge number of constraints. We use our algorithm 5 as the third level algorithm to nd and check the convergency for the columns and then use the algorithm 6 and algorithm 7 to nd and check the convergency for the constraints.

Overall, our original problem is a nonlinear problem with huge variables and huge constraints. After our method, we rst succussed change our problem to MILP robust optimization problem. And then use the linearization and other methods to decompose our problem to just a normal MILP problem which can be solved with cpelx. Our model is a path ow based model, which is the actually model for the BRUE problem. Lou's work [56] is arc ow based model, in their model, the feasible region is smaller than the actually feasible region. So their optimal solutions are actually an upper bound for the best cases and lower bound for the worst cases. But our model can get the real optimal solution. So by using our model and method, we can get the optimal solutions for the static transportation problem.

As a reminder, section 3 is our static model section, it includes ve subsections. The rst four subsections are the relative subsections for the four cases. The fth subsection is to compare our model with Lou's model [56]. Section 3 is the theorem, which will be used in the algorithm. Section 3 is our algorithm section, we totally have four algorithms, they are relatively to the four level iterations. Section 3 is the experimental results for four node example, nine node example and sioux network. The nal section is the conclusion and future work.

BRUE For the Static Network Models

We proposed totally four conditions for our static models. In rst two conditions, we do not use the pricing strategy to optimize our system. In the last two conditions, we illustrated the pricing strategy to see how the system will work. And for the worst case by using pricing strategy, we will have a robust optimization model.

Case 1, For the best condition without the surplus price

In this case we will make the model for the best condition without using the price to system. The model is in follows,

(B-BRUE):

$$\min \sum_{r} \sum_{(od) \in (OD)} \sum_{w_{od} \in W_{OD}} \sum_{(i;j) \in (I;J)} (a_{ij;w_{od}} \quad g_{ij} \quad x_{r;(od);w_{od}}) \tag{3.1a}$$

$$\text{s.t.} \sum_{w_{od} \in W_{OD}} x_{r;(od);w_{od}} = D_{r;od}; \quad 8r;(od) \in (OD) \tag{3.1b}$$

$$g_{ij} = c_{ij} + k_{ij} \quad f_{ij}^{p}; \quad 8(i;j) \in (I;J) \tag{3.1c}$$

$$f_{ij} = \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{r} (a_{ij;w_{od}} \quad x_{r;(od);w_{od}}); \quad 8(i;j) \in (I;J) \tag{3.1d}$$

$$x_{r;(od);w_{od}} \quad \left(\sum_{(i;j)\in(I;J)} (a_{ij;w_{od}} \quad g_{ij}) \quad U_{od} \quad r \right) \quad 0; \quad 8r;(od) \in (OD); w_{od} \in W_{OD}$$

$$\tag{3.1e}$$

$$\sum_{(i;j)\in(I;J)} (a_{ij;w_{od}} \quad g_{ij}) \quad U_{od} \quad 0; \quad 8(od) \in (OD); w_{od} \in W_{OD} \tag{3.1f}$$

$$x_{r;(od);w_{od}} \quad 0; \quad 8r;(od) \in (OD); w_{od} \in W_{OD} \tag{3.1g}$$

(3.1a) is the objective function contains the summation of all arcs' ows times the time cost on that arc. (3.1b)(3.1c)(3.1d) are the relative constraints for the demand, time cost function and the time cost for arc $(i\,j)$. (3.1e) and (3.1f) are the BRUE constraint. (3.1g) is the constraint to let all of the ows to be nonnegative.

Case 2, For the worst condition without the surplus price

In this case we will make the model for the worst condition without using the price to system. The model is in follows,

(W-BRUE):

$$\max \sum_{r} \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{(i;j)\in(I;J)} (a_{ij;w_{od}} \quad g_{ij} \quad x_{r;(od);w_{od}}) \tag{3.2a}$$

$$\text{s.t.} \sum_{w_{od}\in W_{OD}} x_{r;(od);w_{od}} = D_{r;od}; \quad 8r;(od) \in (OD) \tag{3.2b}$$

$$g_{ij} = c_{ij} + k_{ij} \quad f_{ij}^{p}; \quad 8(i;j) \in (I;J) \tag{3.2c}$$

$$f_{ij} = \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{r} (a_{ij;w_{od}} \quad x_{r;(od);w_{od}}); \quad 8(i;j) \in (I;J) \tag{3.2d}$$

$$x_{r;(od);w_{od}} \quad M \quad z_{r;(od);w_{od}} \tag{3.2e}$$

$$x_{r;(od);w_{od}} \quad M \quad (1 \quad z_{r;(od);w_{od}}) + d \tag{3.2f}$$

$$\sum_{(i,j)\in(I,J)} (a_{ij,w_{od}} \quad g_{ij}) \quad U_{od} \quad r \quad M \quad (1 \quad z_{r,(od),w_{od}}), \forall r, (od) \in (OD), w_{od} \in W_{OD} \tag{3.2g}$$

$$\sum_{(i,j)\in(I,J)} (a_{ij,w_{od}} \quad g_{ij}) \quad U_{od} \quad 0, \quad \forall (od) \in (OD), w_{od} \in W_{OD} \tag{3.2h}$$

$$x_{r,(od),w_{od}} \quad 0, \quad \forall r, (od) \in (OD), w_{od} \in W_{OD} \tag{3.2i}$$

We can know that actually in these two models, the only difference should be just the rst one is to make minimization of the objective function and the second one is to make maximization of the objective function. But we make another form for the second case. In the rst case, our model for the BRUE constraint is nonlinear constraint. But in the second case, we change our model to a linear model by illustrating the big M method. But actually they show the same constraints. (3.2e)(3.2f)(3.2g)(3.2h) are the constraints to ful ll this. Here is a small positive value.

Case 3, For the best condition with the surplus price

In this case we will make the model for the best condition by using the price to system. The model is in follows,

(BP-BRUE):

$$\min_{b} \min_{x,z} \sum_{r} \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{(i,j)\in(I,J)} (a_{ij,w_{od}} \quad g_{ij} \quad x_{r,(od),w_{od}}) \tag{3.3a}$$

$$\text{s.t.} \sum_{w_{od}\in W_{OD}} x_{r,(od),w_{od}} = D_{r,od}, \quad \forall r, (od) \in (OD) \tag{3.3b}$$

$$g_{ij} = c_{ij} + k_{ij} \quad f_{ij}^{p}, \quad \forall (i,j) \in (I,J) \tag{3.3c}$$

$$f_{ij} = \sum_{(od)\in(OD)} \sum_{w_{od}\in W_{OD}} \sum_{r} (a_{ij,w_{od}} \quad x_{r,(od),w_{od}}), \quad \forall (i,j) \in (I,J) \tag{3.3d}$$

$$x_{r;(od);w_{od}} \quad M \quad z_{r;(od);w_{od}} \tag{3.3e}$$

$$x_{r;(od);w_{od}} \quad M \quad (1 \quad z_{r;(od);w_{od}}) + d \tag{3.3f}$$

$$\sum_{(i;j)2(I;J)} (a_{ij;w_{od}} \quad g_{ij}) + b \quad U_{od} \quad r \quad M \quad (1 \quad z_{r;(od);w_{od}}); 8r;(od) \ 2 \ (OD); w_{od} \ 2 \ W_{OD} \tag{3.3g}$$

$$\sum_{(i;j)2(I;J)} (a_{ij;w_{od}} \quad g_{ij}) + b \quad U_{od} \quad 0; \quad 8(od) \ 2 \ (OD); w_{od} \ 2 \ W_{OD} \tag{3.3h}$$

$$x_{r;(od);w_{od}} \quad 0; \quad 8r;(od) \ 2 \ (OD); w_{od} \ 2 \ W_{OD} \tag{3.3i}$$

This is add the price to the system, we add the minimization of b to the objective function, and also add the price b to the relative constraint. (3.3g)(3.3h) are the constraints have relation with beta

Case 4, For the worst condition with the surplus price

In this case we will make the model for the worst condition by using the price to system. The model is in follows, it is a robust optimization model.

(WP-BRUE):

$$\min_{b} \max_{x;z} \sum_{r} \sum_{(od)2(OD)} \sum_{w_{od}2W_{OD}} \sum_{(i;j)2(I;J)} (a_{ij;w_{od}} \quad g_{ij} \quad x_{r;(od);w_{od}}) \tag{3.4a}$$

$$\text{s.t.} \sum_{w_{od}2W_{OD}} x_{r;(od);w_{od}} = D_{r;od}; \quad 8r;(od) \ 2 \ (OD) \tag{3.4b}$$

$$g_{ij} = c_{ij} + k_{ij} \quad f_{ij}^{p}; \quad 8(i;j) \ 2 \ (I;J) \tag{3.4c}$$

$$f_{ij} = \sum_{(od)2(OD)} \sum_{w_{od}2W_{OD}} \sum_{r} (a_{ij;w_{od}} \quad x_{r;(od);w_{od}}); \quad 8(i;j) \ 2 \ (I;J) \tag{3.4d}$$

$$x_{r;(od);w_{od}} \quad M \quad z_{r;(od);w_{od}} \tag{3.4e}$$

$$x_{r;(od);w_{od}} \quad M \quad (1 \quad z_{r;(od);w_{od}}) + d \tag{3.4f}$$

$$\sum_{(i;j)2(I;J)} (a_{ij;w_{od}} \quad g_{ij}) + b \quad U_{od} \quad r \quad M \quad (1 \quad z_{r;(od);w_{od}}); 8r; (od) \; 2 \; (OD); w_{od} \; 2 \; W_{OD} \tag{3.4g}$$

$$\sum_{(i;j)2(I;J)} (a_{ij;w_{od}} \quad g_{ij}) + b \quad U_{od} \quad 0; \quad 8(od) \; 2 \; (OD); w_{od} \; 2 \; W_{OD} \tag{3.4h}$$

$$x_{r;(od);w_{od}} \quad 0; \quad 8r; (od) \; 2 \; (OD); w_{od} \; 2 \; W_{OD} \tag{3.4i}$$

This is a robust optimization model, which we need to rst make the minimization for the price p and then make the maximization for the path ows x.

As we given before the ows on arc $(i; j)$ are $f_{ij} = \sum_r \sum_{od2OD} \sum_{w_{od}2W_{OD}} a_{ij;w_{od}} \quad x_{r;(od);w_{od}}$. The time function for the arc $(i; j)$ is $T(ij) = b_{ij} + k_{ij} \quad f_{ij}$. And $T(w_{od}) = \sum_{(ij)2(IJ)} a_{ij;w_{od}} \quad T(ij)$, $T(w_{od})$ is the time cost for the path w_{od}. Now the total system time cost function $f(x) = \sum_{(ij)2(IJ)} T(ij) \quad f_{ij}$.

We can get the result

Gradient of $f(x)$:

$$\tilde{N}_{x_i} f(x) = 2 \quad T(w_{od}) \quad \sum_{(ij)2(IJ)} a_{ij;w_{od}} \quad b_{ij} \tag{3.5}$$

To be simplify, we suppose in our system we only have one group of people who have the same boundedly rational coef cient t and one (OD) pair. There are totally n possible pathes in the

system. The time cost for each path denoted as $\{T(1); T(2); \quad ; T(k); T(k+1); \quad ; T(n)\}$. Only the rst k path has ows. Now the relative formulations becomes

Simplify formulations :

$$f_{ij} = \sum_{i=1}^{n} a_{ij;i} \quad x_i \tag{3.6a}$$

$$T(ij) = b_{ij} + k_{ij} \quad f_{ij} \tag{3.6b}$$

$$T(i) = \sum_{(ij)2(IJ)} a_{ij;i} \quad T(ij) \tag{3.6c}$$

$$f(x) = \sum_{(ij)2(IJ)} \left(\sum_{i=1}^{n} a_{ij;i} \quad x_i \quad b_{ij} + k_{ij} \sum_{i=1}^{n} a_{ij;i} \quad x_i \right) \tag{3.6d}$$

$$\tilde{N}_{x_i} f(x) = 2 \quad T(i) \sum_{(ij)2(IJ)} a_{ij;i} \quad b_{ij} \tag{3.6e}$$

In order to use the algorithms in the following, we want to rst show the theorem 1, 2, 3

Theorem 1. The optimal solution for the maximization problem must have the property that

$$\max_{1 \quad i;j \quad k} \{T(i) \quad T(j)\} = r$$

For the non-trivial conditions. Proof.

First, because of the constraints for the BRUE restrictions we must have

$$\max_{1 \quad i;j \quad k} \{T(i) \quad T(j)\} \quad r$$

Second, suppose if

$$\max_{1 \quad i;j \quad k} \{ |T(i) \quad T(j)| \} < r$$

Then Without loss of generality, we can suppose

$$T(1) \quad \frac{1}{2} \sum_{(ij) \in (IJ)} a_{ij;1} \; b_{ij} \qquad T(2) \quad \frac{1}{2} \sum_{(ij) \in (IJ)} a_{ij;2} \; b_{ij} \qquad\qquad T(k) \quad \frac{1}{2} \sum_{(ij) \in (IJ)} a_{ij;k} \; b_{ij}$$

For as > 0, we can let

$$\begin{aligned} x^0(k) &= x(k) + s \quad 0; \\ x^0(1) &= x(1) \quad s \quad 0; \\ x^0(i) &= x(i) \quad 0; \quad 8i = 2;3; \quad ;k \quad 1;k+1; \quad ;n \end{aligned}$$

We denote

$$4\,T(i) = T^0(i) \quad T(i)$$

as the difference of the time cost for the path i when we change the solution from x to x^0. Then we can know that

$$4\,T(i) = \sum_{(ij) \in (IJ)} k_{(ij)} \; a_{ij;i} \; a_{ij;1} \; (\quad s) + \sum_{(ij) \in (IJ)} k_{(ij)} \; a_{ij;i} \; a_{ij;k} \; s = K(i) \; s$$

Where $K(i)$ is the relative constant coef cient for the path i

$$K(i) = \sum_{(ij) \in (IJ)} k_{(ij)} \; a_{ij;i} \; (a_{ij;k} \quad a_{ij;1}) \; s$$

We can know that $4\ T(1) \quad 0; 4\ T(k) \quad 0$, this is because

$$a_{ij;1}\ (a_{ij;k} \quad a_{ij;1}) \ = \ a_{ij;1}\ (a_{ij;k} \quad 1) \quad 0$$

$$a_{ij;k}\ (a_{ij;k} \quad a_{ij;1}) \ = \ a_{ij;k}\ (1 \quad a_{ij;1}) \quad 0$$

$$k_{(ij)} \quad 0 \quad ; \quad s > 0$$

Then

$$4\ T(1) \ = \ K(1)s \quad 0$$

$$4\ T(k) \ = \ K(k)s \quad 0$$

We know that

$$T^0(i) \quad T^0(j) \ = \ jT(i) + 4\ T(i) \quad T(j)\ 4 \quad T(j)j \quad jT(i) \quad T(j)j + j4\ T(i)\ 4 \quad T(j)j$$

Then

$$\max_{1\ i;j\ k} f\ T^0(i) \quad T^0(j)\ g \qquad \max_{1\ i;j\ k} f jT(i) \quad T(j)j + j4\ T(i)\ 4 \quad T(j)jg$$

$$\max_{1\ i;j\ k} f jT(i) \quad T(j)jg + \max_{1\ i;j\ k} f j4\ T(i)\ 4 \quad T(j)jg$$

And because

$$j4\ T(i)\ 4 \quad T(j)j = j(K(i) \quad K(j))j\ s$$

So

$$\max_{1\ i;j\ k} f j4\ T(i)\ 4 \quad T(j)jg = f \max_{1\ i;j\ k} f j(K(i) \quad K(j))jgg\ s = Ks$$

Where

$$K = \max_{1 \le i;j \le k} \{|K(i) - K(j)|\}$$

is a constant value, as we suppose before,

$$\max_{1 \le i;j \le k} \{|T(i) - T(j)|\} < r$$

So we can always find small enough s to let

$$\max_{1 \le i;j \le k} \{|T^0(i) - T^0(j)|\} < r + Ks$$

$$\max_{1 \le i;j \le k} \{|T^0(i) - T^0(j)|\} \le r$$

It means that the current solution x^0 are also a feasible solution to the primal BRUE problem. And as we know, if now the solution x is the optimal solution for the maximization problem. We can get the optimal value as

$$\begin{aligned} f(x^0) &= f(x) + \tilde{\nabla} f(x)\,(x^0 - x) \\ &= f(x) + 2\,T(k)\left(\frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;k}\,b_{ij}\,s\right) - 2\,T(1)\left(\frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;1}\,b_{ij}\,s\right) \\ &> f(x) \end{aligned}$$

It is contradict to the conclusion x is the optimal solution for the maximization problem. So the former suppose

$$\max_{1 \le i;j \le k} \{|T(i) - T(j)|\} < r$$

is not right. So we can just have

$$\max_{1 \le i;j \le k} \{|T(i) - T(j)|\} = r$$

Proved

Theorem 2. For minimization problem, if

$$\max_{1 \le m,l \le n} \sum_{(i,j) \in (I,J)} a_{ij;m} \; a_{ij;l} \; b_{ij} < 2r \tag{3.7a}$$

then we can always find an optimal solution has the following property

$$T(m) \quad \frac{1}{2} \sum_{(i,j) \in (I,J)} a_{ij;m} \; b_{ij} = \overline{T}; \quad \forall \{ m | x_m > 0 \} \tag{3.7b}$$

$$\max_{1 \le i,j \le k} \{ | T(i) \quad T(j) | \} < r \tag{3.7c}$$

$$T(l) \quad \frac{1}{2} \sum_{(i,j) \in (I,J)} a_{ij;l} \; b_{ij} \quad \overline{T}; \quad \forall \{ l | x_l = 0 \} \tag{3.7d}$$

Else if

$$\max_{1 \le m,l \le n} \sum_{(i,j) \in (I,J)} a_{ij;m} \; a_{ij;l} \; b_{ij} \quad 2r \tag{3.7e}$$

then

$$\max_{\{ m,l | x_m, x_l > 0 \}} | T(m) \quad T(l) | = r \tag{3.7f}$$

Proof.

If the formulation (3.7a) happens, suppose the condition (3.7b) is not right. And now the optimal solution is x. If now

$$\max_{\{ m,l | x_m, x_l > 0 \}} | T(m) \quad T(l) | < r$$

Then similar as the prove in Theorem 1, without loss of generality, we can find $\{ p,q | x_p, x_q > 0 \}$ to

let

$$T(p) \quad \frac{1}{2} \sum_{(ij)2(IJ)} a_{ij;p} \; b_{ij} > T(q) \quad \frac{1}{2} \sum_{(ij)2(IJ)} a_{ij;q} \; b_{ij}$$

and to let

$$\begin{aligned} x^0(p) &= x(p) \quad s \quad 0; \\ x^0(q) &= x(q) + s \quad 0; \\ x^0(i) &= x(i) \quad 0; \quad 8i = 1;2;3; \quad ;n; \text{and } i \in p;q \end{aligned}$$

with

$$4\ T(p) \quad 0; 4\ T(q) \quad 0$$

From the prove of Theorem 1, now the solution x^0 is also a feasible solution for the minimization problem.

The new objective value is

$$\begin{aligned} f(x^0) &= f(x) + \tilde{N} f(x) \ (x^0 \quad x) \\ &= f(x) + 2 \ T(p) \ \frac{1}{2} \sum_{(ij)2(IJ)} a_{ij;p} \; b_{ij} \ (\quad s) + 2 \ T(q) \ \frac{1}{2} \sum_{(ij)2(IJ)} a_{ij;q} \; b_{ij} \ s \\ &< f(x) \end{aligned}$$

Contradict to the conclusion x is the optimal solution for the minimization problem.

Else if

$$\max_{\{m;l \mid x_m;x_l > 0\}} \mid T(m) \quad T(l) \mid = r$$

We denote the pair as $(p;q)$. Without loss of generality, we can suppose

$$T(p) \quad \frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;p}\ b_{ij} > T(q) \quad \frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;q}\ b_{ij} \tag{3.8}$$

We also make the same change for the variable x. Then if $T(p) \quad T(q)$, we have

$$T(p) \quad T(q) = r\ ;\ 4\ T(p) \quad 0;\ 4\ T(q) \quad 0$$

so now $T^0(p) \quad T^0(q) \quad r$.

Else if $T(p) \quad T(q)$, then we can know that $T(q) \quad T(p) = r$, from condition (3.8), we can get

$$r = T(q) \quad T(p) < \frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;q}\ b_{ij} \quad \frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;p}\ b_{ij}$$

contradict to our primal suppose formulation (3.7a). So this condition can not be happen. x^0 is feasible for the problem. Then similar as before, we can know that x is not the optimal solution which is contradict to the former suppose. So in any cases we discussed, we must have the condition (3.7b).

Suppose condition (3.7c) is not right, then

$$\max_{1\ i;j\ k} \{|T(i) \quad T(j)|\} = r$$

To let

$$\max_{1\ i;j\ k} \{|T(i) \quad T(j)|\} = |T(p) \quad T(q)|$$

Because of (3.7b), we will have

$$\sum_{(ij)\in(IJ)} a_{ij;p} \quad a_{ij;q} \quad b_{ij} = 2r$$

Which is contradict with (3.7a). So (3.7c) is proved.

Suppose the condition (3.7d) is not right. Then for some $m \in \{j : x_j = 0\}$,

$$T(l) - \frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;l}\, b_{ij} < \bar{T} = T(m) - \frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;m}\, b_{ij} \tag{3.9}$$

Where m the index that to let $x_m > 0$. We can find small enough s to get the following transforms

$$\begin{aligned} x^0(l) &= s > 0; \\ x^0(m) &= x(m) - s > 0; \\ x^0(i) &= x(i) > 0; \quad \forall i = 1, 2, 3, \ldots, n, \text{ and } i \neq l, m \end{aligned}$$

If $T(l) \geq T(m)$, obviously, we have the solution x^0 is also feasible. Else if $T(l) < T(m)$ From (3.9) we can get

$$T(l) - T(m) < \frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;l}\, b_{ij} - \frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;m}\, b_{ij} < r$$

And because of (3.7c) and the prove of Theorem 1, we know that x^0 is feasible. And obviously

$$f(x^0) < f(x)$$

It is contradict to x is the optimal solution. So the condition (3.7d) is right.

The prove of (3.7f) is similar with the prove of Theorem 1. We do not list the details here.

Theorem 3. For the minimization problem, when the r is increasing, the objective function value will be strictly decreasing for $r \leq \bar{r}$, and when $r > \bar{r}$, the objective function value will keep the

same. Where

$$\bar{r} = \frac{1}{2} \max_{m,l \in n} \sum_{(ij) \in (IJ)} a_{ij;m} \quad a_{ij;l} \quad b_{ij} \tag{3.10}$$

Proof. When $r > \bar{r}$, it is just the condition(3.7a), actually the optimal solution has no relation with r, in such cases, the BRUE constraints actually are redundant. So even though the r is still increasing, the objective value will keep the same. When $r > \bar{r}$, it is just the condition(3.7f), now we can know that when r increasing, the feasible region is also strict enlarged, and the enlarged part of the feasible region is also be used, so the objective value is strictly increasing.

Lemma 1.

Counter part for the Difference Between Path Flow And Link-based Flow Distribution

By using the example in Lou's work [56], the data was shown in Table 3.1. The network picture was shown in 3.2.

Figure 3.1: Parameters For the Four Nodes Example

Figure 3.2: Four Nodes Flow Distribution

Link	Time function(T(ij))	Path	Relative Variables
(1,3)	$3+f_{13}$	1-3-4	x1
(1,2)	$7+f_{12}$	1-3-2-4	x2
(2,3)	$0+f_{23}$	1-2-4	x3
(2,4)	$5+f_{24}$	1-2-3-4	x4
(3,2)	$0+f_{32}$		
(3,4)	$2+f_{34}$		

If we set the ow capacity for each path to be 1. Then we can get the solution for case 1 of this problem as it shown in the table. But if we use the model presented in Lou's work. It will be infeasible for the potential value of the node 3 and 2. It is because if we set the potential of node 3 and 2 as $x; y$. Then we will have the following constraint.

$$\text{s.t.}\ 9 \quad 5 \quad x \tag{3.11a}$$

$$7 \quad 4 \quad 4 \quad x \tag{3.11b}$$

$$8 \quad 6 \quad y \tag{3.11c}$$

$$6 \quad 5 \quad 4 \quad y \tag{3.11d}$$

Obviously, this is no feasible solution for this part. So These two model are not equals at least for the minimum problem.

Algorithm

Actually for our robust optimization problem, we can simplify to write our our model as follows,

(WP-BRUE):

$$\min_{b} \max_{x;z} f(x; z) = x^T Dx + dx + e \tag{3.12a}$$

$$\text{s.t.}\ Ax + Bz + Cb \quad R \tag{3.12b}$$

(WP-BRUE):

$$\min_{b} \max_{x;z} f(x;z) = x^T Dx + dx + e \qquad (3.13a)$$

$$\text{s.t} \; Ax + Bz + Cb \quad R \qquad (3.13b)$$

To simplify, actually the problem WP-BRUES, WP-BRUEM has a lot of constraints and variables like the follows.

(BRANCHP):

$$\max_{x;z} f(x;z) = e^T x \qquad (3.14a)$$

$$\text{s.t} \; A^n x + B^n z \quad R^n \qquad (3.14b)$$

We can use our algorithm 3 to rst make the linearization of our objective function. Then use the algorithm 4 to make the robust part min-max to a master problem and a sub problem we can solve these two problems separately by using the algorithm 5. Then at last after the four level iterations, we can get our result for our case 4 problem. And for other three cases, actually we can just use part of the algorithms we posted to solve it.

To simplify state the problem for B-BRUE and W-BRUE, we can write the problem as the follows,

(P):

$$\min_{x_{l_n};z_{l_n}} \sum_{l} \sum_{n} c_{l_n}^T x_{l_n} \qquad (3.15a)$$

$$\text{s.t.} \sum_{l} \sum_{n} A_{l_n;l_n} x_{l_n} + \sum_{l} \sum_{n} B_{l_n;l_n} z_{l_n} \quad b_{l_n}; \forall l_n \tag{3.15b}$$

$$\sum_{n} E_{l;l_n} x_{l_n} = D_l; \forall l \tag{3.15c}$$

Where $A_{l_n;l_n} \in R^{l_n \times l_n}$, because this problem has huge number of l_n, we can not solve it directly, so we want to solve a subproblem of it, we just consider about the l_k paths in the system. We call it SP problem. And we define that $l_k \in J; l_n = l_k \in J^0$, J is the set that $\{x_{l_k} > 0; z_{l_k} = 1 | l_k \in J\}$

(SP):

$$\min_{x_{l_k}; z_{l_k}} \sum_{l} \sum_{k} c_{l_k}^T x_{l_k} \tag{3.16a}$$

$$\text{s.t.} \sum_{l} \sum_{k} A_{l_k;l_k} x_{l_k} + \sum_{l} \sum_{k} B_{l_k;l_k} z_{l_k} \quad b_{l_k}; \forall l_k \tag{3.16b}$$

$$\sum_{k} E_{l;l_k} x_{l_k} = D_l; \forall l \tag{3.16c}$$

(SPC):

$$\min_{x_{l_k}; z_{l_k}} \sum_{l} \sum_{k} c_{l_k}^T x_{l_k} \tag{3.17a}$$

$$\text{s.t.} \sum_{l} \sum_{k} A_{l_k;l_k} x_{l_k} + \sum_{l} \sum_{k} B_{l_k;l_k} e_{l_k} \quad b_{l_k}; \forall l_k \tag{3.17b}$$

$$\sum_{k} E_{l;l_k} x_{l_k} = D_l; \forall l \tag{3.17c}$$

Where e_{l_k} is a vector has l_k elements with 1.

As the definition of the set J above, we know that $SP = SPC$. And we also establish two other problems. called SPC_1 and SPC_2.

(SPC_1):

$$\min_{x_{l_k};z_{l_k}} \sum_{l} \sum_{k} c_{l_k}^T x_{l_k} + \sum_{l} c_{l_{k+1}} x_{l_{k+1}} \tag{3.18a}$$

$$\text{s.t.} \sum_{l} \sum_{k} A_{l_k;l_k} x_{l_k} + \sum_{l} \sum_{k} B_{l_k;l_k} e_{l_k} +$$

$$\sum_{l} a_{l_k;l_{k+1}} x_{l_{k+1}} + \sum_{l} b_{l_k;l_{k+1}} \quad b_{l_k}; \forall l_k \tag{3.18b}$$

$$\sum_{k} E_{l;l_k} x_{l_k} = D_l; \forall l \tag{3.18c}$$

(SPC_2):

$$\min_{x_{l_k};z_{l_k}} \sum_{l} \sum_{k} c_{l_k}^T x_{l_k} \tag{3.19a}$$

$$\text{s.t.} \sum_{l} \sum_{k} A_{l_k;l_k} x_{l_k} + \sum_{l} \sum_{k} B_{l_k;l_k} e_{l_k} \quad b_{l_k}; \forall l_k \tag{3.19b}$$

$$\sum_{l} a_{l_{k+1};l_k} x_{l_k} + \sum_{l} b_{l_{k+1};l_k} e_{l_k} \quad b_{l_{k+1}}; \tag{3.19c}$$

$$\sum_{k} E_{l;l_k} x_{l_k} = D_l; \forall l \tag{3.19d}$$

$(SPCN_1)$:

$$\min_{x_{l_n};z_{l_n}} \sum_{l} \sum_{n} c_{l_n}^T x_{l_n} \tag{3.20a}$$

$$\text{s.t.} \sum_{l} \sum_{n} A_{l_k;l_n} x_{l_n} + \sum_{l} \sum_{n} B_{l_k;l_n} z_{l_n} \quad b_{l_n}; \forall l_k \tag{3.20b}$$

$$\sum_{n} E_{l;l_n} x_{l_n} = D_l; \forall l \tag{3.20c}$$

(SPCN$_2$):

$$\min_{x_{l_k}; z_{l_k}} \sum_{l} \sum_{k} c_{l_k}^T x_{l_k} \tag{3.21a}$$

$$\text{s.t.} \sum_{l} \sum_{k} A_{l_n;l_k} x_{l_k} + \sum_{l} \sum_{k} B_{l_n;l_k} e_{l_k} \quad b_{l_k}; \forall l_n \tag{3.21b}$$

$$\sum_{k} E_{l;l_k} x_{l_k} = D_l; \forall l \tag{3.21c}$$

And we also use the following model to choose the minimum reduce cost of the problem to determine which path we will choose to add the system.

(RC):

$$\min_{a_{ij}} \sum_{(ij)} [(l_{ij} \quad 1/2c_{ij}) a_{ij} \quad p^T l_{ij} a_{ij}] \tag{3.22a}$$

$$\text{s.t.} \sum_{j} a_{ij} = \sum_{j} a_{ji}; \forall i \in o; i \in d \tag{3.22b}$$

$$\sum_{j} a_{oj} = 1; \sum_{j} a_{jd} = 1; \tag{3.22c}$$

$$a_{ij} \in \text{Bin}; \forall (ij) \tag{3.22d}$$

Where p is the dual value for problem SPC.

And we will also use the algorithm 6 to find out which constraints we need to add to our model

SPC_2 for our system.

Theorem 4. If $SPC_1 = SPC_2$, then $SPC = P$. Where the new series of columns to add is the path with the maximum reduce cost of problem RC. And the new series of constraints to add to the system is the time cost for the path with the minimum time cost except that l_k pathes that already existed after current iteration for OD pair l.

Proof. As we know we will have the following relation $SPC_2 \quad SPC \quad SPC_1$. This is because first the feasible region for SPC is actually a subset with SPC_1 for which to let the variable x_{l_k} to be zero in SPC, so for minimization problem, $SPC \quad SPC_1$. Second, the only difference of the feasible region with SPC and SPC_2 is that SPC_2 has a new serious of constraints. So $SPC_2 \quad SPC$.
Now if $SPC_1 = SPC_2$, it means $SPC_1 = SPC = SPC_2$. The new serious of columns were chose to add from SPC to SPC_1 is from the maximum reduce cost, so if $SPC_1 = SPC$, it means that $SPC = SPCN_1$. And the new serious of constraints is the constraints that violate most for the left pathes. If $SPC_2 = SPC$, it means that the most possibly violation constraints are still within the current feasible region. So now $SPC = SPCN_2$ and then $SPCN_1 = SPC = SPCN_2$. Also as we know $SPCN_1 \quad P \quad SPCN_2$. Then now we will have $SPCN_1 = P = SPC = SPCN_2$. So $SPC = P$. □

As we illustrated above we totally have four levels of optimizations, we find actually we do not need the accuracy too much in the lower levels when the higher levels are just in the beginning several iterations. It means for the first several iterations of the higher levels we just need coarse control by calculating the lower levels. But together with the iterations become larger especially the last several iterations for the outer level, we need fine control by calculating the inner levels, that means we need to set the small tolerance for the check of convergency of the lower iterations.

Algorithm 3 LINEARIZATION

Step 0. Select $x(1) \in X$ such that

$$A^n x + B^n z + C^n b \quad R^n.$$

Set $k = 1$.

Step 1. Solve the following robust mixed linear problem:

$$\underline{T(x)} = \min_{b} \max_{x,z} \; x^{k^T} D^n x^k + d^T x^k (D^n x^k)^T (x \quad x^k)$$
$$\text{s.t.} \; A^n x + B^n z + C^n b \quad R^n$$
$$x \quad 0; z \in \text{Bin}$$

and let $x = y^k; b^k; z^k$ as the optimal solution, $p^k = y^k \quad x^k$ is the resulting search direction.

Step 2. Convergency Check:
Let $LBD = \max\{LBD; \underline{T(x^k)}\}$. If

$$\frac{T(x^k) \quad LBD}{LBD} < e;$$

then stop and $(b^k; x^k; z^k)$ is the solution. Else

Step 3. Line search.
Find a step length l^k which solves the following problem,

$$\min\{T(x^k + l p^k) | 0 \quad l \quad 1\}.$$

Update $x^{k+1} = x^k + l^k p^k$. $k = k + 1$, and go to step 1

Result

Simple Example

First we solve our models under a simple example, the data is from Lou's article [56]. it just has four nodes and six arcs in the system. The relative gure and data are shown in Figure 3.2 and Table 3.1, and we add an allowance for the surplus price is that the bound for the price is in $[e1, 1]$

Algorithm 4 Robust Transform

Step 0: Select $b(1) \in B$: Set $m = 1$. $UB = M$; $LB = -M$

Step 1: Solve WP-BRUES:

$$\text{(WP-BRUES):}$$

$$\max_{x,z} T(x) = e^T x \tag{3.23a}$$

$$\text{s.t. } A^n x + B^n z + C^n b_m \le R^n \tag{3.23b}$$

We can get x^{m+1}, z^{m+1} and p^{m+1}. Set $UB = \min\{UB; T(x^{m+1})\}$.

Step 2: Solve WP-BRUEM:

$$\text{(WP-BRUEM):}$$

$$\min_{b} T(\tilde{x}) = e^T \tilde{x} \tag{3.24a}$$

$$\text{s.t. } A^n \tilde{x} + B^n \tilde{z} + C^n b \le R^n \tag{3.24b}$$

$$e^T \tilde{x} \le e^T \tilde{x}^j; \quad 1 \le j \le m \tag{3.24c}$$

$$A^n \tilde{x}^j + B^n \tilde{z}^j + C^n b \le R^n; \quad 1 \le j \le m \tag{3.24d}$$

$$A^{nT} p^j \ge e^T \tag{3.24e}$$

$$\tilde{x}^j (A^{nT} p^j - e^T) = 0 \tag{3.24f}$$

$$p^j (R^n - A^n \tilde{x}^j - B^n \tilde{z}^j - C^n b) = 0 \tag{3.24g}$$

We can get b^{m+1} and $\tilde{x}$. Let $LB = T(\tilde{x})$. If

$$\frac{UB - LB}{LB} < \epsilon;$$

then stop and $b^{m+1}; x^{m+1}k; z^{m+1}$ is the solution. Else $m = m + 1$, go to step 1.

The results are shown in table 3.1, in this table the meaning of iteration in row k means the relative iterations by using the algorithms 1, i is algorithm 2 is j, algorithm 3 is k. And $\tilde{i}$ means the approximate times for the iteration of that algorithm, because for different outer iterations, we will

Algorithm 5 Convergency for Big Problem

Step 0: Set $k = 1$.

Step 1: Solve the problem SPC, if all of the variable $x_{l_k} > 0$. Go to step 3, else go to step 2.

Step 2: Delete all of the variables with the optimal value of 0 from the set J, then go to step 1.

Step 3: Solve the problem RC, insert the new founded path to set J.

Step 4: Solve the problem SPC_1 to get the lower bound of SPC as LB, and solve the problem SPC_2 to get the upper bound of SPC as UB. If

$$\frac{UB - LB}{LB} < e$$

stop, else go to step 1.

Algorithm 6 Add Constraint

In this algorithm we want to find out the shortest path for the OD pair l except the known path $l_1; l_2; \ldots; l_k$.

Step 0: Set p=1. Make the order of $l_1; l_2; \ldots; l_k$ from lower to higher. Without loss of generality, we can assume $l_1 \leq l_2 \leq \ldots \leq l_k$. Set the time for the path l_i as T_{l_i} for $i = 1; 2; \ldots; k$.

Step 1: Find the pth shortest path between OD pair l. The time for this path is $T_{l_p^0}$

Step 2: If $T_{l_p^0} = T_{l_p}$, $p = p + 1$. Go to step 1.
Else set $l_{k+1} = l_p^0$, stop

have different inner level iterations to be converge.

We can see from the table that by introducing the pricing strategy, we can make the system work better.

We can get the relative improvement by introducing the price for the best and worst case are relatively 0.38% and 0.82%. The improvement is not very large, this is because we mandatory to set the scale of the price within a certain scale.

Algorithm 7 pth shortest path

In this algorithm we want to find out the pth shortest path for the OD pair l. And set the rth shortest path as s_r

Step 0: Set $q = 1$.

Step 1: If $q \leq p$, set the time cost of one arc on path l_q to be big M for every $r = 1, 2, \ldots, q-1$. Then calculate the shortest path in the new network. and set the newly time cost as T_i for the ith combinations. After we calculate all of the possible combinations of different arcs on l_q, path $1, 2, \ldots, q-1$. We can choose the T_i with the minimum value as T_q and set the relative path of it is l_q. $q = q + 1$. Go to step 1.
Else stop.

Table 3.1: The Results of Four Nodes Example

	Case1	Case2	Case3	Case4
Total cost	49.82	51.31	49.63	50.89
Iteration	1*1*3	4*1*3	1*1*3	5*6*3
Time spend(s)	0.76	1.93	0.82	60

Check the Theorems

In this example, there are totally four pathes, we set them as p_1, p_2, p_3, p_4. We set the D=30. And the relatively $\frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;1} b_{ij} = 2.5$, $\frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;2} b_{ij} = 4$, $\frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;3} b_{ij} = 6$, $\frac{1}{2}\sum_{(ij)\in(IJ)} a_{ij;4} b_{ij} = 4.5$. Then

$$\bar{r} = \max_{1 \leq m, l \leq n} \sum_{(ij)\in(IJ)} a_{ij;m} a_{ij;l} b_{ij} = 3.5$$

The calculation results for different r are shown in table 3.2.

Table 3.2: The Results of Four Nodes Example

r	3.3	3.4	3.5	3.6
x_1	15.8625	15.8375	15.8125	15.8125
x_2	0.125	0.125	0.125	0.125
x_3	14.0125	14.0375	14.0625	14.0625
x_4	0	0	0	0
T_1	36.85	36.8	36.75	36.75
T_2	38.25	38.25	38.25	38.25
T_3	40.15	40.2	40.25	40.25
T_4	38.875	38.875	38.875	38.875
T_1 $\frac{1}{2}$å$_{(ij)2(IJ)}$ $a_{ij;1}$ b_{ij}	34.35	34.3	34.25	34.25
T_2 $\frac{1}{2}$å$_{(ij)2(IJ)}$ $a_{ij;2}$ b_{ij}	34.25	34.25	34.25	34.25
T_3 $\frac{1}{2}$å$_{(ij)2(IJ)}$ $a_{ij;3}$ b_{ij}	34.15	34.2	34.25	34.25
T_4 $\frac{1}{2}$å$_{(ij)2(IJ)}$ $a_{ij;4}$ b_{ij}	34.375	34.375	34.375	34.375
f(x)	1151.916	1151.908	1151.906	1151.906

All of the data accords with the theorems.

Nine Nodes Example

The data we use is still from the article [56], and the the network is shown in gure 2. We can see from the result that the relative improvement by introducing the price for the best and worst case are relatively 1.82% and 2.01%

Figure 3.3: The nine nodes network

Table 3.3: The Results of Nine Nodes Example

	Case1	Case2	Case3	Case4
Total cost	2347.15	2532.68	2305.74	2481.89
Iteration	1*1	7*1	1*1	6*11
Time spend(s)	1.02	5.88	1.32	180

Sioux Fall System

Table 3.4 is the result with two original places and two destinations, so totally have 4 OD pairs. Figure 3.4 gives us the compare for different boundedly rationality coefficient influence to our system. Such results are got by not using the algorithm 6 and 7. We just still check the path for

the constraints that get from the column generation. But actually such method can not guarantee us this is a convergency result for our initially problem. This just give us a comparison result. And the calculation time for Case 4 are just about 2 hours.

Table 3.4: The Results for Sioux Fall network with 4 OD pairs

Total Cost	Case1	Case2	Case3	Case4
r = 0:05	2347.66	2446.37	2311.49	2370.36
r = 0:1	2312.59	2502.38	2297.68	2386.76
r = 0:2	2298.28	2715.69	2295.46	2462.38

Figure 3.4: The Sioux Fall Network

So we still want to add our algorithm 6 and 7 to our system. These results are shown in table 3.5, in such results we can nd that, all of the best cases of our problem we have higher results compare

to the former method, and all of the worst cases of our problem we can get lower results. This is because when we use the former method, actually we just check the rightness for the column generation, so our results actually is the results for SPC_1. Here for the best case, it is the lower bound for the real results, and for worst case it is the upper bound for the real results. The gure 3.5 shows the relations for the new method. And we give an upper bound of the number of pathes for nding the new pathes to add to check the convergency for the constraints as 15. We can nd from the two tables that the most relative difference of the two methods is 4.0%. It is not a big difference, but still the new method is more close to the real result. But the calculation time for case 4 is about 33 hours. It is more than the former method.

Table 3.5: The Results for Sioux Fall Network With 4 OD Pairs By Add Constraints

Total Cost	Case1	Case2	Case3	Case4
r = 0:05	2352.45	2444.58	2313.22	2365.18
r = 0:1	2314.29	2495.26	2304.43	2382.47
r = 0:2	2306.76	2703.72	2299.28	2453.28

Figure 3.5: The Sioux Fall Network 2

Then we nd out that because we have totally four levels of iterations, so actually for the beginning several iterations of the outer levels, actually we do not need to require the high accuracy for the inner level. By introducing this to our algorithms. We change the number of 15 we mentioned above to 5 for the rst ten iterations of our Linearization level, and still keep 15 for the last iterations of the rst level. Then we get some new results below, we can know from the table 3.6 that the most relative different between these two methods is only 0.04%. But this can save a lot of calculation time for our problem. The calculation time for case 4 is about 21 hours. But how to get the most ef cient and accuracy for setting the convergency, it still need us to have further research. Here is just an illustration that we can get the similar results compared by using the same convergency principles.

Table 3.6: The Results for Sioux Fall Network With 4 OD Pairs By Add Constraints With Different Accuracy

Total Cost	Case1	Case2	Case3	Case4
$r = 0.05$	2353.16	2444.29	2313.76	2365.88
$r = 0.1$	2313.79	2496.32	2304.12	2381.86
$r = 0.2$	2306.22	2703.34	2299.76	2453.43

Figure 3.6: The Sioux Fall Network 3

Then this is the results with multiple OD pairs, the pairs were shown in [52]. We use $r = 0.1$. And for each level, we set the tolerance to be 2%, then results were shown in table 3.7. We did not use the algorithm for this problem. And the time for calculation is about 10 hours.

Table 3.7: The Results for Sioux Fall network with multiple OD pairs

Total Time Cost	Case1	Case2	Case3	Case4
	33468.1	35776.3	32867.4	34368.7

CHAPTER 4: INFORMATION NETWORK CASCADING ANDNETWORK RE-CONSTRUCTION WITH BOUNDEDLY RATIONALUSER BEHAVIORS

NOMENCLATURE

A. Sets, Indices for the Equilibrium Model

N	Set of users index by i,j,k
L	Set of information indexed by l
i	Index of information provider

B. Parameters

p_{jl}	Information post plan of user j
d_{ki}	The value for the connection bene t to user i if followed by user k
v_{kl}	Value of information l to user k
b_{kl}	Unit boring value for user k to receive information l for multiple times
T_{jl}	Threshold for user j to re-post information l
$\hat{x}_{ij}$	Initial network connection for user i follows to user j
r_j	BRUE coef cient for user j
c_l	Posting cost for information l
B_i	Budget for information provider i

C. Variables

p_{jl} Binary variable for user j whether to post or re-post information l

x_{kj} Binary variable for user k whether to follow user j

z_{kl} Number of times for user k to receive information l

g_{kl} Binary variable for user k whether to receive information l

U_k Total information utility for user k

U_{kl} Information utility for user k from information l

D. Function

$F_{kl}(\ )$ Relation function for U_{kl} respect to z_{kl} for user k with information l

$G(\ )$ Linear threshold function for user's re-post decision p_{jl} respect to information provider's post plan p_{il}

Introduction

With the development of information technology, the social media platform plays a vital role in most people's life. For some commercial users or non-pro t organizations, their pro ts or in u-ences can increase by using the social media system [57, 65]. Especially for some commercial users that highly depend on the social media platform such as News Media, YouTuber or We Media Organization. Lots of people are willing to expand their network connection in social media in order to expand their in uences. The number of followers will dramatically in uence their pro ts or in uences.

The behaviour of user's connectivity to a information provider is primarily in uenced by two aspects [90]: the content of information posted by the information provider[89], and the personality of the user who follows the information provider. In order to expand the connections of the information provider, we need to study for what kind of content to post and also study the human

personality and human behaviours. The increasing rate of the follower will highly depend on what kind of information they post. We call these users need to make choice for information post plans as information provider. In this paper, we will optimize the information provider's information post plan in order to expand the network connections of it. Here, the information post plan means in one time period the information provider need to decide to post and not post what kind of information. The other part need to consider is different people will have different personalities, their interesting items can varies widely. And different people will also have different criteria for connectivity action.

The in uence of the information to one user can only be activate if this user can receive the information. We need to consider about the information cascading route. Linear threshold model [33, 19, 14] and independent cascading model [46] are two widely used models for information cascading. in this paper, linear threshold propagation model is used for information cascading. This model was rst proposed by Granovetter [34] to describe the people's behaviour. It means when the linear summation of the in uence of one user's followees exceed the threshold the this user. Then this user will become active. In our case, it means when some of one user's followees re-post the information and the linear summation of these followees in uence exceed the threshold of this user, then this user will also re-post this information.

After information cascades in the network, each user in the network will have a decision whether they have willing to change their followees. Users can get utility from each information, they want to have a network that can get more information they want to get. But they still want to avoid multiple times to receive the same information. because it is actually a redundant for them to get new information. We apply the concept bounded rationality user equilibrium (BRUE) as the decision principle of user's for actions of connectivity. It means one users want to choose connective schedule that can help them to achieve higher utility for different information but not need the maximum utility.

This idea of BRUE originally comes from Simon's Theory [77] in 1957. In this paper, it tells us that human behavior will has its bounded rationality. In 1972, Simon published another paper [79] which gives us an fundamental illustration about the theory of bounded rationality. The rationality of human beings will have the style of their behaviour's utility to achieve a percentage of the ideal goals, within the limits by given conditions and constraints. Simon also continuously worked on bounded rationality [80, 81] to expand the application of the theory.

The concept of bounded rationality can also be used in many elds, such as the energy system [96], psychology [44], military [69], transportation [58, 24, 56]and so on [70, 36, 27]. But as far as we know, fewer people used it on information network system. We suppose that in the information network system users' action of connectivity will obey BRUE. It means one user do not need to get the maximum information utility they can theoretically get. They just need their utilities are greater then a percentage of the maximum utility. They may execute any connection plan's utility that ful ll such criteria.

Some researchers used game theory model in social media network to determine the user's decision [82, 85]. But in the real world, the BRUE model should be more close to the natures compare to game theory model. First, the users in the information networks system will not take care about the little difference for their utility function. Second, in the information network system the utility function is not an exactly function, no one can know the exactly utility function value by using the information network, it is just an appropriate value. Third, based on the reinforcement learning [43], the users in the information networks system will also obey the BRUE principle. Because there should take some time or steps to get to the optimal condition. And before it gets the optimal condition, the topology already changed. So it is actually But in the information networks system, there does not have long enough time to let the equilibrium occurs before the next post come out. Compare to the game theory model, BRUE model should be used for the users' behavior in the information networks system. Some recent research tells us that bounded rationality of

individual users will in uence the information network. Kasthurirathna and Piraveenan [45] made the simulation for a number of strategic games. Then they regenerated the network so that the network on average converged towards Nash equilibrium, despite the bounded rationality of nodes. The link between bounded rationality distributions and social structure is important in explaining social phenomena.

We generate a three-level mathematical optimization model. The rst level is to optimize the information post plan of information provider in order to maximize its connections. The second level is to optimize the human behaviours of other users under BRUE. It has two formats. In optimistic condition, we maximize human behaviours for the connections of information provider. But in pessimistic condition, we minimize these variables. The reason we have two conditions is because, as we discussed before, by introducing BRUE, users' behaviours will drop in an uncertainty set. We have interesting in how best and how worst this uncertainty will in uence the information provider's network connections. That is the reason we study the optimistic and pessimistic conditions for BRUE. The third level is to calculate the maximum information utility for one user can get, which need to be used in second level for BRUE constraints.

We solve a small-scale synthetic network by exact algorithms. But for large-scale network, the calculation time is increasing exponentially. We tackle this problem by using large neighbourhood search (LNS) algorithms. It is a heuristic algorithms [50] used to solve large-scale problem. It is an effect way to nd a good solution quickly when the time to nd the global optimal solution is too long. The main idea of this method is to block the local optimal solution and then nd its neighbourhood to get a new solution. Even though sometimes the second solution is not as better as the rst one. But by using such ways, sometimes the new solution can get rid of one local optimal solution. Then we can nd another local optimal solution that is better than the rst local solution we nd.

Math Model Formulation

We propose the following model to information network system. Our objective is to maximize our information provider's connections by controlling its information post plan. With different post plan, rstly, the information cascading process will be different. Secondly After information cascades, some users may have a choice to connect to a new followee or disconnect to an exiting followee. Different information post plan of our information provider will lead to different number of its followees. We use the linear threshold principle to determine users' information post behavior. And use BRUE for users to simulation the network reconstruction after information cascading.

Linear Threshold Model for Information Cascading

We use the linear threshold propagation model to determine whether or not one user decides to re-post the information when this user receive it several times. When the summation of the in uence of one user's followers who post the information exceeds this users threshold, it will choose to re-post this information. We will give the detail of the linear threshold constraint inside the model CR later. Figure 4.1 gives one example of the procedure for information cascading process and the nal connection of the network after the information cascades.

Figure 4.1(a) is the initial network of follower's linking network, the link and arrow between node 3 and node 5 means user 3 follows user 5. Some links between two users have two arrows mean these two users follow with each other. In this example our information provider is node 10 in black. And we just simulate for the cascading of one kind of information. Figure 4.1(b) is the rst step cascading, by principle of linear threshold the in uence of node 10 is greater than the threshold of node 5 and node 11. Then after node 10 post this information, node 5 and node 11 re-post it. We

(a) Initial Graph (b) First Step Cascading (c) Second Step Cascading

(d) Third Step Cascading (e) New Followers (f) New Unfollowers

Figure 4.1: Cascading for the information and the nal connection network

mark the re-post node in red color. Figure 4.1(c) is the second step of cascading, we can nd that now node 3 and node 8 also re-post this information. Even though node 3 already follow node 10 directly in the initial network, but the in uence of node 10 does not beat the threshold of node 3. At that time the in uence of 10 to 3 is less than the threshold of node 3. After the rst step node 5 also re-post this information and node 3 can receive this information from both node 5 and node 10. And currently the summation of the in uence of node 5 and node 10 is over the threshold of node 3, so node 3 re-post this information in the second step. Figure 4.1(d) is the third step of cascading, the similar reason for node 9 and node 7 to re-post the information. And after gure 4.1(d), the information cascading will stop. Because the node 1,2,4,6 will not re-post this information any more. And the network become stable. Figure 4.1(e) adds two new followers to our target node 10 after the cascading based on our BRUE model. Figure 4.1(f) shows that node 9 determine to un-follow node 10 based on the BRUE model. So from the principle of linear threshold, if we know the post plan of the information provider, we can know the information cascading route and procedure. Our next step is to optimize for the action of connection for other users in the system,

which will follow the BRUE constraints.

BRUE Model

BRUE is the math model's equilibrium constraints from the bounded rationality. We suppose one user in the system have multiple choices, for choice i has utility $U(i)$. With out loss of generality, we can set the choice i has the optimal utility value. Then BRUE tells us that for any choice i has the following property will be deemed as the possible choice for this user.

$$U(i) \quad r \quad U(i \ ): \tag{4.1}$$

Where r is called the bounded rationality coef cient. And we must have r 1 because of the optimality of the choice i . From this constraint, we can know that by introducing BRUE to our math model, we will have an uncertainty feasible region for the users. And when $r = 1$, it is the perfect rationality user equilibrium (PRUE). And it is actually the nash equilibrium. Beacuse it means the user can only accept the plan's utility to be the maximum utility, and it has no motivation to move to another paln. Game theory nash equilibrium is a special condition of BRUE when $r = 1$

In this paper, after information cascades, users may change their connections. We assume the way of users' choice to connect or disconnect will obey BRUE constraints, which is also the nature of human beings.

Pessimistic Condition

The following model (CP) works for our information provider to maximize the connections by determine their post plan. It is constructed under the pessimistic condition by introducing the

BRUE constraints. pessimistic condition means under BRUE constraint the behaviour of the other users in the system act in the worst case for our information provider. The other users' choice to connect or dis-connect will lead to the minimization the number of the information provider's connections.The rst level is to nd the best choice for the information provider to maximize the possible worst case. In this model it also has the third level, it is to maximize for the information utility of each user in the system except the information provider. This information utility need to be used in BRUE constraint. But this value also depends on the variables in the rst level.

(CP_I):

$$\max_{p_{il}} \min_{x_{ki}} BP(x_{ki}) = \sum_{k \in N; k \neq i} d_{ki} x_{ki} \tag{4.2a}$$

$$\text{s.t.} \quad U_k = \sum_l U_{kl}; \qquad \forall k \in N; \tag{4.2b}$$

$$U_{kl} = F_{kl}(z_{kl}); \qquad \forall k \in N; \forall l \in L \tag{4.2c}$$

$$z_{kl} = \sum_{j \in N; j \neq k} x_{kj} \; p_{jl}; \qquad \forall k \in N; \forall l \in L \tag{4.2d}$$

$$p_{jl} = G(p_{il}); \qquad \forall j \in N; j \neq i; \forall l \in L \tag{4.2e}$$

$$x_{ki} \quad \sum_l \sum_{k' \in N; k' \neq i; k' \neq k} p_{k'l} \quad x_{k'k} + \hat{x}_{ki}; \qquad \forall k \in N; k \neq i; \forall l \in L \tag{4.2f}$$

$$x_{ij} \in \{0, 1\}; \qquad \forall i, j \in N; i \neq j \tag{4.2g}$$

$$U_k \quad U_k \quad r_k; \qquad \forall k \in N; k \neq i \tag{4.2h}$$

$$U_k = \max_{x'_{kj}} U'_k; \qquad \forall k \in N; k \neq i \tag{4.2i}$$

$$\text{s.t.: } (4.2b') \quad (4.2g') \tag{4.2j}$$

In model CP_I, the objective function is the total connection bene t the information provider

get from the network.

The first level's decision variable p_{il} is a binary variable to indicate whether the information provider i will post information l to the system, this kind of variable can be controlled by the information provider.

In the second level x_{ij} is a binary variable, when it is equal to 1, it means user i follows user j in the network. else $x_{ij} = 0$. Constraint (4.2b) gives the total information utility can get for user k equals the summation of the utility from each information in the system. We denote the function as $F_{kl}(\ )$. Where

$$F_{kl} = \begin{cases} v_{kl} \quad (z_{kl} \quad 1) \quad b_{kl}; & 8 z_{kl} \quad 1 \\ 0; & z_{kl} = 0 \end{cases}$$

Two examples for the utility from the information respect to the number of times of one user to receive that information is showed in figure 4.2. Figure 4.2(a) has the parameter $b_{kl} > 0$ and figure 4.2(b) has $b_{kl} < 0$. Actually This function can be linearized, it is given later in (LN).

The relation of the variable p_{jl} and variable p_{il} is restricted by linear threshold principle, we denote their relation in function $G(\ )$. This function can not simply write out in formula. We just show the mechanism to determine p_{jl} from p_{il} in algorithm 8 ALG-LT, where T_{kl} is the information re-post threshold for user k with information l. We can notice that if we get the value of p_{il}, we can directly get the value of p_{jl} by linear threshold principle.

Constraint (4.2d) gives that for user k, the frequency of information l it gets equals to the number of its followee who re-post information l. It will be a mixed linear constraint by given p_{il}.

Constraint (4.2f) shows that user k can not have the choice to follow the information provider i if it does not follow user i originally and there has no followee of user k re-post any information generated by information provider i. $\hat{x}_{ki}$ is the originally connection from user k to user i.

Constraint (4.2h) is the BRUE constraint. It gives out that user k can accept any follow-unfollow plan for which information utility drop within the BRUE gaps. Where r_k is the BRUE coef cient for user k. Constraint (4.2i) is the third level problem. It calculates the maximum information utility that user k can get in the system. The constraints of the third level has the same formula compare to the constraint (4.2b)-(4.2g) in the second level. But they did not share the same variable x_{jk} and U_k. We should replace all of the relative variable in the second level with new variable x^0_{jk} and U^0_k in third level. This is the reason we write the constraints in third level (4.2b^0) (4.2g^0) .

(a) Decreasing utility function

(b) Increasing utility function

Figure 4.2: Relation of Information Utility to Times User Receive this information

In addition, we propose another information utility function of all user $k \in N$ for information l. It shows in gure 4.2(b). It means when a user receive the same information multiple, it's utility is increasing. In this case, we consider that if one user receive the information from other user's re-post, it means the other user likes this information. Then this information can have commonality among the user's friend circle. Then the initial value of the information will be larger if this user receive the information more times. For example, if one user has a lot of friends who re-post one information of the result of super bowl, then this user should feel this information will have more utility value to him/her. Because if he/she get this information, it is more easy to chat about it